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A note on the arithmetic of differential equations

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In this note we give a method for computing the differential Galois group of some linear second-order ordinary differential equations using arithmetic information, namely the p-curvatures.

1. INTRODUCTION

Let K be a number field and consider a finite extension $F/K(x)$, where x is an indeterminate, with derivation $D = d/dx$. To a linear differential equation $Ly = \sum_{i=0}^{n} c_i D^i y = 0, c_i \in F$, one associates its differential Galois group G, which is a linear algebraic subgroup of GL_n defined over F and isomorphic over $\bar{F}$ to a group defined over $\bar{K}$.

A conjecture of Katz [K3], which generalizes a conjecture of Grothendieck, predicts that the Lie algebra of G is the smallest F-Lie subalgebra of the Lie algebra of GL_n, whose reduction modulo primes p contains the p-curvature of L, for all sufficiently large p. The p-curvature of L is the $n \times n$ matrix $A_p (\bmod p)$ where $(D^p y, D^{p+1} y, \ldots, D^{p+n-1} y)^t = A_p(y, Dy, \ldots, D^{n-1} y)^t$. Katz has shown ([K3]) that the p-curvatures of L all belong to the Lie algebra of G. We will show, in some cases where we have an a priori restriction on G, that the fact that its Lie algebra contains the p-curvatures is enough to determine G. This gives an affirmative answer, in these cases, to the question posed at the end of the introduction of [K4].

From now on, assume that $n = 2$, that is L is a second-order linear differential operator. We will also assume that c_1/c_2 is the logarithmic derivative of an element of F. As is well-known, this is equivalent to the Galois group G of L be a subgroup of SL_2. Let us write

$$A_p = \begin{pmatrix} a_p & b_p \\ a_{p+1} & b_{p+1} \end{pmatrix}$$

From the fact that c_1/c_2 is a logarithmic derivative, it follows that the p-curvature of the determinant of $Ly = 0$ is always zero, which means that the trace of the p-curvature of $Ly = 0$ is zero, that is $b_{p+1} = -a_p$.

For an example that will be useful in the sequel, take the equation $D^2y = ay$, where a has expansion $\alpha x^k + \cdots$, with $k > 0$ at a place P of F/K above $x = \infty$, so a has a pole there. Then $D^n y = a_n y + b_n Dy$, where $a_2 = a, b_2 = 0$ and, for $n > 2$, $a_{n+1} = Da_n + ab_n, b_{n+1} = Db_n + a_n$. It follows by induction that the a_n, b_n have the following expansion at P:

$$a_{2n} = \alpha^n x^{kn} + \cdots, b_{2n} = kn(n-1)\alpha^{n-1}x^{k(n-1)-1} + \cdots,$$
$$a_{2n+1} = kn^2\alpha^n x^{kn-1} + \cdots, b_{2n+1} = \alpha^n x^{kn} + \cdots.$$

It follows that, for a prime $p = 2n + 1$,

$$(*) \qquad A_p = \begin{pmatrix} kn^2\alpha^n x^{kn-1} + \cdots & \alpha^n x^{kn} + \cdots \\ \alpha^{n+1}x^{k(n+1)} + \cdots & kn(n+1)\alpha^n x^{kn-1} + \cdots \end{pmatrix}.$$

Another result that will be useful in the sequel is the following lemma.

Lemma. *Let p be a prime sufficiently large and consider the equation $Ly = 0$ modulo p and assume it has a solution $y \neq 0$ with $u = Dy/y$ separable algebraic over $\mathbf{F}_p(x)$ and that A_p has trace zero. Then u satisfies $b_p u^2 + 2a_p u - a_{p+1} = 0$.*

Proof. Since u is separable algebraic over $\mathbf{F}_p(x)$, we have $D^p u = 0$. Since D^p is a derivation we get $D^{p+1}y = D^p(uy) = uD^p y$. On the other hand, by definition of A_p, $D^p y = a_p y + b_p Dy = y(a_p + b_p u), D^{p+1}y = a_{p+1}y + b_{p+1}Dy = y(a_{p+1} + b_{p+1}u)$. Thus, $a_{p+1} + b_{p+1}u = u(a_p + b_p u)$ and using that $b_{p+1} = -a_p$, we get the equation stated in the lemma.

2. PROPER SUBGROUPS OF SL_2

The list of proper algebraic subgroups of SL_2, up to conjugation, is well-known and we will go through the list pointing out facts relevant to our purposes. Our arguments in this section are based on the work of van der Put [P].

2.1. G finite

In this case all the solutions to the equation $Ly = 0$ are algebraic. This leads to infinitely many Dy/y which are also algebraic and the equation of the lemma therefore has infinitely many solutions, which proves that the p-curvature is zero.

2.2. $G \cong \mathbf{G}_m$

In this case G is conjugate to the group of diagonal matrices with determinant 1. The action of G on the space of solutions of $Ly = 0$ has two invariant lines, generated by y_1, y_2, say. Since the lines are invariant, the logarithmic derivatives $u_i = Dy_i/y_i$ are invariant under G, which means that the u_i are in F but are not logarithmic derivatives, for otherwise we would be in the case G finite.

We then obtain from the Lemma the following two relations among the entries of the p-curvature $b_p u_i^2 + 2a_p u_i - a_{p+1} = 0, i = 1, 2$. Recall that we also have $b_{p+1} = -a_p$, since the p-curvature has trace zero. These relations are easily seen to be independent.

2.3. G extension of $\mathbf{Z}/2$ by $\mathbf{G}_m$

In this case G is conjugate to the group of diagonal and antidiagonal matrices with determinant 1. This case is similar to the previous case, except that now the u_i are in a quadratic extension E/F and are conjugate over F, since the $\mathbf{Z}/2$ must permute the lines invariant under the subgroup of G isomorphic to $\mathbf{G}_m$.

2.4. G extension of $\mathbf{G}_m$ by $\mathbf{G}_a$

In this case G is conjugate to the group of triangular matrices with determinant 1. In this case there is an unique invariant line under G in the solution space of $Ly = 0$, generated by y_1, say. As before, the logarithmic derivative $u_1 = Dy_1/y_1$ is in F and we get a relation $b_p u_1^2 + 2a_p u_1 - a_{p+1} = 0$ as well as $b_{p+1} = -a_p$.

2.5. $G \cong \mathbf{G}_a$

In this case G is conjugate to the group of triangular matrices with both diagonal entries equal to 1. In this case G acts trivially on the invariant line so there is a solution of $Ly = 0$, say y_1, in F. As before, $u_1 = Dy_1/y_1$ is also in F and this gives a relation $b_p u_1^2 + 2a_p u_1 - a_{p+1} = 0$. But we do not get all relations among the entries of A_p this way. In order to get all relations we use that y_1 is in F to get $D^p y_1 = D^{p+1} y_1 = 0$. This gives two relations $a_p y_1 + b_p D y_1 = a_{p+1} y_1 + b_{p+1} D y_1 = 0$ and, again as before, we have $b_{p+1} = -a_p$.

As an application of the above, we expand an argument of van der Put [P] for the Airy equation (and correct a small error there) in order to reprove a theorem of Katz.

Theorem 1. *The equation $D^2 y = ay$, where a is a polynomial of odd degree, has Galois group SL_2.*

Proof. Suppose G is not SL_2. Notice that, from (*) above, we get in particular that the p-curvature is not zero so G is not finite. In all other cases, $Ly = 0$ has a solution for which $u = Dy/y$ is algebraic. So from the lemma we get the relation $b_p u^2 + 2a_p u - a_{p+1} = 0$, which is a quadratic equation for u. Again from (*), we get the top terms of the a_p, b_p, a_{p+1} and this gives that the discriminant of the quadratic equation for u, $4(a_p^2 + b_p a_{p+1})$, is a polynomial of degree kp, which is odd for p odd. Therefore the quadratic equation cannot have a rational function as root, since the discriminant is not a square. So the Galois group can only be conjugate to (2.3). To rule out (2.3), we proceed as in [P]. From [P] 4.1 (our b_p is f there), b_p satisfies a third order equation with polynomial coefficients $D^3b_p - 4aDb_p - 2Dab_p = 0$. If the Galois group is assumed to be (2.3), u satisfies a quadratic equation over $K(x)$ and since it also satisfies $b_pu^2 + 2a_pu - a_{p+1} = 0$, we get that $2a_p/b_p = -Db_p/b_p$ is independent of p. However, b_p has degree $(p-1)/2$ (from (*)) and, because of the differential equation it satisfies, b_p cannot have any triple zeros ([P] erroneously asserts the zeros are simple but gives counterexamples a paragraph earlier!). Thus $-Db_p/b_p$ has at least $[(p-1)/4]$ poles and thus cannot be independent of p. This contradiction completes the proof.

3. GLOBALLY NILPOTENT EQUATIONS

As before we consider a second-order equation $Ly = 0$ with Galois group G contained in SL_2. We say that $Ly = 0$ is globally nilpotent if its p-curvatures are nilpotent for all sufficiently large p. Katz [K3] has shown that factors of the Gauss-Manin connection on the cohomology of families of algebraic varieties are globally nilpotent. In particular, the Gauss hypergeometric equation $x(x-1)D^2y + ((a+b+1)x - c)Dy + aby = 0$ is globally nilpotent, a result which is also proved directly in [M]. Katz [K3] has shown that globally nilpotent equations have regular singular points and rational exponents. Dwork [D] conjectured that the only second-order equations over $K(x)$ which are globally nilpotent are obtained from Gauss hypergeometric equation by a change of variable or have a rational solution.

In this section we will compute the Galois group of some second order globally nilpotent equations using the p-curvatures. More generally, we will consider equations whose p-curvatures are nilpotent for infinitely many p. We note that, for the Gauss hypergeometric equation, our result (Theorem 2 below) follows from [BH].

As shown by Honda [H], the nilpotence of A_p implies that $Ly = 0$ has a non-zero solution y_1 in the reduction of F modulo p. Then, as before, $u_1 = Dy_1/y_1$ satisfies the quadratic equation $b_pu^2 + 2a_pu - a_{p+1} = 0$, as does any other algebraic Dy/y with $Ly = 0$, by the Lemma. However, the discriminant of the quadratic equation is $4(a_p{}^2 + b_pa_{p+1}) = -4\det A_p = 0$, by the nilpotence of A_p. So this quadratic equation has only one solution if it is not identically zero, i.e. if $A_p \neq 0$. Hence the only possibilities, if $A_p \neq 0$, for the Galois group are SL_2,(2.4) and (2.5). In both cases (2.4) and (2.5), there a unique $u \in F$, of the form $Dy/y, Ly = 0$. and so, from the above, $u_1 = u$.

Suppose that $Ly = 0$ has m singularities, all regular, and denote by $\rho_1', {}_1'', \ldots, \rho_m', \rho_m''$ their respective exponents. We say that $Ly = 0$ satisfies the *exponent restriction* if for all choices of ρ_i from $\rho_i', \rho_i'', i = 1, \ldots, m$, we have that $\sum \rho_i$ is not a nonpositive integer. If $Ly = 0$ satisfies this exponent restriction, then it is well-known that the action of G on the space of solutions of $Ly = 0$ is irreducible. This can be proved as follows. Suppose by contradiction that y_1 is a non-zero solution of $Ly = 0$ generating an invariant line. Then, the valuation of

y_1 is 0 or 1 at the regular points. Let k be the number of regular points where the valuation of y_1 is 1. The valuation of y_1 is either ρ_i' or ρ_i'' at the i-th singular point and therefore, by the residue theorem applied to dy_1/y_1 (which is a rational function, by hypothesis) $\sum \rho_i + k = 0$ for the choice of ρ_i corresponding to the valuations of y_1. This contradicts the exponent restriction.

The action of the Galois group in cases (2.4) and (2.5) is reducible. Those cases are thus ruled out by the exponent restriction and we obtain the following theorem. Note that the hypothesis of regular singularities follows if we assume global nilpotence, by Katz's result [K3].

Theorem 2. *Let $Ly = 0$ be a second order differential equation with regular singular points whose p-curvatures are nilpotent and non-zero for infinitely many primes p, whose Galois group is a subgroup of SL_2 and which satisfies the above exponent restriction. Then the differential Galois group of $Ly = 0$ is the whole of SL_2.*

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