Indag. Mathem., N.S., 18 (3), 339–348 September 24, 2007

On the motive of certain subvarieties of fixed flags ☆

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Communicated by Prof. J.P. Murre at the meeting of May 21, 2007

ABSTRACT

We compute the Chow motive of certain subvarieties of the Flags manifold and show that it is an Artin motive.

INTRODUCTION

Let $\mathbf{G}$ be a connected algebraic semisimple group defined over an algebraically closed field $\mathbb{K}$, with universal separable covering. We denote by U the variety of unipotent elements of $\mathbf{G}$ and by B a Borel subgroup of $\mathbf{G}$. It is well known (see [6]) that if $\mathbf{B} = \mathbf{G}/B$ and

$$Y := \{(x, gB) \in U \times \mathbf{B} \mid g^{-1}xg \in B\}$$

then

$$\pi : Y \to U$$

is a desingularization, π denotes the natural projection.

Consider the case $\mathbf{G} = SL_n$, then it is easy to see that the variety of complete flags is $\mathbf{B} = \mathbb{F}$. For this case we have that for any unipotent element x the fiber

MSC: 19E15, 14C25

☆ Partially supported by DGAPA IN105905.

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$\pi^{-1}(x)$ is isomorphic to the variety of fixed flags. Moreover, J.A. Vargas (see [7]) has given a description of a dense open set for every irreducible component of the fiber, and N. Spaltenstein (see [4]) has constructed a stratification of the fiber, which unfortunately is not completely compatible with the decomposition into the irreducible components.

The purpose of this work is to describe the motive of the irreducible components of the fibers $\pi^{-1}(x)$ when x is of type (p,q). Observe that it is interesting to know the geometry and K-theory of the fibers of the desingularization since algebraic singularities appear within the unipotent variety. It is also important for applications such as computing zeta functions and counting points over finite fields.

The paper is divided as follows. In Sections 1 and 2 we introduce basic notation and the stratification given by Spaltenstein. In Section 3 it is given a description of the irreducible components of the fiber of x for any unipotent element x of type (p,q). In Section 4 we use the above description to compute the Chow motive, and show that the image of the Chow motive of any irreducible component into the Voevodski's category is an Artin motive. We also compute the motive of some irreducible components of a slightly more general type, showing that these motives are extension by Artin motives of the motive of a product of flag varieties.

1. PRELIMINARIES

Let $\mathbf{G}$ be the group SL_n with coefficients in an algebraically closed field $\mathbb{K}$. Consider a Borel subgroup B of $\mathbf{G}$ and T a maximal torus on B. If V is a $\mathbb{K}$-vector space of dimension n, then the variety $\mathbf{B}$ is isomorphic to the variety of complete flags $\mathbb{F} = \mathbb{F}(V)$.

For any unipotent element $x \in U \subset \mathbf{G}$, we denote the fiber $\pi^{-1}(x)$ by $\mathbb{F}_x$. One says that x is of type $(\lambda_1, \dots, \lambda_s)$ if the Jordan canonical form of x consists exactly of s blocks of sizes $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$.

If we write $x = 1 + n$ where n is the nilpotent part of x, then there is a basis

$$\{e_{i,j} \mid 1 \leqslant j \leqslant s, 1 \leqslant i \leqslant \lambda_j\}$$

of V adapted to x in the sense that $n(e_{i,j}) = e_{i-1,j}$, where $e_{0,j} = 0$ for every j. Therefore we can write $V = V_1 \oplus \cdots \oplus V_{\lambda_1}$ and $n : V_{\lambda_1} \to V_{\lambda_1 - 1} \to \cdots \to V_1 \to 0$, where V_i is the space generated by $\{e_{i,j}\}$ with i fix.

It is well known that if $x \in U \subset SL_n$ is unipotent of type $(\lambda_1, \dots, \lambda_s)$, then the fiber $\mathbb{F}_x$ has as many irreducible components as standard tableaux of type $(\lambda_1, \dots, \lambda_s)$. The shape of a standard tableau is as shown in Fig. 1, where the numeration strictly decreases from the top to the bottom and from left to right. Spaltenstein constructed a stratification of $\mathbb{F}_x$. We will use, and explain, the particular case where x is of type (p,q).

Given $x \in U \subset SL_n$ of type (p,q) and a $\mathbb{K}$-vector space V of dimension n, consider a basis $\{e_{i,j} \mid 1 \leqslant j \leqslant 2\,, 1 \leqslant i \leqslant \lambda_j\}$ of V adapted to x, as mentioned above. Given a number $t \in \{1,2\}$ we define some subsets of $\mathbb{F}_x$ as follows:

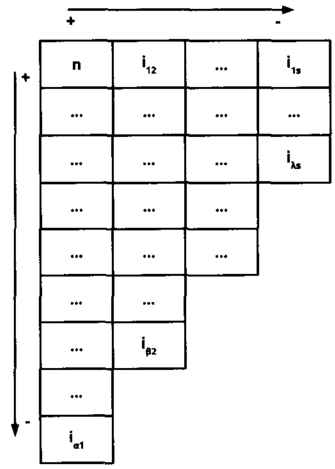

Figure 1. Standard tableau of type $(\alpha, \beta, \dots, \lambda)$.

1. For $t = 1$, let $X_1 = X_1(V)$ be the set of flags $F_1 \subset \cdots \subset F_n \in \mathbb{F}$ such that $F_1 := \langle e_{1,1} \rangle$.
2. For $t = 2$, let $X_2 = X_2(V)$ be the set of flags $F_1 \subset \cdots \subset F_n \in \mathbb{F}$ such that $F_1 := \langle a e_{1,1} + e_{1,2} \rangle$ for some number $a \in \mathbb{K}$ not necessarily different from zero.

We also define inductively the sets

$$X_{ij}(V) := \{F_1 \subset \cdots \subset F_n \in X_j(V) \mid F_2/F_1 \subset \cdots \subset F_n/F_1 \in X_i(V/F_1)\}.$$

Applying the same process to $X_{ij}(V)$, we get sets of the form $X_{ijk}(V)$ and so on. It is not difficult to see that after n times we end up with a locally closed subset of flags which actually belongs to $\mathbb{F}_x$, since a flag $F := F_1 \subset \cdots \subset F_n$ is in $\mathbb{F}_x$ if and only if $n(F_i) \subset F_{i-1}$ for every i. Moreover, the sets we obtain form a stratification of $\mathbb{F}_x$.

It is clear from the construction that all spaces of the stratification are affine spaces and that the affine strata of maximal dimension (which is precisely q if x is of type (p, q)) are open disjoint sets. You can also count the number of irreducible components of $\mathbb{F}_x$, which coincides with the number of standard tableaux of the given type.

Remark 1.1. The natural projection from $X_1(V)$ to $\mathbb{F}(V/F_1)$ is an isomorphism.

Remark 1.2. Given a nilpotent element $x \in U$ of type $(\lambda_1, \dots, \lambda_s)$, a standard tableau σ as above, a $\mathbb{K}$-vector space V of dimension $n = \lambda_1 + \cdots + \lambda_s$ and a basis $\{e_{i_1,\dots,i_s}\}$ of V adapted to x, there is a maximal affine space among those obtained from Spaltenstein's stratification of $\mathbb{F}_x$, which can be associated to σ.

Let $\psi : \{1, \dots, n\} \to \{1, \dots, s\}$ be the function defined as $\psi(k) = i$, if k appears in the ith column of σ, counting from left to right. Now, let us consider the stratum $X_\sigma := X_{\psi(n)\cdots\psi(1)}$, then X_σ is of maximal dimension and the corresponding irreducible component will be denoted by Y_σ.

2. DECOMPOSABLE IRREDUCIBLE COMPONENTS

Let $A := \{a_1, \ldots, a_r\}$ be a totally ordered set and we assume that $a_1 < \cdots < a_r$. Let $\phi : A \twoheadrightarrow \{1, \ldots, s\}$ be a surjective map. Then the pair (A, ϕ) induces a tableau $\sigma := \sigma(A, \phi)$ as follows.

The tableau σ has s columns, the ith column of the tableau has $b_i := |\phi^{-1}(i)|$ boxes and one fills them up according to ϕ, i.e. starting with r and then going down to 1. The number k should appear in the column $\phi(k)$. The numbers appearing in a column are in decreasing order, and the tableau will be of type $(b_1, \ldots, b_s)$.

Not every tableau obtained in this way is a Young tableau, unless $b_1 \geqslant \cdots \geqslant b_s$, but it might be no standard. For instance, the tableau associated to $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = 2$ and $\phi(5) = 3$ is a Young tableau but it is not standard, whereas the tableau associated to $\phi(1) = \phi(4) = 2$, $\phi(2) = 3$ and $\phi(5) = \phi(3) = 1$ is a standard Young tableau.

If ϕ is a decreasing bijection and $|A| > 1$ we say that the pair (A, ϕ) is of flag type.

Remark 2.1. Any standard Young tableau of type $(\lambda_1 \ldots \lambda_s)$ is obtained from the pair $(\{1, \ldots, n\}, \psi)$, where $\psi : \{1, \ldots, n\} \to \{1, \ldots, s\}$ is the function described in Remark 1.2.

Definition 2.2. *A standard Young tableau σ of type $(\lambda_1 \ldots \lambda_s)$, with $n = \lambda_1 + \cdots + \lambda_s$, is called* decomposable *if there is a partition $\{1, \ldots, n\} = (\bigsqcup_k A_k) \sqcup (\bigsqcup_t B_t)$ by totally ordered sets such that the pair $(A_k, \psi|_{A_k})$ is of flag type for all k, i.e. if $|A_k| = m_k$ then $\psi|_{A_k}$ is a decreasing bijection between A_k and $\{1, \ldots, m_k\}$, and the pair $(B_t, \psi|_{B_t})$ induces a standard Young tableau of type (p_t, q_t) for every t, where ψ is as above. In particular,* $\mathrm{im}\,(\psi|_{B_t}) = \{1, 2\}$.

Remark 2.3. A pair (A, ϕ) is of flag type if and only if the associated standard tableau σ is of type $(1, \ldots, 1)$, which corresponds to the identity in SL_n. In this case there is only one irreducible component, namely the whole flag variety.

3. IRREDUCIBLE COMPONENTS OF $\mathbb{F}_X$ FOR X OF TYPE (P, Q)

The description of the irreducible components of type (p, q) given here is already in [1]. However, the present proof is better than the one that appears there, and it is also important to incorporate the description here for the sake of completeness.

Let $x = 1 + n$ be a unipotent element of type (p, q) in SL_n with nilpotent part n, V a $\mathbb{K}$-vector space of dimension $n = p + q$ and $\mathbb{F} = \mathbb{F}(V)$ the variety of complete flags on V.

We have the following related three lemmas.

Lemma 3.1. *The set*

$$A = \{F := F_1 \subset \cdots \subset F_n \in \mathbb{F} \mid n^i(F_k) \subset F_m\}$$

is closed in $\mathbb{F}$ *for all fixed* $i, k, m \in \mathbb{N}$. *We set* $F_0 = 0$.

Lemma 3.2. *The set*

$$L = \{F := F_1 \subset \cdots \subset F_n \in \mathbb{F} \mid n^i(F_k) \subset S\}$$

is closed in $\mathbb{F}$ *for all fixed* $i, k \in \mathbb{N}$. S *is a fixed subspace of* V.

Lemma 3.3. *The set*

$$H = \{F := F_1 \subset \cdots \subset F_n \in \mathbb{F} \mid \dim(F_r + n^k(F_r)) \leqslant d\}$$

is closed in $\mathbb{F}$ *for all fixed* $r, k, d \in \mathbb{N}$.

For these lemmas one shows that the corresponding set (either A, L or H) is a determinantal set and therefore algebraic.

Let $\pi_r : \mathbb{F} \to \mathbb{G}r(1,n) \times \cdots \times \mathbb{G}r(r,n)$ be the composition of the natural embedding of $\mathbb{F}$ in $\mathbb{G}r(1,n) \times \cdots \times \mathbb{G}r(n-1,n)$ followed by the projection to the first r factors.

Theorem 3.4. *Let* σ *be a standard tableau of type* (p,q), *and* Y_σ *the corresponding irreducible component of* $\mathbb{F}_x$. *Let* $1 \leqslant r \leqslant n$ *be a natural number and set* $Y_\sigma(r) := \pi_r(Y_\sigma)$. *We set* $Y_\sigma(0)$ *to be a fixed point. Let* f_r *be the natural projection* $Y_\sigma(r) \to Y_\sigma(r-1)$, *then for all* $p \in Y_\sigma(r-1)$ *one has*

$$f_r^{-1}(p) = \begin{cases} 1pt & \text{if } r \text{ appears in the left column of } \sigma, \\ \mathbb{P}^1 & \text{if } r \text{ appears in the right column of } \sigma. \end{cases}$$

In particular, $Y_\sigma(r) \to Y_\sigma(r-1)$ *is either a* $\mathbb{P}^1$*-bundle or an isomorphism for every* $1 \leqslant r \leqslant n$.

Proof. Let us assume that there exist a 2-dimensional Hermitian $\mathbb{K}$-vector space W (see II below). The more general situation follows in a similar way, but it does not give more light into the geometry of the problem, which is our interest.

As mentioned before, there is an open subset X_σ of the irreducible component Y_σ. Consider the image $X_\sigma(r) := \pi_r(X_\sigma)$ of this set under the map π_r. It is not hard to observe that the fibers of f_r restricted to $X_\sigma(r)$ are isomorphic to a point or to an affine line, depending on whether r appears in the left or in the right column of σ. Now, we need to consider the map f_r as a map from $Y_\sigma(r)$, and for this we need to give a good description of Y_σ. We will do these in four steps, which involve different cases. The three last lemmas will be used.

I. Fix a basis $\{e_{i,j}\}$ of V adapted to x as before, and let σ be a standard Young tableau of type (p,q). The map $\psi : \{1,\dots,n\} \to \{1,2\}$ defined in Remark 1.2 induces a partition $\{1,\dots,n\} = (\bigsqcup_{k=1}^{t} A_k) \sqcup (\bigsqcup_{k=1}^{t} B_k)$ such that $A_k \subset \psi^{-1}(1)$ and $B_k \subset \psi^{-1}(2)$ are made up by consecutive integers for all k. For $1 \leqslant i \leqslant t$ define $a_i = |A_i|$, $b_i = |B_i|$, $s_j = \sum_{i<j} a_i$ and $S_j = \sum_{i<j} b_i$. We also set $a_0 = 0$ and $b_0 = 0$.

Because σ is a standard tableau, it is clear that either $A_1 < B_1 < A_2 < B_2 < \cdots < A_t$ and $B_t = \emptyset$ or $B_1 < A_1 < B_2 < A_2 < \cdots < B_t < A_t$, where $C < D$ means $c < d$ for all $c \in C$ and all $d \in D$.

Without loss of generality we can assume $B_1 < A_1 < \cdots < B_t < A_t$. Indeed, if $A_1 < B_1$ then $A_1 = \{1,\dots,a_1\}$, in this case $p - a_1 > q$ since σ is a standard tableau and the numeration inside of it decreases to the right. Therefore, for every flag $F_1 \subset \cdots \subset F_n \in X_\sigma$ one has $F_i = \operatorname{im} n^{p-i}$ for $1 \leqslant i \leqslant a_1$. But these are closed conditions and so they are fulfilled by all flags in Y_σ. In this situation we have that remark 1.1 implies that $Y_\sigma \cong Y_{\sigma'}(V/(\operatorname{im} n^{p-a_1}))$; where σ' is the standard Young tableau induced by $(\{a_1+1,\dots,n\}, \psi|_{\{a_1+1,\dots,n\}})$.

II. We first consider the following case.

$\sigma =$

n	q
...	...
...	1
...	
q+1	

Let W be a Hermitian vector space of dimension 2 with basis $\{w_1, w_2\}$. For every point $(P_1,\dots,P_q) \in \mathbb{P}(W)^q$ write $P_i = (a_i : b_i)$. Let $R_i = (c_i : d_i)$ be the point in $\mathbb{P}(W)$ that represents the orthogonal complement of the subspace $\langle a_i w_1 + b_i w_2 \rangle \subset W$. For all $1 \leqslant j \leqslant q$ consider the vectors (in V)

$$P_{1,j} = a_1 e_{j,1} + b_1 e_{j,2},$$
$$R_{1,j} = c_1 e_{j,1} + d_1 e_{j,2}$$

and for all i, j with $i > 1$ and $i + j \leqslant q + 1$ define the following vectors:

$$P_{i,j} = a_i R_{i-1,j} + b_i P_{i-1,j+1},$$
$$R_{i,j} = c_i R_{i-1,j} + d_i P_{i-1,j+1}.$$

Now, with the notation described above, and for $1 \leqslant k \leqslant q$, let Y_k be the set giving by the following conditions. $F_1 \subset \cdots \subset F_n \in Y_k$ if and only if there is a point $(P_1, \ldots, P_q) \in \mathbb{P}(W)^k \times [\mathbb{P}(W) - \{(1:0)\}]^{q-k}$ such that

(1) $F_m = \langle P_{1,1}, \ldots, P_{m,1} \rangle$ for all $1 \leqslant m \leqslant q$.
(2) $F_{q+t} = \langle P_{1,1}, \ldots, P_{q,1}, R_{q,1}, \ldots, R_{q-t+1,t} \rangle$ for all $1 \leqslant t \leqslant q$.
(3) $F_s = \operatorname{Ker} n^s$ for all $2q \leqslant t \leqslant n$.

Observe that if the P_i's are different from $(1:0)$ for all i, then

$$\langle P_{1,1}, \ldots, P_{q,1}, R_{q,1}, \ldots, R_{q-t+1,t} \rangle = \langle P_{1,1}, \ldots, P_{q,1}, e_{1,1}, \ldots, e_{1,t} \rangle$$

for all t, therefore $Y_0 = X_\sigma$. This situation corresponds to: $F_1 \subset \cdots \subset F_n \in X_\sigma$ if and only if

(1) $F_1 \subset \operatorname{Ker} n - \langle e_{1,1} \rangle$,
(2) $n(F_i) \subset F_{i-1}$ but $n(F_i) \not\subset F_{i-2}$ for every $2 \leqslant i \leqslant q$,
(3) $\operatorname{Ker} n^t \subset F_{q+t}$ for every $1 \leqslant t \leqslant p$.

Similarly the sets Y_k for $1 \leqslant k < q$ satisfy $F_1 \subset \cdots \subset F_n \in Y_k$ if and only if these conditions are fulfilled:

(1) $F_m \subset \operatorname{Ker} n^m$ for all $1 \leqslant m \leqslant k$,
(2) $n(F_i) \subset F_{i-1}$ but $n(F_i) \not\subset F_{i-2}$ for every $k+1 \leqslant i \leqslant q$,
(3) $\operatorname{Ker} n^t \subset F_{q+t}$ for every $1 \leqslant t \leqslant p$.

Finally the set Y_q satisfies $F_1 \subset \cdots \subset F_n \in Y_k$ if and only if the three following conditions are fulfilled:

(1) $F_m \subset \operatorname{Ker}\ n^m$ for all $1 \leqslant m \leqslant k$,
(2) $n(F_i) \subset F_{i-1}$, for all $1 \leqslant i \leqslant n$,
(3) $\operatorname{Ker} n^t \subset F_{q+t}$ for every $1 \leqslant t \leqslant p$.

where we have set $F_0 = 0$.

Now Y_q contains $X_\sigma = Y_0$ and is irreducible by construction, moreover it is actually a closed set because of Lemmas 3.1, 3.2 and 3.3. Therefore $Y_q = Y_\sigma$.

The natural projection $f_k : Y_\sigma(k) \to Y_\sigma(k-1)$ is nothing more than the map

$$F_0 \subset \cdots \subset F_{k-1} \subset F_k \mapsto F_0 \subset \cdots \subset F_{k-1}$$

and, because of the construction of Y_q, the fiber of this map is $\mathbb{P}^1$ for all $1 \leqslant k \leqslant q$. On the other hand,

$$\langle P_{1,1}, \ldots, P_{q,1}, R_{q,1}, \ldots, R_{q-k+1,k} \rangle = \langle P_{1,k+1}, \ldots, P_{q-k,k+1} \rangle \oplus \operatorname{Ker} n^k \tag{3.1}$$

for all $1 \leqslant k \leqslant q$, therefore f_{q+k} is an isomorphism for $1 \leqslant k \leqslant q$ since the space F_{q+k} is already determined by the space F_{q+k-1}. Finally, we have that $F_{2q+t} = \ker n^{q+t}$ for all $t \geqslant 0$, and the theorem follows in this case.

III. Let us assume that $s_j < S_j$ for all $1 \leqslant j < t$. We follow the notation introduced in I and II. We define Y_k in a similar way than in the other cases. Namely, $F_1 \subset \cdots \subset F_n = V \in Y_k$ if and only if there exist a point $(P_1, \ldots, P_q) \in \mathbb{P}(W)^k \times (\mathbb{P}(W) - \{(1:0)\})^{q-k}$ such that

(1) $F_{S_j+m} = \operatorname{Ker} n^{s_j} \oplus \langle P_{1,s_j+1}, \ldots, P_{S_j-s_j+m,s_j+1}\rangle$ for all $1 \leqslant m \leqslant b_j$, and $1 \leqslant j \leqslant t$.
(2) $F_{S_{j+1}+s_j+m} = \operatorname{Ker} n^{s_j} \oplus \langle P_{1,s_j+1}, \ldots, P_{S_{j+1}-s_j,s_j+1}, R_{S_{j+1}-s_j,s_j+1}, \ldots, R_{S_{j+1}-s_j-m+1,s_j+m}\rangle$ for all $1 \leqslant m \leqslant a_j$, and $1 \leqslant j \leqslant t$.
(3) $F_s = \operatorname{Ker} n^s$ for all $2q \leqslant t \leqslant n$.

One sees immediately that $X_\sigma = Y_0 \subset \cdots \subset Y_q$. Observe that Y_q is irreducible by construction, and similarly as in I, it can be described by the following conditions: $F_1 \subset \cdots \subset F_n \in Y_k$ if and only if

(1) $\operatorname{Ker} n^{s_j} \subset F_{S_j+m} \subset \operatorname{Ker} n^{S_j+m}$ for all $1 \leqslant m \leqslant b_j$, for all $1 \leqslant j \leqslant t$,
(2) $\operatorname{Ker} n^{s_j+m} \subset F_{S_{j+1}+s_j+m} \subset \operatorname{Ker} n^{S_{j+1}+m}$ for all $1 \leqslant m \leqslant a_j$, for all $1 \leqslant j \leqslant t$,
(3) $\operatorname{Ker} n^m = F_m$ for all $2q \leqslant m \leqslant p$,

where condition (2) is a consequence of the following equality

$$\begin{aligned}(3.2)\qquad &\operatorname{Ker} n^s \oplus \langle P_{1,s+1}, \ldots, P_{m,s+1}, R_{m,s+1}, \ldots, R_{m-t+1,s+t}\rangle \\ &\quad = \operatorname{Ker} n^{s+t} \oplus \langle P_{1,s+t+1}, \ldots, P_{m-t,s+t+1}\rangle,\end{aligned}$$

where $1 \leqslant t \leqslant m$ and $s + t + 1 \leqslant q$.

As in II, it follows from the construction of Y_q that the fiber of f_k is isomorphic to $\mathbb{P}^1$ if $k \in B_s$, for some s. We have that either from condition (3) or from Eq. (3.2) above, one gets that f_k will be an isomorphism if $k \in A_s$ for some s.

IV. We follow the notation introduced in III. Consider a standard tableau σ such that $s_j \not< S_j$ for some $1 \leqslant j \leqslant t$. Let j_0 be the smallest index such that $s_j \geqslant S_j$. Since the A_i's and the B_i's are made up by consecutive numbers, and for every index $k > m$ the numbers appearing in B_k and A_k are bigger than those appearing in A_m or B_m by applying induction one shows that $\max\{A_m\} = s_{m+1} + S_{m+1}$ and $\max\{B_m\} = s_m + S_{m+1}$ for every m, in particular $\max\{A_{j_0}\} \geqslant 2S_{j_0}$. Moreover, since j_0 was minimal then $\max\{B_{j_0}\} < 2S_{j_0}$, i.e. $2S_{j_0} \in A_{j_0}$. Therefore, for every flag $0 = F_0 \subset \cdots \subset F_n \in X_\sigma$ one has $F_{2S_{j_0}} = \operatorname{Ker} n^{S_{j_0}}$. Since this is a closed condition, this is also true for all flags in the irreducible component Y_σ, and therefore $Y_\sigma \cong Y_{\sigma'}(\operatorname{Ker} n^{S_{j_0}}) \times Y_{\sigma''}(V/\operatorname{Ker} n^{S_{j_0}})$, where σ' is the standard tableau induced by the pair $([1, \ldots, 2S_{j_0}]; \psi|_{[1,\ldots,2S_{j_0}]})$, and σ'' is the standard tableau induced by the pair $([2S_{j_0}+1, \ldots, n]; \psi|_{[2S_{j_0}+1,\ldots,n]})$, both of them of type (a, b) for some $a \geqslant b$. The proposition follows by induction on the dimension.

If X is a scheme and G a group, the G-torsors on X for the étale cohomology are parametrized by $H^1_{et}(X, G)$. The exact sequence

$$0 \to \mathbb{G}_m \to GL_2 \to PGL_2 \to 0$$

gives a map

$$\delta : H^1_{et}(X, PGL_2) \to H^2_{et}(X, \mathbb{G}_m) = Br(X).$$

Moreover, $z \in \operatorname{Ker} \delta \Leftrightarrow z$ can be extended to a GL_2 torsor on X, i.e. if z can be extended to a vector bundle on X for the étale topology. Therefore, in order to show that a torsor that corresponds to an element $z \in H^1_{et}(X, PGL_2)$ is the projective bundle associated to a rank 2 vector bundle on X, it is enough to show that its image in $Br(X)$ is zero. In this chapter we will deal with varieties over an algebraically closed field $\mathbb{K}$ of characteristic zero and therefore the Brauer group of X coincide with the "geometric" Brauer group for this case.

Theorem 4.1. *Let σ be a standard tableau of type (p, q) and Y_σ be the corresponding irreducible component of $\mathbb{F}_x$. Then the motive $h(Y_\sigma)$ is isomorphic to $(1 + L)^q$. In particular, it is an Artin motive.*

Proof. Since the open cell $X_{\psi(1),\dots,\psi(n)} \subset Y_\sigma$ is an affine space, it follows from the proof of Theorem 3.4 that all the varieties $Y_\sigma(r)$ are rational and so $Br(Y_\sigma(r)) = 0$ for all r, therefore $Y_\sigma(r + 1) \to Y_\sigma(r)$ is either an isomorphism or the projective bundle associated to a rank 2 vector bundle on $Y_\sigma(r)$, in this case we have $h(Y_\sigma(r+1)) \cong (1+L) \otimes h(Y_\sigma(r))$, see [2]. Since $\dim Y_\sigma = q$ the conclusion follows by induction. □

Remark 4.2. Following the notation of Remark 1.2, if Y_σ is an irreducible component for which

$$\psi(t) = \begin{cases} 2 & \text{for } t \text{ odd,} \\ 1 & \text{for } t \text{ even} \end{cases}$$

then $Y_\sigma \cong (\mathbb{P}^1)^q$, and the multiplicative structure of the corresponding motive is clear. In general one needs to find sections of the maps $Y_\sigma(r + 1) \to Y_\sigma(r)$ that correspond to the bundle $\mathcal{O}_{Y_\sigma(r+1)}(1)$ and compute their autointersection numbers to explicitly find a normalized rank 2 vector bundle which induces the $\mathbb{P}^1$-bundle over $Y_\sigma(r)$ and, lastly, being able to actually compute the multiplicative structure of the motive, see [2].

Corollary 4.3. *If σ is a decomposable standard tableau then its motive is an Artin motive.*

Proof. If σ is decomposable then the irreducible component Y_σ is isomorphic to a product of towers of $\mathbb{P}^1$-bundles over flag varieties. Since flag varieties are rational, then all the Brauer groups involved are zero. Moreover, since the flag varieties are themselves towers of projective bundles associated to vector bundles, the motive that is obtained is an Artin motive. □

ACKNOWLEDGEMENTS

The authors want to thank Pedro Dos Santos, Bruno Kahn, Jochen Heinloth, James Lewis and Stefan Müller-Stach for some useful discussions and suggestions, as well as the referee, specially for his comments on the proof of Theorem 3.4. We are particularly indebted to Herbert Kurke for pointing out a mistake in a previous version of this article.

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(Received January 2007)