Color-bounded hypergraphs, IV: Stable colorings of hypertrees

Csilla Bujtás, Zsolt Tuza

Department of Computer Science, University of Pannonia, H–8200 Veszprém, Egyetem u. 10, Hungary
Computer and Automation Institute, Hungarian Academy of Sciences, H–1111 Budapest, Kende u. 13–17, Hungary

Abstract

We consider vertex colorings of hypergraphs in which lower and upper bounds are prescribed for the largest cardinality of a monochromatic subset and/or of a polychromatic subset in each edge. One of the results states that for any integers \( s \geq 2 \) and \( a \geq 2 \) there exists an integer \( f(s, a) \) with the following property. If an interval hypergraph admits some coloring such that in each edge \( E_i \) at least a prescribed number \( s_i \leq s \) of colors occur and also each \( E_i \) contains a monochromatic subset with a prescribed number \( a_i \leq a \) of vertices, then a coloring with these properties exists with at most \( f(s, a) \) colors. Further results deal with estimates on the minimum and maximum possible numbers of colors and the time complexity of determining those numbers or testing colorability, for various combinations of the four color bounds prescribed. Many interesting problems remain open.

1. Introduction

The classical theory of graph and hypergraph coloring assumes vertex colorings where every edge contains two vertices of different colors. In this paper we continue the systematic study of a much more general model – that includes many further ones studied in the literature as particular cases – in which four types of conditions may be prescribed for each edge, two of them concerning the multiplicity of the most frequent color inside the edge and two others for the number of colors occurring on the edge.

Many important results show that various problems, which are hard in general, can be handled in an elegant way on interval graphs and chordal graphs. Those classes are exactly the intersection graphs of interval hypergraphs and hypertrees (arboreal hypergraphs), respectively. Our theorems indicate that already on these two fundamental hypergraph classes, the combinations of coloring conditions mentioned above lead to much more complex situations than in the traditional model.

The bulk of our paper deals with interval hypergraphs. After the formal definitions given in the rest of this introduction, in Section 2 we present tight bounds on the smallest and largest possible numbers of colors when just one type of restrictive color bounds is given. On the other hand, for more than one restrictive type of color bounds, already the existence of a proper coloring is hard to decide in most cases (Section 3). Nevertheless, some conditions still admit an efficient solution (Section 4.1) and a finiteness theorem can be proved (Section 4.2) for the pair of conditions which, for structurally unrestricted hypergraphs, is known to be the most general one [5]. The possible numbers of proper colorings of hypertrees are investigated in Section 5, and some of the challenging open problems are collected in the concluding section.

1.1. Preliminaries

Here we introduce definitions and notation used throughout the paper, and give more background information from the literature.

* Research supported in part by the Hungarian Scientific Research Fund, OTKA grant T-049613.
* Corresponding author.

E-mail address: bujtas@dcs.vein.hu (C. Bujtás).

© 2009 Elsevier B.V. All rights reserved.

doi:10.1016/j.disc.2009.07.014
Hypergraph coloring. We use the notation \( \chi(\mathcal{H}) = (X, \mathcal{E}) \) for a hypergraph with a finite vertex set \( X = \{x_1, \ldots, x_n\} \) and an edge set \( \mathcal{E} = \{E_1, \ldots, E_m\} \). It will be assumed that each edge \( E_i \subseteq X \) is nonempty.

In a general sense, an unrestricted vertex coloring, that we shall simply call coloring in this paper, is a mapping \( \varphi : X \to \mathbb{N} \). Given \( \varphi \), a set \( Y \subseteq X \) is monochromatic if \( \varphi(y) = \varphi(y') \) for all \( y, y' \in Y \), and polychromatic if \( \varphi(y) \neq \varphi(y') \) for any two distinct \( y, y' \in Y \). The largest cardinality of a monochromatic and of a polychromatic subset of a set \( A \subseteq X \) will be denoted by \( \mu(A) \) and \( \pi(A) \), respectively. Hence, writing \( \varphi(A) \) for the set of colors appearing in \( A \), the equality \( \pi(A) = |\varphi(A)| \) holds for any set \( A \) of vertices.

In the bulk of hypergraph coloring theory, monochromatic edges are excluded. In Voloshin's more general 'mixed hypergraph' model \cite{17}, such non-monochromatic edges are called D-edges, and edges of another type (called C-edges) may also occur, which are required to be non-polychromatic. These conditions precisely mean two vertices of distinct colors in a D-edge and of a common color in a C-edge.

**Color-bound functions and stably bounded hypergraphs.** In the most general coloring model considered here, the cardinality of largest monochromatic/polychromatic subset will be required to be within a range specified for each edge. Formally, we assume that four functions, called color-bound functions, are given on the edge set:

\[
s, t, a, b : \mathcal{E} \to \mathbb{N}.
\]

To simplify notation, for every \( E_i \in \mathcal{E} \) we write

\[
s_i := s(E_i), \quad t_i := t(E_i), \quad a_i := a(E_i), \quad b_i := b(E_i).
\]

We assume throughout that the inequalities

\[
1 \leq s_i \leq t_i \leq |E_i|, \quad 1 \leq a_i \leq b_i \leq |E_i|
\]

are valid for all edges \( E_i \). Then, \( \varphi : X \to \mathbb{N} \) will be called a proper coloring if

\[
s_i \leq \pi(E_i) \leq t_i \quad \text{and} \quad a_i \leq \mu(E_i) \leq b_i \quad \text{for all } E_i \in \mathcal{E}.
\]

That is, the number of colors in \( E_i \) has to be between \( s_i \) and \( t_i \); moreover, at least one color has to occur at least \( a_i \) times and no color is allowed to occur more than \( b_i \) times in \( E_i \). The extremal values of those color bounds will be indicated in subscript as \( s_{\text{max}} := \max_{1 \leq i \leq m} s_i, \ t_{\text{min}} := \min_{1 \leq i \leq m} t_i, \) etc.

The terminology stably bounded hypergraph stands for \((s, t, a, b)\)-ly bounded, and if the hypergraph and the color-bound functions are given together, it can be written in the form of a six-tuple \( \mathcal{H} = (X, \mathcal{E}, s, t, a, b) \).

**Functional subclasses.** Any conditions of the form \( s_1 = 1, t_1 = |E_1|, a_1 = 1, b_1 = |E_1| \) mean no restriction on \( E_1 \). Hence, if one or more of the color-bound functions are nonrestrictive on the entire edge set, they can be omitted. We shall use Capital letters to indicate essential conditions. In this way, \((S,T)\)-hypergraph means that \( a_i = 1 \) and \( b_i = |E_i| \) hold for all edges; and \( S\)-hypergraph further means \( t_i = |E_i| \) for all \( i \). To indicate those types, one can write \( \mathcal{H} = (X, \mathcal{E}, s, t) \) or \( \mathcal{H} = (X, \mathcal{E}, s) \). This terminology and notation can be extended for all nonempty combinations of the four color-bound functions analogously. Let us note that every \( S\)-hypergraph is an \((S,T)\)-hypergraph at the same time; i.e., \( t \) is not required to be nontrivial for the latter.

Let us mention some subclasses which have already appeared in the literature. The \( S\)-hypergraphs extend the concept of ‘classical’ proper coloring (where \( s \equiv 2 \)) in a very natural way, and their chromatic polynomials have recently been studied in \cite{7}. The class of \( B\)-hypergraphs, for which some approximation algorithms have been designed, occurs in the context of scheduling theory \cite{1,12,13}. The mixed hypergraphs \cite{15–17} can be described with the conditions \( s_i \in \{1, 2\} \) and \( t_i \in \{|E_i|−1, |E_i|\} \) or, equivalently, with \( a_i \in \{1, 2\} \) and \( b_i \in \{|E_i|−1, |E_i|\} \) for all \( 1 \leq i \leq m \). The \((S,T)\)-hypergraphs, also called color-bound hypergraphs, were first introduced in \cite{2} with a somewhat different terminology and then studied in \cite{3,4}.

**Colorability, feasible sets, chromatic spectrum.** Some stably bounded hypergraphs \( \mathcal{H} = (X, \mathcal{E}, s, t, a, b) \) admit no proper colorings at all. If \( \mathcal{H} \) has at least one proper coloring, it is called colorable; and otherwise it is uncolorable.

Assume that \( \mathcal{H} \) is colorable. By a \( k \)-coloring we mean a proper coloring with exactly \( k \) colors; that is, a coloring \( \varphi : X \to \mathbb{N} \) with \( |\varphi(X)| = k \). Adopting terminology from mixed hypergraphs, the set

\[
\Phi(\mathcal{H}) := \{k \mid \mathcal{H} \text{ has a } k\text{-coloring}\}
\]

is termed the feasible set of \( \mathcal{H} \). The values

\[
\chi(\mathcal{H}) := \min \Phi(\mathcal{H}), \quad \Upsilon(\mathcal{H}) := \max \Phi(\mathcal{H})
\]

are called the lower chromatic number and upper chromatic number of \( \mathcal{H} \), respectively.

Each \( k \)-coloring \( \varphi \) of \( \mathcal{H} \) induces a color partition \( X = X_1 \cup \cdots \cup X_n \), where the \( X_i \) are the maximal monochromatic subsets of \( X \). The number of proper color partitions with precisely \( k \) nonempty classes will be denoted by \( r_k \). The chromatic spectrum of \( \mathcal{H} \) is defined as the \( n \)-tuple \( (r_1, r_2, \ldots, r_n) \). In a more flexible way, one may omit the zeroes from the end of the sequence, hence restricting it to \( (r_1, r_2, \ldots, r_{\Upsilon(\mathcal{H})}) \).

An integer \( k \) is called a gap in the feasible set or in the chromatic spectrum if \( \chi(\mathcal{H}) < k < \Upsilon(\mathcal{H}) \) and \( k \not\in \Phi(\mathcal{H}) \).

**Interval hypergraphs and hypertrees.** We shall consider two important classes of hypergraphs. A hypergraph \( \mathcal{H} \) is called an interval hypergraph if there exists a linear order on its vertex set, such that every edge of \( \mathcal{H} \) consists of consecutive vertices. If \( \mathcal{H} \) is of this type, we shall assume that the labeling \( x_1, \ldots, x_n \) of its vertices corresponds to the linear order required.
i.e., every edge of \( \mathcal{G} \) is an ‘interval’ of the form \( \{x_i, x_j \} : = \{x_i \mid i \leq \ell \leq j\} \). Vertices \( x_i \) and \( x_j \) will be referred to as the ends of \( \{x_i, x_j\} \). By the size of \( \{x_i, x_j\} \) we mean the number of its elements, that is \( j - i + 1 \).

More generally, \( \mathcal{G} \) is termed a hypergraph (often called arborescent hypergraph in the literature) if there exists a tree graph \( T \) on the vertex set of \( \mathcal{G} \), such that each edge of \( \mathcal{G} \) induces a subtree in \( T \). This \( T \) is called the host tree of \( \mathcal{G} \). Also, for an interval hypergraph, the sequence \( x_1, \ldots, x_n \) may be viewed as a path graph, which is then called the host path, and the edges of \( \mathcal{G} \) are its subpaths.

In notation, we shall write \( \mathcal{I} \) for an interval hypergraph and \( T \) for a hypertree.

2. Tight bounds for one restrictive function

If just one of the color-bound functions \( s, t, a, b \) is restrictive, then the hypergraph always is colorable. In particular, an \( S \)- or a \( B \)-hypergraph is properly colored if each vertex has its dedicated color, so in these cases the goal is to determine the lower chromatic number. For interval hypergraphs this can be done easily, as shown in the first subsection. On the other hand, \( T \)- and \( A \)-hypergraphs are 1-colorable, hence the main question concerns the upper chromatic number. A structural description of optimal colorings of interval \( T \)-hypergraphs and an efficient coloring algorithm will be given in the second subsection, while the problem remains open for \( A \)-hypergraphs.

Let us mention that the feasible set is necessarily gap-free for any one color-bound function. Even more generally, splitting any non-singleton color class of a proper \( k \)-coloring \((k < n)\) into two parts in any \((S,B)\)-hypergraph, a proper \((k+1)\)-coloring is obtained; moreover, given a proper \( k \)-coloring \((k > 1)\) of any \((T,A)\)-hypergraph, the identification of any two colors yields a proper \((k - 1)\)-coloring. Thus, in either case, no gaps can occur in the chromatic spectrum.

2.1. Lower chromatic number and periodic colorings

Proposition 1. If \( \mathcal{I} = (X, E, s) \) is an interval \( S \)-hypergraph, then

\[
\chi(\mathcal{I}) = s_{\text{max}}
\]

and, if the value of \( s_{\text{max}} \) and the number \( n \) of vertices are known, then a proper coloring can be determined in linear time, by determining the color of any one vertex in constant time.

Proof. Clearly, in every proper coloring of \( \mathcal{I} \) there occur at least \( s_{\text{max}} \) colors. On the other hand, the periodic coloring \( \varphi : X \rightarrow \{0, 1, \ldots, s_{\text{max}} - 1\} \) that assigns the color

\[
\varphi(x_j) := j \, \text{(mod } s_{\text{max}}) \quad \text{for } j = 1, \ldots, n
\]

is a proper \( s_{\text{max}} \)-coloring of \( \mathcal{I} \), since every edge \( E_i \) contains exactly \( \min\{s_{\text{max}}, |E_i|\} \geq s_i \) colors.

Proposition 2. If \( \mathcal{I} = (X, E, b) \) is an interval \( B \)-hypergraph, then

\[
\chi(\mathcal{I}) = \max_{1 \leq |E_i| \leq b} \left\lceil \frac{|E_i|}{b_i} \right\rceil
\]

and, if the value of \( \max_{1 \leq |E_i| \leq b} \left\lceil |E_i|/b_i \right\rceil \) and the number \( n \) of vertices are known, then a proper coloring can be determined in linear time, by determining the color of any one vertex in constant time.

Proof. Let us denote \( k := \max_{1 \leq |E_i| \leq b} \left\lceil |E_i|/b_i \right\rceil \). Since no color can occur more than \( b_i \) times inside any edge \( E_i \), each \( E_i \) has to contain at least \( \left\lceil |E_i|/b_i \right\rceil \) colors, thus \( \chi(\mathcal{I}) \geq k \) holds. To show that \( k \) colors suffice, we consider the periodic coloring

\[
\varphi(x_j) := j \, \text{(mod } k) \quad \text{for } j = 1, \ldots, n.
\]

Any edge \( E_i \) contains at most \( \left\lceil |E_i|/k \right\rceil \leq b_i \) vertices from each of the \( k \) color classes, hence a proper \( k \)-coloring is obtained and \( \chi(\mathcal{I}) = k \) holds, indeed.

Remark 1. An important corollary of the above observations is that the actual edges of the given \( S \)- or \( B \)-hypergraph need not be known for determining the lower chromatic number. Moreover, a proper \( \chi \)-coloring can be found from the order \( x_1, \ldots, x_n \) of the vertex set without knowing the location of edges.

Proposition 4 in Section 4 will be a common generalization for \((S,B)\)-hypergraphs.

2.2. Upper chromatic number

Lemma 1. For every interval \( T \)-hypergraph there exists a \( \chi \)-coloring in which every color class induces a subpath in the host path.

Proof. Consider a proper \( \chi \)-coloring \( \varphi \) of \( T \)-hypergraph \( \mathcal{I} \), and determine the smallest subscript \( j \) where the color \( \varphi(x_j) \) differs from the previous color \( \varphi(x_{j-1}) \) but repeats the color of an earlier vertex \( x_{\ell} \), for some \( \ell < j - 1 \). If there exists no such \( j \), the coloring \( \varphi \) already satisfies the requirements. Now, let the colors \( \alpha = \varphi(x_j) \) and \( \beta = \varphi(x_{j-1}) \) be switched on the interval \( [x_j, x_{\ell}] \). The obtained coloring \( \varphi^* \) uses exactly \( \chi \) colors. We claim that it is a proper coloring of \( \mathcal{I} \). Indeed, if an edge \( E_i \) is entirely contained either in the interval \( [x_1, x_{j-1}] \) or in \( [x_j, x_n] \), the number of colors on \( E_i \) remains the same. On the other
hand, if $E_i$ contains both vertices $x_{i-1}$ and $x_i$, then the coloring $\varphi$ uses both colors $\alpha$ and $\beta$ on $E_i$, whilst $\varphi^*$ uses either both or only $\beta$. Since there has been no change in the occurrences of other colors, $|\varphi^*(E_i)| \leq |\varphi(E_i)| \leq t_i$ holds for every edge $E_i$, hence $\varphi^*$ properly colors each edge of $\mathcal{I}$.

Repeatedly applying this recoloring procedure, we get larger and larger values for the subscript $j$, thus after a finite number of steps a $\overline{\chi}$-coloring is obtained in which every color class induces a subpath of the host path. □

**Proposition 3.** There is an algorithm which determines the upper chromatic number $\overline{\chi}$ and finds a $\overline{\chi}$-coloring of an interval $T$-hypergraph $\mathcal{I} = (X, E, t)$ in $O(\sum_{i=1}^{m} |E_i|)$ steps.

**Proof.** We design a greedy coloring algorithm which colors the vertices of the interval $T$-hypergraph $\mathcal{I}$ in their natural order $x_1, x_2, \ldots, x_m$. The partial coloring obtained after coloring the first $i$ vertices will be denoted by $\varphi_i$ and, correspondingly, $|\varphi_i(E_i)|$ stands for the number of different colors in $\{x_1, x_2, \ldots, x_i\} \cap E_i$.

Let us set $\varphi_1(x_1) = 1$ and, for $i = 2, \ldots, n$, define $\varphi_i$ as follows:

- $\varphi_i(x_i) = \varphi_{i-1}(x_i)$ if $\ell < i$.
- $\varphi_i(x_i) = \varphi_{i-1}(x_i - 1) + 1$ if the inequality $|\varphi_{i-1}(E_i)| < t_i$ holds for every edge $E_i$ containing $x_i$.
- $\varphi_i(x_i) = \varphi_{i-1}(x_i - 1)$ otherwise, that is, if there exists an edge $E_j \ni x_i$ for which $|\varphi_{i-1}(E_j)| = t_j$.

One can see that this algorithm results in a proper coloring $\varphi = \varphi_m$ of $\mathcal{I}$. Now, assume for a contradiction that the number $k$ of different colors in $\varphi$ is smaller than the possible maximum $\overline{\chi}$. By Lemma 1 there exists a $\overline{\chi}$-coloring $\varphi^*$ where each color class induces a subpath of the host path. Moreover, it can be assumed, without loss of generality, that $\varphi^*$ uses the colors $1, 2, \ldots, \overline{\chi}$ in increasing order on the host path. Since $\overline{\chi} > k$, there is a smallest $i$ for which $\varphi^*(x_i) > \varphi(x_i)$.

Since both colorings have proper color classes, this means that $\varphi(x_i) = \varphi(x_{i-1})$ whilst $x_i$ has a new color in $\varphi^*$. The former equality can be satisfied only if $|\varphi_{i-1}(E_i)| = t_i$ holds for an edge $E_i = [x_i, x_{i+1}] \ni x_i$ in the $i$th step of the algorithm. By the choice of $j$, the inequality $\varphi^*(x_j) \leq \varphi(x_j)$ is valid, because the colors appear in $\varphi$ in increasing order; moreover, all colors $\varphi^*(x_j) + 1, \ldots, \varphi^*(x_i) - 1$ occur in $\varphi^*(E_i)$ not earlier than in $\varphi$.

This yields the contradiction

$$|\varphi^*(E_i)| \geq \varphi^*(x_i) - \varphi(x_i) + 1 > \varphi(x_i) - \varphi(x_{i+1}) + 1 = |\varphi_{i-1}(E_i)| = t_i,$$

and hence there is no proper coloring of $\mathcal{I}$ with more than $k$ colors.

By proper book-keeping the values $|\varphi_i(E_i)|$ can be computed, increasing $|\varphi_{i-1}(E_i)|$ by 1 whenever $x_i \in E_j$ gets a color different from $\varphi_{i-1}(x_{i+1})$, and at the same time it can be checked whether this counter reaches $t_i$. If it does, for at least one $j$, then $\varphi_{i+1}(x_{i+1}) = \varphi(x_i)$ is forced for the next step, and otherwise $x_{i+1}$ can get the next new color. These operations require time proportional to the number of edges incident with $x_i$, therefore the running time is $O(\sum_{i=1}^{m} |E_i|)$. □

### 3. Time complexity of colorability

As opposed to the case of just one restrictive color-bound function, two types of conditions together yield substantial complications. Although ($S, B$)- and ($T, A$)-hypergraphs always are colorable, all the other restrictive function-pairs admit uncolorable systems already for interval hypergraphs. Thus, the first natural question concerns the complexity of deciding whether the hypergraph has at least one proper coloring.

For the colorability of mixed interval hypergraphs and, more generally, of mixed hypertrees, there is a simple necessary and sufficient condition [14], and the characterization leads to a linear-time algorithm to decide colorability for any input mixed hypertree. In sharp contrast to this, here we prove that the colorability problem is NP-complete for interval ($T, B$)-, ($A, B$)-, and ($S, A$)-hypergraphs, but the complexity status for ($S, T$) remains open. The NP-hardness results can be extended for the lower chromatic number, too. This will be proved in [6], where also linear-time algorithms will be given for testing colorability in some subclasses of stably bounded interval hypergraphs.

In the proof of NP-completeness we shall apply a reduction from the 3-PARTITION problem, which is well known to be NP-hard in the strong sense, see e.g. [8].

**3-PARTITION**

**Instance:** A set $W = \{w_1, w_2, \ldots, w_{3m}\}$ of $3m$ positive integers and an integer $y$ such that $\sum_{i=1}^{3m} w_i = ym$ and $y/4 < w_i < y/2$ for all $1 \leq i \leq 3m$.

**Question:** Can $W$ be partitioned into $m$ disjoint subsets, such that every subset has exactly the same sum $y$?

The bounds $y/4$ and $y/2$ ensure that if $W$ has a partition corresponding to the prescribed conditions, each class contains precisely three integers.

**Theorem 1.** The colorability problem is NP-complete on interval ($T, B$)-, ($A, B$)-, and ($S, A$)-hypergraphs.

**Proof.** The problems are clearly in NP for each of the three classes, since it can be tested in $O(\sum_{i=1}^{m} |E_i|)$ time whether or not a given coloring is proper. Hence, for the proof it will suffice to show NP-hardness. We shall do this by reduction from 3-PARTITION, constructing interval hypergraphs $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ of the types under consideration, which are colorable if and only if the instance $W = \{w_1, w_2, \ldots, w_{3m}\}$ admits a solution.

**($T, B$) – Construction of $\mathcal{H}_1.** Let the vertex set $X$ be the disjoint union of $3m$ blocks, $X = X_1 \cup \cdots \cup X_{3m}$, where each $X_i$ contains precisely $w_i$ vertices; hence, $|X| = ym$. Every block $X_i$ is taken as an edge with color bound $t_1(X_i) = 1$, forcing that each
block should be monochromatic. Moreover, the whole vertex set $X$ is also considered as an edge with bounds $t_1(X) = m$ and $b_1(X) = y$.

$$c_1 = |X_i| \leq i \leq 3m| \cup \{X\}, \quad \mathcal{H} = (X, c_1, t_1, b_1).$$

Since $|X| = ym$, the required conditions $t_1(X) = m, b_1(X) = y$ can be satisfied if and only if $\mathcal{H}$ is colored with exactly $m$ colors and each of them is assigned to exactly $y$ vertices. In this case each color class is the union of three blocks, for which the sum of their sizes equals $y$. Therefore, the interval $(T, B)$-hypergraph $\mathcal{H}$ has a proper coloring if and only if the given set $W$ has a solution.

**Proposition 1** - Construction of $\mathcal{H}$. In this construction we begin with the vertex set $X$ of $\mathcal{H}$ containing the $3m$ blocks $X_i$, and extend it with $m$ new blocks

$$M = M_1 \cup \cdots \cup M_m \text{ where } |M_j| = jym \text{ for all } 1 \leq j \leq m.$$ 

The following edges and $a_2$-bounds will force that each block is monochromatic:

$$e_j = |X_i, M_j| \leq i \leq 3m, 1 \leq j \leq m \text{ and } a_2(M_j) = |M_j| = jym \text{ for all } 1 \leq i \leq 3m, 1 \leq j \leq m.$$ 

The set $M$ is also taken as an edge with bound $s_2(M) = m$, that is the blocks $M_j$ must have mutually distinct colors.

The crucial point of the construction is the definition of the following edges:

$$E_j = X \cup \bigcup_{i=1}^{j} M_i \text{ with } a_2(E_j) = jym + y \text{ for all } 1 \leq j \leq m.$$ 

Let

$$e_j = |E_j| \leq j \leq m, \quad \mathcal{H}_2 = (X \cup M, e_j \cup e_j \cup \{M\}, s_2, a_2).$$ 

The prescribed number $jym + y$ of monochromatic vertices in $E_j$ is larger than the maximum size of sets $X \cup M_k$ for $1 \leq k < j$. Consequently, any proper coloring of $\mathcal{H}_2$ requires at least $y$ vertices from $X$ which have the color of the block $M_j$, for every $1 \leq j \leq m$. Since the set $X$ contains exactly $ym$ vertices, this condition can be satisfied for all $j$ if and only if $X$ is colored with precisely $m$ colors and each of them occurs precisely $y$ times on $X$. Moreover, any solution of instance $W$ determines a color partition for $X$, which can be extended appropriately to the whole vertex set. This leads to the conclusion that the interval $(S, A)$-hypergraph $\mathcal{H}_2$ is colorable if and only if $W$ admits a solution.

**Proposition 2** - Construction of $\mathcal{H}_3$. In the $(S, A)$-hypergraph $\mathcal{H}_2$, $M$ is the only edge whose $s_2$-bound is restrictive. This can be replaced with $m + 1$ new edges, where the color-bound function $b_2$ is used:

$$h_j = \bigcup_{i=1}^{j+1} M_i \text{ with } b_2(h_j) = (j + 1)y, \text{ for all } 1 \leq j \leq m - 1.$$ 

These new edges ensure that the monochromatic block $M_{j+1}$ cannot have a common color with any of the previous blocks from $M$. Hence, the effect is equivalent to the omitted edge $M$ of $\mathcal{H}_2$. By this modification, an interval $(A, B)$-hypergraph $\mathcal{H}_3$ is obtained whose proper colorings correspond to those of $\mathcal{H}_2$. Thus, $\mathcal{H}_3$ is colorable if and only if the instance $W$ has a solution. All the above reductions from 3-Partition to interval hypergraph colorability take time proportional to $\sum_{i \in W} w_i$. Since the 3-partition problem is NP-hard in the strong sense, it does not admit any pseudo-polynomial algorithms with running time polynomial in $\sum_{i \in W} w_i$. This proves NP-hardness of the colorability problem for each class of interval $(T, B)$-, $(A, B)$-, and $(S, A)$-hypergraphs.

**Corollary 1.** The colorability problem is NP-complete on each subclass of stably bounded interval hypergraphs with at least three restrictive functions. In particular, the colorability problem is NP-complete on stably bounded interval hypergraphs in general.

**4. Bounded $\chi$ for interval hypergraphs**

In this section we address the problem of finding universal upper bounds on the lower chromatic number. In this context we investigate the coloring properties of interval hypergraphs, under various conditions. The main result of the section is Theorem 2, where we prove that the lower chromatic number is bounded over the entire class of interval $(S, A)$-hypergraphs for any fixed values of $s_{max}$ and $e_{max}$. A similar theorem has already been proved for interval $(S, T)$-hypergraphs in [4], for which it has been shown that $\chi = s_{max}$ always holds.

**4.1. Some simple upper bounds**

Let us begin with the following common generalization of Propositions 1 and 2. It gives an exact formula determining the lower chromatic number of interval $(S, B)$-hypergraphs.
Proposition 4. The lower chromatic number of any interval \((S,B)\)-hypergraph \(I\) is
\[
\chi(I) = \max \left\{ s_{\max}, \max_{E_i \in I} \left[ \frac{|E_i|}{b_i} \right] \right\}
\]
and, if the value of \( \max \left\{ s_{\max}, \max_{1 \leq i \leq n} \left[ \frac{|E_i|}{b_i} \right] \right\} \) and the order \(x_1, \ldots, x_n\) of vertices are known, then a proper coloring can be determined in linear time, by determining the color of any one vertex in constant time.

Proof. By definition, the number of colors in every proper coloring of \(I\) is at least \(k := \max \left\{ s_{\max}, \max_{E_i \in I} \left[ \frac{|E_i|}{b_i} \right] \right\} \). On the other hand, if we color the vertices in their natural order with \(k\) distinct colors periodically, that is
\[\psi(x_j) := j \mod k \quad \text{for } j = 1, \ldots, n,\]
then each edge \(E_i\) gets \(\max\{k, |E_i|\} \geq s_i\) colors and the largest monochromatic subset of \(E_i\) has exactly \(\left\lfloor \frac{|E_i|}{k} \right\rfloor \leq b_i\) vertices. Hence, the lower chromatic number equals \(k\). □

Next, we give a general upper bound for the lower chromatic number of stably bounded interval hypergraphs of bounded rank, without any assumption on the color-bound functions \(s, t, a, b\). This result will also imply an upper bound on \(\chi\) for interval \((T, B)\)-hypergraphs in terms of the maximum values \(s_{\max}\) and \(b_{\max}\) of the restrictive functions.

Proposition 5. If \(I\) is a colorable stably bounded interval hypergraph of rank \(r\), then \(\chi(I) \leq r\). Moreover the chromatic spectrum of \(I\) cannot have a gap between \(r\) and \(\chi(I)\).

Proof. If there exists no proper coloring of \(I\) with more than \(r\) colors, the assertion holds. In the other case, consider any proper \(k\)-coloring \(\varphi\) of \(I\) on the vertex set \(X = \{x_1, x_2, \ldots, x_n\}\), for which \(k > r\). Determine the smallest subscript \(i\) such that the interval \([x_i, x_{i+1})\) contains exactly \(r + 1\) colors. Then, choose the largest subscript \(j\) such that \([x_j, x_{j+1})\) still contains all those \(r + 1\) colors. By these conditions, the inequality \(j + r \leq i\) is surely valid, and the interval \([x_{j+1}, x_{j-1})\) does not contain vertices of colors \(\alpha := \varphi(x_j)\) and \(\beta := \varphi(x_i)\).

We shall prove that if the colors \(\alpha\) and \(\beta\) are transposed on the interval \([x_i, x_{i+1})\), the obtained coloring \(\varphi'\) is proper. Indeed, since \([x_{j+1}, x_{j-1})\) has at least \(r - 1\) vertices, there is no edge intersecting both \([x_i, x_{j+1})\) and \([x_j, x_{j-1})\). Hence, every edge \(E_i\) is either entirely contained in \([x_i, x_{i+1})\) and there is no change in its coloring, or \(E_i\) is entirely contained in \([x_j, x_{j+1})\), which means that all the operations of \(\alpha\) and \(\beta\) have been switched inside \(E_i\). That is, \(\pi(E_i)\) and \(\mu(E_i)\) remain unchanged in either case.

Applying this procedure repeatedly, the value of subscript \(i\) gets larger and larger. Thus, after a finite number of iterations, some color gets eliminated and a \((k - 1)\)-coloring is obtained. (In each step, only the current color \(\alpha\) can disappear.)

If we start with a \(\chi\)-coloring \(\varphi\), the algorithm yields proper colorings with \(\chi - 1, \chi - 2, \ldots, r\) colors. Therefore, \(\chi(I) \leq r\) and there is no gap between \(r\) and \(\chi\). □

Proposition 6. If \(I\) is a colorable interval \((T, B)\)-hypergraph, then \(\chi(I) \leq t_{\max}b_{\max}\).

Proof. If \(I\) is colorable, then each edge \(E_i\) satisfies the inequality-chain
\[|E_i| \leq t_i b_i \leq t_{\max}b_{\max}.\]
Consequently, the hypergraph is of rank \(t_{\max}b_{\max}\), and this implies the claimed upper bound by Proposition 5. □

4.2. Bounded \(\chi\) for fixed \((s_{\max}, a_{\max})\)

Here we prove a general upper bound for all interval \((S,A)\)-hypergraphs, in terms of \(s_{\max}\) and \(a_{\max}\). A lower bound will be given at the end of the subsection for a particular infinite sequence of the pair \((s_{\max}, a_{\max})\).

Theorem 2. There exists a function \(f(s, a)\), such that every colorable interval hypergraph \(I = (X, E, s, a)\) with \(s_{\max} \leq s\) and \(a_{\max} \leq a\) has \(\chi(I) \leq f(s, a)\).

Remark 2. From the proof it will turn out that \(f(s, a) = 2 \frac{\mu(a) - 1}{s} + s - 1\) is a proper choice.

Proof. Given \(s\) and \(a\), let \(I\) be a colorable interval \((S,A)\)-hypergraph satisfying \(s_{\max} \leq s\) and \(a_{\max} \leq a\). Assuming that a proper coloring \(\varphi : X \to \mathbb{N}\) of \(I\) has been chosen, let us introduce the following terminology and notation.

- \(a\)-critical interval and critical color: an interval \(A = [x_i, x_j]\) (not necessarily an edge of \(I\)) with \(\mu(A) \leq a\), such that \(\mu(A) > \mu(A \setminus \{x_i\})\) and \(\mu(A) > \mu(A \setminus \{x_j\})\). In particular, if \(\varphi(x_i) = \varphi(x_{i+1})\), then \([x_i, x_{i+1})\) is \(a\)-critical. Moreover, by definition, the two ends of any \(a\)-critical interval necessarily have the same color, termed its critical color.
- \(\alpha\): the smallest subscript for which \(x_\alpha\) is the last vertex of some color class; that is, \(\alpha = \min \{i \mid \varphi(x_i) \neq \varphi(x_j)\ \text{for all} \ j > i\}\).
Suppose for a contradiction that there are at most two essential colors with the same typeset.

Claim 4. The coloring \( \varphi' \) cannot contain two essential colors with the same type set.

Proof. Suppose for a contradiction that \( T(c') = T(c'') \) holds for two essential colors \( c' \) and \( c'' \). We claim: Transposing \( c' \) and \( c'' \) in the interval \([x_{a+1}, x_n]\), the obtained coloring \( \varphi^* \) of \( I \) is proper.

Let \( x_\ell \) denote the vertex for which \([x_{a+1}, x_\ell] \) is the longest interval not containing colors \( c' \) and \( c'' \). That is, \( x_{\ell+1} \) is the first vertex after \( x_\ell \) having color \( c' \) or \( c'' \). Since \( c' \) and \( c'' \) are essential colors, the interval \([x_{a+1}, x_{\ell+1}]\) has at least \( s \) colors. Moreover, its coloring does not change while transposing \( c' \) and \( c'' \).

In order to verify the claim, we need to show that each \( E_i \in E \) remains properly colored in \( \varphi^* \). There are a few cases to check.

• If \( E_i \subseteq [x_1, x_\beta] \) or \( E_i \subseteq [x_\ell, x_n] \), then either its coloring has not been changed at all by transposing the two colors on \([x_{a+1}, x_n]\) or just \( c' \) and \( c'' \) are switched on it (note that the color of \( x_{a+1} \) is different from \( c' \) and \( c'' \)), having no effect with respect to the color bounds \((s_i, a_i)\).
In the remaining cases \( E_i \) contains the interval \([x_{\alpha}, x_{\alpha+1}]\).
- By the choice of \( \beta \), the edge \( E_i \) has at least \( s+1 \) colors also after the recoloring. Consequently, every \( s \)-bound is fulfilled.
- If \( E_i \) is \( a \)-critical but \( c(E_i) \not\subseteq \{c', c''\}\), then recoloring keeps the \( a_0 \)-condition satisfied.
- From now on we may assume that \( E_i \) is \( a \)-critical in one of the colors \( c', c'' \); say, \( E_i = A' \subseteq A_\alpha \) and \( c(A') = c' \). Since \( c' \) and \( c'' \) have the same type set, there exists an \( A'' \subseteq A_\alpha \) with \( c(A'') = c'' \) where \( A' \) and \( A'' \) are of the same type. Note that \( A' \) and \( A'' \) are overlapping intervals, for otherwise the larger one containing the other would not be critical. But then, after recoloring, the subinterval \( A' \cap A'' \) has exactly the same type \((l, r)\) as originally \( A' \) and \( A'' \) had; and it is \( a \)-critical, since the number of occurrences of other colors remained unchanged. The interval \( A' \cap A'' \) is contained in both \( A' \) and \( A'' \) hence, the original intervals remain properly colored.

Thus, any two \( a \)-critical intervals of the same type \((l, r)\) with critical colors \( c' \) and \( c'' \) become non-critical, but will contain a new, shorter \( a \)-critical interval of the same type in \( \varphi^* \). The latter interval will be called the substitute of the former ones. Similarly, there can be some \( a \)-critical interval \( A \subseteq A_\alpha \) in \( \varphi' \) with critical color different from \( c' \) and \( c'' \), which becomes non-critical in \( \varphi^* \). But every such interval \( A \) will necessarily contain at least one new and shorter substitute (also including \( x_\alpha \) and \( x_{\alpha+1} \)), which is a new \( a \)-critical interval in \( \varphi^* \).

Now, consider a new \( a \)-critical interval \([x_j, x_k]\) in \( \varphi^* \). It has size \( \ell_i := k - j + 1 \), hence the corresponding new term in \( S(\varphi^*) \) is \( 3^{-\ell_i} \). Let us denote by \( S_{jk} \) the set of intervals having \([x_j, x_k]\) as a substitute. There exist exactly two intervals in \( S_{jk} \) having critical colors \( c' \) and \( c'' \), whilst for each remaining interval \([x_j', x_k'] \in S_{jk} \) the following inequalities hold:

\[
1 \leq j' < j - 1, \quad k + 1 \leq k' \leq n, \quad \text{and} \quad 3^{-k' - j' + 1} = 3^{-(j - j')} \cdot 3^{-k} \cdot 3^{-(k' - k)}.
\]

Consequently,

\[
\sum_{A \in S_{jk}} 3^{-|A|} \leq 2 \cdot 3^{-(j+1)} + \left( \sum_{h=1}^{j-1} 3^{-(j-h)} \right) \left( \sum_{h=k+1}^{n} 3^{-(h-k)} \right) 3^{-\ell_i}
\]

\[
\leq 2 \cdot 3^{-\ell_i} + \left( \sum_{h=1}^{\infty} \left( \frac{1}{3} \right)^h \right)^2 \cdot 3^{-\ell_i} < 3^{-\ell_i}.
\]

This means that the new critical intervals of \( \varphi^* \) increase the sum \( S(\varphi') \) by a greater value than the sum of the deleted terms.

Therefore, the presence of two essential colors with the same type set would result in a proper coloring \( \varphi^* \), which corresponds to \( \varphi' \) in the number of used colors and in the value of \( \alpha \), but \( S(\varphi') < S(\varphi^*) \). This contradicts the choice of coloring \( \varphi' \), and hence the proof of Claim 4 is completed. \( \diamond \)

Now we are in a position to complete the proof of the theorem. Let us put \( f(s, a) = 2^{\frac{\alpha(s-1)}{s-1}} + s - 1 \) and suppose for a contradiction that \( \varphi' \) uses exactly \( k > f(s, a) \) colors. By condition (i), this assumption is equivalent to \( \chi(I) > f(s, a) \).

- By Claims 2 and 4, there exist at most \( 2^{\frac{\alpha(s-1)}{s-1}} - 1 \) essential colors, which are critical colors of intervals containing \([x_\alpha, x_{\alpha+1}]\). Moreover, by definition, there exist at most \( s \) non-essential colors.

From this, since \( k > 2^{\frac{\alpha(s-1)}{s-1}} + s - 1 \) has been assumed, we obtain that there exists an essential color \( c^* \) which has not been associated with any \( a \)-critical interval containing \( x_\alpha \). Consider the first vertex in the interval \([x_\alpha, x_\gamma] \), say \( x_\gamma \), whose color is \( c^* \).

- If the color \( c^* \) is replaced with the color of \( x_\alpha \) on the interval \([x_\gamma, x_\alpha] \), a proper coloring is obtained.

Indeed,
- for each edge contained in \([x_1, x_{\gamma-1}] \), the coloring remains unchanged;
- the color of \( x_\alpha \) does not occur in edges contained in \([x_{\alpha+1}, x_\gamma] \), hence the \( (s, a) \) conditions remain satisfied;
- if an edge \( E \) contains both \( x_\alpha \) and \( x_\gamma \), it necessarily involves the \( s \) non-essential colors in \( \varphi' \) and in \( \varphi^* \), too, therefore it satisfies the bound \( s(E) \) which has been assumed to not exceed \( s \). By the choice of color \( c^* \), if \( E \) is an \( a \)-critical interval, its critical color cannot be \( c^* \). Moreover, if its critical color is different from \( c^* \) then the number of occurrences of this color has not changed, so that the \( a \)-bound remains fulfilled.

Hence, the new coloring is proper. We have two cases:
- There occurs some vertex with color \( c^* \) in \([x_1, x_\gamma] \). Then, after recoloring, the number of colors does not change, but the last occurrence of color \( c^* \) is placed before \( x_\gamma \). This contradicts condition (ii) in the choice of \( \varphi' \).
- No vertex has color \( c^* \) in \([x_1, x_\alpha] \). In this case the recoloring would lead to a \((k - 1)\)-coloring, a contradiction to requirement (i).

Consequently, for every interval \((S, A)\)-hypergraph satisfying the conditions of Theorem 2, the bound \( \chi(I) \leq f(s, a) = 2^{\frac{\alpha(s-1)}{s-1}} + s - 1 \) is valid. \( \Box \)

Analyzing the recoloring procedure described in the proof, one can observe that the number of colors either remains the same or decreases by precisely 1. Hence, replacing condition (i) with the assumption that \( \varphi' \) is a proper \( k \)-coloring for a fixed
If \( \chi(s, a) \) is a proper \((k - 1)\)-coloring \( \varphi^a \) can be derived. In this way, we obtain:

**Theorem 3.** If \( \mathcal{I} = (X, \mathcal{E}, s, a) \) is a colorable interval hypergraph with \( s_{\text{max}} \leq s \) and \( a_{\text{max}} \leq a \), then its chromatic spectrum cannot have any gaps above \( 2^{\frac{\chi(s, a) - 1}{s}} + s - 2 \).

With regards to the lower bounds on \( \chi \), we have the following estimate.

**Proposition 7.** For every \( s \geq 2 \) and \( a \geq s + 3 \), we have \( f(s, a) \geq 2s - 1 \) for every proper choice of function \( f(s, a) \).

**Proof.** For every \( s \geq 2 \) we will construct an interval \((S, A)\)-hypergraph \( \mathcal{I}_s \) having \( s_{\text{max}} = s \), \( a_{\text{max}} = s + 3 \) and \( \chi = 2s - 1 \). The vertex set \( X_s \) consists of \( 3s - 1 \) blocks \( B_i \) of the following sizes:

\[
|B_i| = \begin{cases} 
  s + 2 - i & \text{if } 1 \leq i \leq s \\
  1 & \text{if } s + 1 \leq i \leq 2s - 1 \\
  2 & \text{if } 2s \leq i \leq 3s - 1.
\end{cases}
\]

For each \( 1 \leq i \leq s \) and \( 2s \leq i \leq 3s - 1 \), block \( B_i \) is considered as an edge with restrictive bound \( a(B_i) = |B_i| \). Hence, each \( B_i \) must be monochromatic. Moreover, edges \( E_i \) prescribe that every \( s \) consecutive blocks have pairwise different colors:

\[
E_i = \bigcup_{j=i}^{i+s-1} B_j \quad \text{and} \quad s(E_i) = s \quad \text{for all } 1 \leq i \leq 2s.
\]

The following edges force that there must be some blocks having the same color:

\[
H_i = \bigcup_{j=i}^{i+2s-1} B_j \quad \text{and} \quad a(H_i) = a(B_i) + 2 = s + 4 - i \quad \text{for all } 1 \leq i \leq s.
\]

To prove that the obtained interval \((S, A)\)-hypergraph is uniquely colorable with \( 2s - 1 \) colors, we consider the edges \( H_i \). The \( 2s \) blocks contained in \( H_i \) can be partitioned into classes \( \{B_i, \ldots, B_{i+s-1}\} \) and \( \{B_{i+s}, \ldots, B_{i+2s-1}\} \). Both classes consist of blocks having pairwise distinct colors, hence we can identify the colors of at most two blocks. The largest block of the former is \( B_i \), which is of size \( s + 2 - i \), whilst the latter class has blocks with as few as \( 1 \) and \( 2 \) vertices. Therefore, the bound \( a(H_i) = s + 4 - i \) can be achieved only by identification of the colors of \( B_i \) and a two-element block from the second part of \( H_i \).

For \( i = 1 \) this means that \( B_1 \) and the only two-element block \( B_2 \), of the second part must have a common color in every proper coloring. For \( i = 2 \), block \( B_2 \) would have a common color either with \( B_2 \) or \( B_{2s} \), but since \( B_1 \) and \( B_2 \) have different colors, the only possibility is the identification of colors of \( B_2 \) and \( B_{2s} \). Step by step, we obtain that the only proper coloring of \( \mathcal{I}_s \) apart from renumbering of colors is obtained when \( B_i \) and \( B_{i+2s-1} \) receive color \( i \) for each \( 1 \leq i \leq s \), and the blocks \( B_{i+1}, B_{i+2}, \ldots, B_{2s-1} \) remain singletons. Consequently, \( \chi(\mathcal{I}_s) = \chi(\mathcal{I}_2) = 2s - 1 \) for every \( s \geq 2 \), what shows that \( f(s, a) \geq 2s - 1 \) is valid indeed if \( a = s + 3 \). \( \square \)

## 5. Stably bounded hypertrees

The earlier papers [4,5] contain various results concerning stably bounded hypertrees; we summarize them as follows.

**Time complexity:** Testing colorability is an NP-complete problem on the classes of 3-uniform \((S, T)\)-, \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertrees.

**Possible feasible sets:** Let \( F \) be a finite set of positive integers. There exists an \((S, T)\)-hypergraph \( \mathcal{T} \) with feasible set \( \Phi(\mathcal{T}) = F \) if and only if

(i) \( \min(F) = 1 \) or \( \min(F) = 2 \), and \( F \) contains all integers between \( \min(F) \) and \( \max(F) \), or

(ii) \( \min(F) \geq 3 \).

**Possible feasible sets of \( r \)-uniform hypertrees:** For every \( r \geq 4 \), \( F \) is the feasible set of some \( r \)-uniform \((S, T)\)-hypertree if and only if it satisfies (i) or (ii).

**Chromatic spectrum:** Every finite sequence \((r_1, r_2, \ldots, r_t)\) of nonnegative integers with \( r_1 = r_2 = 0 \) is the chromatic spectrum of some \((S, T)\)-hypertree. Moreover, the same is true concerning \( r \)-uniform \((S, T)\)-hypertrees, for an arbitrarily prescribed \( r \geq 4 \).

In this section we prove analogous results for the chromatic spectra and feasible sets of \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertrees.

As noted in [5] and recalled in Section 2, it is easy to see that every one-colorable stably bounded hypergraph has gap-free chromatic spectrum and, for any positive integer \( k \), there exist \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertrees with feasible set \( \{1 \leq i \leq k\} \). Moreover, it was proved in [4] that every two-colorable \((S, T)\)-hypergraph has a gap-free feasible set. But the corresponding questions are still open for \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertrees, we do not have characterization for their possible feasible sets if \( \chi = 2 \).

To prove that the chromatic spectrum is practically unrestricted in several subclasses of non-2-colorable stably bounded hypertrees, we shall refer to a construction from [10], which yields:
Lemma 2 ([10]). Every finite sequence \((r_1 = 0, r_2, \ldots, r_k)\) of nonnegative integers is the chromatic spectrum of a mixed hypergraph having only two-element \(D\)-edges and three-element \(C\)-edges.

We will prove:

Proposition 8. For every finite sequence \((r_3, \ldots, r_k)\) of nonnegative integers with \(r_k > 0\),

(i) for every \(r \geq 4\), there exists some \(T\)-uniform \((T, B)\)-hypertree, and

(ii) for \(r = 4\) and for every \(r \geq 6\), there exist some \(T\)-uniform \((S, A)\)-hypertree and \(r\)-uniform \((A, B)\)-hypertree

whose chromatic spectrum is \((r_1 = 0, r_2 = 0, r_3, \ldots, r_k)\).

Proof. For a given vector \((r_1 = 0, r_2 = 0, r_3, \ldots, r_k)\), let us define \(r'_i := r_{i+1} - 1\) for every \(2 \leq i \leq k - 1\), and consider the mixed hypergraph \(\mathcal{H} = (X, C, D)\) satisfying the conditions of Lemma 2 with sequence \((r'_1 = 0, r'_2, \ldots, r'_k)\).

The vertices of \(\mathcal{H}\) will be denoted by \(x_1, x_2, \ldots, x_n\). To construct stably bounded hypertrees, we take \(r\) copy-vertices \(x_1', x_2', \ldots, x_r'\) for each element of \(X\), and \(r\) further vertices \(z_1, z_2, \ldots, z_r\) having a specified role. Each of the three stably bounded hypertrees \(T_1, T_2, T_3\) that are going to construct will have the \(r(n + 1)\)-element vertex set

\[
X' = \{x_i' \mid 1 \leq i \leq n; 1 \leq \ell \leq r\} \cup \{z_i' \mid 1 \leq \ell \leq r\}.
\]

The host tree contains the edges

\[
\{z_1', x_1', \ldots, x_r'\} \quad \text{for every } 1 \leq i \leq n \text{ and } 1 \leq \ell \leq r - 1.
\]

We define the following edges and bounds for the construction of hypertrees:

\[
Z = \{z_1^\ell \mid 1 \leq \ell \leq r\} \quad \text{with } t(Z) = 1 \quad \text{or } a(Z) = r, \quad \text{and}
\]

\[
X_i = \{x_i^\ell \mid 1 \leq \ell \leq r\} \quad \text{with } t(X_i) = 1 \quad \text{or } a(X_i) = r, \quad \text{for every } 1 \leq i \leq n.
\]

These \(t\)- and \(a\)-bounds also separately force the monochromaticity of edges \(X_i\) and \(Z\). Moreover, we prescribe that the color used on \(Z\) is different from each of the other colors:

\[
E_i = \{x_i^1, z_1^1, \ldots, z_r^1\} \quad \text{with } s(E_i) = 2 \quad \text{or } b(E_i) = r - 1, \quad \text{for every } 1 \leq i \leq n.
\]

If \(D_h = \{x_i, x_j\}\) was a \(D\)-edge in the mixed hypergraph \(\mathcal{H}\), its vertices must have different colors. This can be expressed in stably bounded hypertrees \(T_1, T_2, T_3\) by the edge

\[
D_h' = \{z_1, x_i, x_j, \ldots, x_{r-2}\} \quad \text{with } s(D_h') = 3 \quad \text{or } b(D_h') = r - 2, \quad \text{for every } D_h \in \mathcal{D}.
\]

The 3-element \(C\)-edges of \(\mathcal{H}\) will be differently handled for parts (i) and (ii).

(T, B) – Construction of \(T_1\). Consider a \(C\)-edge \(C_h = \{x_i, x_j, x_p\}\) from \(\mathcal{H}\). This requires at least two vertices with common color. To obtain the \((T, B)\)-hypertree \(T_1\) corresponding to assertion (i), we express this constraint as follows:

\[
C_h' = \{x_i', x_j', x_p', z_1, \ldots, z_r\} \quad \text{with } t(C_h') = 3 \quad \text{for every } C_h \in C.
\]

This edge forbids the case when vertices \(x_i', x_j', x_p'\) would have pairwise different colors. The edge can be defined for every \(r \geq 4\).

Now, the \((T, B)\)-hypertree \(T_1 = (X'', \varepsilon_1, t, a)\) is obtained, where

\[
\varepsilon_1 = \{Z\} \cup \{E_i \mid 1 \leq i \leq n\} \cup \{D_h' \mid D_h \in \mathcal{D}\} \cup \{C_h' \mid C_h \in C\}.
\]

(S, A) and (A, B) – Construction of \(T_2\) and \(T_3\). To treat the \(C\)-edge \(C_h\) in these types of hypertrees, we shall apply two simple relations:

\[
x = \left\lceil \frac{x}{3} \right\rceil + \left\lceil \frac{x + 1}{3} \right\rceil + \left\lceil \frac{x + 2}{3} \right\rceil \quad \text{for every integer } x,
\]

(1)

\[
\left\lceil \frac{x}{3} \right\rceil + \left\lceil \frac{x + 1}{3} \right\rceil > \left\lfloor \frac{x}{2} \right\rfloor \quad \text{for } x = 3 \text{ and every integer } x \geq 5.
\]

(2)

By (1), the following edge \(C_h'^+\) contains exactly \(r\) vertices:

\[
C_h'^+ = \{z_1\} \cup \{x_i' \mid 1 \leq \ell \leq \left\lfloor \frac{r - 1}{3} \right\rfloor\} \cup \{x_i' \mid 1 \leq \ell \leq \left\lceil \frac{r}{3} \right\rceil\} \cup \{x_i' \mid 1 \leq \ell \leq \left\lceil \frac{r + 1}{3} \right\rceil\}.
\]

We prescribe at least \(a(C_h'^+) = \left\lceil \frac{r - 1}{3} \right\rceil\) vertices having a common color in \(C_h'^+\). As a consequence of (2), for \(r = 4\) and for every \(r \geq 6\) this can be satisfied if and only if at least two of \(X_i, X_j, X_p\) have a common color.
Now, we are in a position to define the \((S, A)\)-hypertree \(T_2\) and the \((A, B)\)-hypertree \(T_3\):
\[
\mathcal{E}_2 = \{Z\} \cup \{X_i \mid 1 \leq i \leq n\} \cup \{E_i \mid 1 \leq i \leq n\} \cup \{D'_n \mid D_n \in \mathcal{D}\} \cup \{C^+_n \mid C_n \in \mathcal{C}\},
\]
\[
T_2 = (X', \mathcal{E}_2, s, a), \quad T_3 = (X', \mathcal{E}_2, a, b).
\]

Consider any proper color partition of the constructed stably bounded hypertrees. Due to the edges \(E_i\), the monochromatic set \(Z\) cannot have a common color with any \(x_i'\), hence \(Z\) is a separated color class. If this color class is removed, every remaining set \(X_i\) uniquely defines a color for the vertex \(x_i\) of the mixed hypergraph \(\mathcal{H}\), such that a proper coloring of \(\mathcal{H}\) is obtained. The reverse correspondence can be shown similarly.

This means that the \(\ell\)-colorings of \(T_i (i = 1, 2, 3)\) are in one-to-one correspondence with \((\ell-1)\)-colorings of \(\mathcal{H}\), thus \(r_i = r_i' - 1\) holds for all \(2 \leq \ell \leq k\). The constructed \(T_i\) clearly have no proper 1-coloring, hence \(r_1 = 0\), what completes the proof of (i) and (ii). \(\square\)

As consequences, we obtain characterizations for possible feasible sets and chromatic spectra of \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertrees whose lower chromatic number is at least 3.

**Corollary 2.** Every finite set \(F\) of integers, for which \(\min F \geq 3\) holds, is a feasible set of some \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertree.

**Corollary 3.** Every finite sequence \((r_1, r_2, \ldots, r_k)\) of nonnegative integers with \(r_1 = r_2 = 0\) and \(r_k > 0\) is the chromatic spectrum of some \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypertree.

### 6. Open problems

In the previous sections we have already mentioned several open problems concerning stably bounded interval hypergraphs and hypertrees. It may be useful to collect them here and to put a few related comments. For just one restrictive function, the interval \(S\)-, \(T\)-, and \(B\)-hypergraphs have been handled efficiently in Section 2. The following class is missing:

**Problem 1.** Is there a polynomial-time algorithm to determine the upper chromatic number of interval \(A\)-hypergraphs?

The method of \(S\)- and \(B\)-hypergraphs has been extended also for their ‘colorable’ combination, \((S, B)\). The other pair \((T, A)\), however, is unsolved so far and it is not even clear whether it is of the same complexity as the case of \(A\)-hypergraphs.

**Problem 2.** What is the complexity of determining the upper chromatic number of interval \((T, A)\)-hypergraphs?

For three of the four other pairs, namely for interval \((T, B)\)-, \((A, B)\)-, and \((S, A)\)-hypergraphs, the colorability problem is NP-complete (Theorem 1). In fact, there is a unique subset of \(s, t, a, b\) for which the complexity status has not been determined so far.

**Problem 3.** What is the complexity of testing the colorability of interval \((S, T)\)-hypergraphs?

Interestingly enough, this missing case is the one for which the strongest result is known for the lower chromatic number [4], that means the equality \(\chi = s_{\max}\).

For interval \((S, A)\)-hypergraphs we succeeded in proving an upper bound on \(\chi\) in terms of \(s_{\max}\) and \(a_{\max}\) (Theorem 2). But this bound does not imply that \(\chi\) can be determined in polynomial time. Moreover, it is not clear whether the bound should necessarily tend to infinity with \(a_{\max}\). The situation is interesting because in principle a large \(a_{\max}\) should not force us to use many colors.

**Problem 4.** Is the lower chromatic number bounded above by a function of \(s\), independently of \(a_{\max}\), for all colorable interval \((S, A)\)-hypergraphs with \(s_{\max} \leq s\)?

**Problem 5.** Find tight bounds on the lower chromatic number of colorable interval \((T, A, B)\)-hypergraphs and their subclasses, possibly depending on the extremal values of \(a, t, and \left\lceil \frac{E_1}{b_1} \right\rceil\).

By Proposition 5, the chromatic spectrum of a stably bounded interval hypergraph cannot contain gaps above its rank. What is more, the whole chromatic spectrum of any interval \((S, T)\)-hypergraph is gap-free [4]. But the following questions are still open:

**Problem 6.** Do there exist interval \((S, A)\)-, \((T, B)\)-, and \((A, B)\)-hypergraphs whose chromatic spectra contain gaps?

Similar questions are open for further interesting classes of hypergraphs, too. For instance, the lower chromatic number of circular \((S, T)\)-hypergraphs – hypergraphs having a cycle as host graph – is at most \(2s_{\max} - 1\) and there can be no gaps above \(2s_{\max} - 1\) in their feasible sets [4]; but it is not known whether any gaps can occur in them, except that the chromatic spectrum of circular mixed hypergraphs is gap-free [11]. Nothing is known about other classes of stably bounded circular hypergraphs.
Another interesting class is that of planar hypergraphs. There is a ‘mixed’ example [9] which is 2-colorable and 4-colorable but not 3-colorable, and actually ‘3’ is the only possible gap. But it is an open problem to investigate the possible gaps in more general classes (color-bounded, stably bounded, etc.) of planar hypergraphs.

Due to a theorem of [4], the chromatic spectrum of any 2-colorable ($S$, $T$)-hypertree is gap-free. It seems natural to raise the analogous questions concerning the other structure classes.

**Problem 7.** Does every 2-colorable ($S$, $A$)-, ($T$, $B$)-, and ($A$, $B$)-hypertree have a gap-free chromatic spectrum?

By Proposition 8, if the hypertree has neither 1-colorings nor 2-colorings, then the conclusion is not true because there are no restrictions on the possible numbers of colors.

Many open problems remain for chromatic spectra, too. The ones most directly related to our results in Section 5 are:

**Problem 8.** Characterize the possible chromatic spectra of classes of 3-uniform hypertrees.

**Problem 9.** Characterize the possible chromatic spectra of 5-uniform ($S$, $A$)- and ($A$, $B$)-hypertrees.

**References**


