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Pentagons vs. triangles

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Abstract

We study the maximum number of triangles in graphs with no cycle of length 5 and analogously, the maximum number of edges in 3-uniform hypergraphs with no cycle of length 5.

Keywords: Extremal graphs; Extremal hypergraphs; Cycles

0. Introduction

Erdős [1] stated several conjectures in extremal graph theory related to triangles and pentagons, on the pentagon-like structure or the number of pentagons in triangle-free graphs. We recall just the two most famous ones.

Conjecture 1. The number of cycles of length 5 in a triangle-free graph of order n is at most $(n/5)^5$ and equality holds for the blown-up pentagon if $5|n$.

The best known upper bound about $1.03(n/5)^5$ is proved in [4], but Füredi announced an improvement to $1.01(n/5)^5$ or maybe to $1.001(n/5)^5$.

Conjecture 2. Every triangle-free graph of n vertices can be made bipartite by deleting at most $n^2/25$ edges, which is sharp again for the blown-up pentagon if $5|n$.

The best estimate is proved in [2], namely that $n^2/(18+\delta)$ edges are sufficient, where δ is a small constant.

In this paper, we study the natural, less studied converse of the problem: what can we say about the number of triangles in a graph not containing any pentagon. We will see that pentagon-free 3-uniform hypergraphs cannot have too many "triangle" hyperedges either.

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1. Graphs

Theorem 1. *If G is a graph of order n not containing any C_5 then the number of triangles in G is at most* $(5/4)n^{3/2} + o(n^{3/2})$.

Theorem 1 is sharp apart from the constant coefficient as the following example shows.

Example 1. Let G_0 be a C_4-free bipartite graph on $n/3 + n/3$ vertices with about $(n/3)^{3/2}$ edges. Double each vertex in one of the color classes and add an edge joining the old and the new copy. (We call these edges monochromatic.) Let G denote the resulting graph. The number of edges in G is $2(n/3)^{3/2} + o(n^{3/2})$. Clearly, the number of triangles in G is the number of edges in G_0. Furthermore, G does not contain any C_5 since the girth of G_0 is 6: suppose $v_1v_2v_3v_4v_5v_1$ is a cycle. It must contain a monochromatic edge since G_0 remains bipartite even after doubling some vertices. Let, say, v_1v_2 be a monochromatic edge. Clearly, the number of monochromatic edges in this pentagon is odd, and if it is 3 or 5 then the pentagon contains two consecutive monochromatic edges, which is impossible since the monochromatic edges are independent. Thus v_1v_2 is the only monochromatic edge in this pentagon. Since the old and the new copies have the same neighborhood in the other color class, we may assume that v_3, v_4, v_5 are old copies and v_1 is the old copy out of v_1 and v_2. Then $v_1v_3v_4v_5v_1$ is a C_4 in G_0, a contradiction.

Proof of Theorem 1. Let G be a C_5-free graph of n vertices.

Claim 1. *The number of K_4's in G is at most* $(1/8)n^{3/2} + o(n^{3/2})$ *and these K_4's are pairwise edge-disjoint.*

Proof. Notice that if two K_4's share a common edge then there is a pentagon in the union of these K_4's, and so in G. So, the K_4's are pairwise edge-disjoint. Now, take four edges from each K_4, a triangle and an arbitrary fourth edge. We prove that the graph G_0 of these edges is C_4-free. If $v_1v_2v_3v_4v_1$ is a four-cycle in this graph such that the four edges are from four distinct K_4's then let $v \neq v_1, v_2$ be a vertex in the K_4 containing v_1v_2. Then $v \neq v_3, v_4$ since the K_4's are pairwise edge-disjoint and so $v_1vv_2v_3v_4v_1$ is a pentagon in G, a contradiction. If $v_1v_2v_3v_4v_1$ is a four-cycle in this graph such that say, v_1v_2 and v_3v_4 are from the same K_4 then all the four edges are from the very same K_4, since they are pairwise edge-disjoint, but it contradicts the choice of the four edges from the K_4. Finally, if $v_1v_2v_3v_4v_1$ is a four-cycle in this graph such that v_1v_2 and v_2v_3 are from the very same K_4 then let v be the fourth vertex of this K_4. Then $vv_2v_3v_4v_1v$ is a pentagon in G, a contradiction. Thus, G_0 contains at most $(1/2)n^{3/2} + o(n^{3/2})$ edges and G contains at most $(1/8)n^{3/2} + o(n^{3/2})$ K_4's.

If e is an edge in a K_4 then every triangle containing e is contained in this K_4, otherwise G contains a pentagon. So, e is contained in two triangles. Delete one edge from each K_4 in G and let G_1 denote the resulting K_4-free graph. Then the number of destroyed triangles is exactly twice the number of K_4's in G, i.e. at most $(1/4)n^{3/2}$.

We prove that G_1 contains at most $n^{3/2} + o(n^{3/2})$ triangles.

Claim 2. *Let $v_1v_2v_3$ be a triangle in G_1. Then at most one of the edges of this triangle is contained in some other triangles too.*

Proof. Suppose that, say, $v_1v_2v_4$ and $v_1v_3v_5$ are triangles that $v_4, v_5 \notin \{v_1, v_2, v_3\}$. Then $v_4 \neq v_5$ since G_1 is K_4-free and $v_1v_4v_2v_3v_5v_1$ is a pentagon in G_1, a contradiction. □

Choose an edge e from each triangle T such that the other edges of T are contained in exactly one triangle, T. Let E_0 denote the set of these edges and let $H = (V(G), E_0)$. Notice that E_0 contains exactly one edge of every triangle. If not, say, the edge set of T contains two elements of E_0, then one of these edges represents another triangle. But then this other triangle contains this edge too, and we would have had to choose it from $E(T)$, a contradiction.

Let $V(G) = x_1, x_2, \ldots, x_n$, let E_{x_i} denote the set of edges x_ix_j in E_0 such that $j > i$ and let S_x denote the set of vertices v such that vwx is a triangle with $wx \in E_x$. Notice that $S_x \cap N_H(x) = \emptyset$, since every triangle contains exactly one element of E_0. Furthermore, $\sum_{x \in V(G)} |S_x|$ is the number of triangles in G_1.

Claim 3. $|S_{x_i} \cap S_{x_j}| \leqslant 1$ *for* $x_i \neq x_j \in V(G)$, $j > i$.

Proof. We prove the claim by contradiction. Suppose that, say, $u, v \in S_{x_i} \cap S_{x_j}$. Then $x_j u$ is in a triangle $x_j u w$ such that $x_j w \in E_0$. Now $w \neq v$ since $S_{x_j} \cap N_H(x_j) = \emptyset$ and $w \neq x_i$ since $x_i x_j \notin E_{x_j}$ by $j > i$. Then $x_i u w x_j v$ is a pentagon in G, a contradiction.

By Claim 3, every pair of vertices are covered by at most one set S_x, and so,

$$\sum_{x \in V(G)} \binom{|S_x|}{2} \leqslant \binom{n}{2}.$$

Then,

$$\sum_{x \in V(G)} \binom{|S_x|}{2} \geqslant n \left(\frac{\sum_{x \in V(G)} |S_x|}{\frac{n}{2}} \right)$$

by the inequality of arithmetic and quadratic means (convexity argument). Thus,

$$\frac{\sum_{x \in V(G)} |S_x|}{n} \left(\frac{\sum_{x \in V(G)} |S_x|}{n} - 1 \right) \leqslant n - 1$$

and the number of triangles in G_1, $\sum_{x \in V(G)} |S_x| \leqslant n^{3/2} + \mathrm{O}(n)$.

2. 3-uniform hypergraphs

After having proved the estimate on the triangles in a pentagon-free graph, it is plausible to study the number of edges in a pentagon-free 3-uniform hypergraph. For this purpose, let us define the 5-cycles in a hypergraph. (Other definitions are possible too!)

Definition. We say that distinct edges h_1, h_2, h_3, h_4, h_5 constitute a 5-cycle if there are distinct vertices v_1, v_2, v_3, v_4, v_5 such that $v_1, v_2 \in h_1$, $v_2, v_3 \in h_2$, $v_3, v_4 \in h_3$, $v_4, v_5 \in h_4$, $v_5, v_1 \in h_5$. (There is no restriction on the other possible incidences!) We say that v_1, v_2, v_3, v_4, v_5 are the (not necessarily unique) vertices of this 5-cycle.

It is remarkable that the analogue of Theorem 1 can be proved for 3-uniform hypergraphs.

Theorem 2. *Let* (V, H) *be a* 3*-uniform hypergraph of n vertices not containing any* 5*-cycle. Then*

$$|H| \leqslant \sqrt{2} n^{3/2} + 4.5n.$$

Remark. It is very remarkable that Lazebnik and Verstraete [5] proved the same order of magnitude upper bound for 3-uniform hypergraphs with no 3- or 4-cycle.

Theorem 2 is also sharp apart from the constant as the natural modification of Example 1 shows.

Example 2. Let G_0 be a C_4-free graph on $n/3 + n/3$ vertices with $(n/3)^{3/2} + \mathrm{o}(n^{3/2})$ edges. Double each vertex in one of the color classes to obtain triples from each edge. Let H denote the resulting hypergraph. Clearly, the number of 3-edges in H is the number of edges in G_0. Furthermore, H does not contain any C_5 since the girth of G_0 is 6: suppose that v_1, v_2, v_3, v_4, v_5 are the vertices of a 5-cycle. Then there are two consecutive vertices in this sequence (or v_5 and v_1) belonging to the same color class. W.l.o.g., we may assume that v_1 and v_5 are such vertices. Then, the images of v_1, v_2, v_3 and v_4 in G_0 constitute a C_4, a contradiction.

Proof of Theorem 2. By contradiction, assume that (V, H_0) is a 3-uniform hypergraph of n vertices not containing any 5-cycle such that

$$|H_0| > \sqrt{2} n^{3/2} + 4.5n.$$

Then deleting the vertices of degree less than $\sqrt{2}n^{1/2}+4.5$ one by one, we do not decrease the average degree $d(H_0)$, and finally, we get a hypergraph H such that the minimum degree $\delta(H)$ is at least $\sqrt{2}n^{1/2}+4.5$ and the average degree $d(H)$ is at least $3\sqrt{2}n^{1/2}+13.5$.

Take an arbitrary vertex x of degree at least $3\sqrt{2}n^{1/2}+13.5$.

Let $N(x)$ be the set of vertices (not equal to x) contained in a hyperedge containing x and let G_1 denote the graph of the edges $e-x$ for the edges of H containing x. Notice that the number of edges in G_1 is $d(x)\geqslant 3\sqrt{2}n^{1/2}+13.5$.

Claim 4. *The average degree in G_1 is at most* 2.

Proof. Suppose not. Then, we may delete all vertices of degree 1 one by one, the average degree will not decrease and we get a graph G_1' with minimum degree 2 or more, average degree more than 2, and we may assume that G_1' is connected since otherwise take a component with average degree more than 2. The graph G_1' has a cycle C. If it has length 6 or more then G_1' contains a six vertex path $x_1x_2x_3x_4x_5x_6$. Let e_i be the edge of H containing x_i, x_{i+1}, x for $i=1,\ldots,5$. Then these edges constitute a pentagon on the vertices x, x_2, x_3, x_4, x_5. If C has five edges, then the hyperedges containing these edges constitute a pentagon in H. If $C=x_1x_2x_3x_4$ then it is not the whole graph since it has average degree 2. W.l.o.g., we may assume that G_1' contains an edge x_4x_5 where $x_5=x_2$ is possible. Then the hyperedges $xx_4x_1, xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_5$ constitute a pentagon on the vertices x, x_1, x_2, x_3, x_4. If $C=x_1x_2x_3$ then it is not the whole graph either since it has average degree 2. W.l.o.g., we may assume that G_1' contains an edge x_3x_4 and since the minimum degree is at least 2, there is an edge x_4x_5 where x_5 might be x_1 or x_2. Then the hyperedges $xx_3x_1, xx_1x_2, xx_2x_3, xx_3x_4, xx_4x_5$ constitute a pentagon on the vertices x, x_1, x_2, x_3, x_4. □

By Claim 4, $m=|N(x)|\geqslant 3\sqrt{2}n^{1/2}+13.5$ and the number of hyperedges having at least one vertex in $N(x)$ (with multiplicities) is $\sum_{y\in N(x)}d(y)\geqslant m(\sqrt{2}n^{1/2}+4.5)$.

Consider the hyperedges e not containing x but containing at least two elements of $N(x)$. Let G_2 be the graph on $N(x)$ with the following edge set: for every such a hyperedge e, if $e\cap N(x)$ has two elements then take it as an edge, if it has 3 elements (i.e. $e\subseteq N(x)$) then take any two elements of it as an edge.

Claim 5. *The graph G_2 does not contain a path P_7 of seven vertices and so it has at most* $2.5m$ *edges by the Erdős–Gallai theorem [3].*

Proof. Suppose $v_1v_2\cdots v_7$ is path in G_2. Then there is a hyperedge e_1 containing x and v_4. Then e_1 cannot contain both v_1 and v_7 since it has only three elements. W.l.o.g., assume that $v_1\notin e_1$. Then there is another hyperedge e_2 containing x and v_1. Furthermore let e_3, e_4, e_5 be some hyperedges containing the vertex pairs $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}$, respectively. Then the hyperedges e_2, e_3, e_4, e_5, e_1 constitute a pentagon on the vertices x, v_1, v_2, v_3, v_4, a contradiction. □

Notice that several hyperedges may define the very same edge in G_2, i.e. the edges of G_2 may have multiplicities.

Claim 6. *The sum of the multiplicities greater than one is at most $n-1$. Furthermore, at most m of these hyperedges are contained in $N(x)$.*

Proof. Suppose not. Consider the hyperedges defining these edges with multiplicities at least 2, with total multiplicities at least n. For each of these at least n hyperedges, take the third vertex not in the defined edge of G_2. These n vertices are different from x, so two of them are equal, say, y and let $e_1=v_1v_2y$ and $e_2=v_3v_4y$ be these hyperedges. Then these hyperedges are distinct, so either v_1, v_2, v_3, v_4 are distinct, or at most two vertices, say v_1 and v_4 are equal.

Case 1: $v_4=v_1$.

Since $v_1\in N(x)$, so there is a hyperedge e_3 containing x and v_1. Then $\{v_2, v_3\}\nsubseteq e_3$. W.l.o.g., assume that $v_2\notin e_3$. Then let e_4 be a hyperedge containing x and v_2 and let $e_5\neq e_2$ be another hyperedge containing v_1 and v_3. Then the hyperedges e_1, e_4, e_3, e_5, e_2 constitute a pentagon on vertices y, v_2, x, v_1, v_3, a contradiction.

Case 2: v_1, v_2, v_3, v_4 are pairwise distinct.

Since $v_1 \in N(x)$, so there is a hyperedge e_3 containing x and v_1. Then $\{v_3, v_4\} \not\subseteq e_3$. W.l.o.g., assume that $v_4 \notin e_3$. Then let e_4 be a hyperedge containing x and v_4 and let $e_5 \neq e_1$ be another hyperedge containing v_1 and v_2. Then the hyperedges e_1, e_5, e_3, e_4, e_2 constitute a pentagon on vertices y, v_2, v_1, x, v_4, a contradiction. □

By Claims 5 and 6, the contribution of hyperedges containing at least two vertices of $N(x)$ but not containing x to the sum of the degrees of the vertices in $N(x)$ is less than $7.5m + 3m + 2(n - 1 - m)$. Since at most m hyperedges contain x and two elements of $N(x)$, so, the number of hyperedges having exactly one vertex in $N(x)$ but not containing x is more than $m(\sqrt{2}n^{1/2} + 4.5) - 8.5m - 2n > 4(n - m)$. Let E_1 denote the set of these hyperedges.

Claim 7. *There are no hyperedges $e_1, e_2, e_3 \in E_1$ such that $|e_1 \cap e_2 \cap e_3| = 2$ and $e_1 \cap e_2 \cap e_3 \cap N(x) = \emptyset$.*

Proof. Suppose that there are vertices $y, z \notin N(x) \cup x$ and hyperedges $e_1, e_2, e_3 \in E_1$ such that $y, z \in e_1 \cap e_2 \cap e_3$. Let v_i be the vertex of e_i in $N(x)$ for $i = 1, 2, 3$. There is hyperedge e_4 containing x and v_1. Then e_4 cannot contain both v_2 and v_3. W.l.o.g., we may suppose it does not contain v_2. Now, take a hyperedge $e_5(\neq e_4)$ containing x and v_2. Then the hyperedges e_1, e_4, e_5, e_2, e_3 constitute a pentagon on y, v_1, x, v_2, z, a contradiction. □

Let G_3 denote the graph of edges of form $e_i - N(x)$ for $e_i \in E_1$ on the vertex set $V - N(x) - x$. Then by Claim 7, G_3 has more than $2(n - m - 1)$ edges. Deleting the vertices of degree 2 or less, the average degree will not decrease and we get a subgraph G_4 with minimum degree at least 3 and average degree more than 4.

By Erdős–Gallai theorem, G_4 contains a path $v_1v_2v_3v_4v_5v_6$. Let e_i denote a hyperedge containing v_i and v_{i+1} for $i = 1, 2, 3, 4, 5$. Let y and z denote the vertex in $N(x) \cap e_1$ and $N(x) \cap e_5$, respectively. If $y = z$ then the hyperedges e_1, e_2, e_3, e_4, e_5 constitute a pentagon on the vertices $v_1, v_2, v_3, v_4, v_5, v_6$, a contradiction.

So, we may assume that $y \neq z$. Now, let w denote the vertex in $N(x) \cap e_3$. Suppose that w is not equal to either y or z. Then $w \in N(x)$, so there is a hyperedge e_6 containing x and w. Then either y or z is not contained in e_6. W.l.o.g., assume that $y \notin e_6$. Then there exists a hyperedge $e_7(\neq e_6)$ containing y and x and the hyperedges e_7, e_1, e_2, e_3, e_6 constitute a pentagon on the vertices x, y, v_2, v_3, w, a contradiction. So, we may assume that either $w = y$ or $w = z$.

W.l.o.g., we may assume that $w = y$, i.e., $e_3 = \{y, v_3, v_4\}$. If there are distinct hyperedges e_6 and e_7 containing $\{x, y\}$ and $\{x, z\}$, respectively, then we are done as above. So, we may assume that the only hyperedge containing these vertex pairs is $e_6 = \{x, y, z\}$.

Since G_4 has minimum degree at least 3, there is an edge v_3v_7 in G_4 such that $v_7 \neq v_2, v_4$ and there is a hyperedge $e_7 = v_3v_7u$ defining this edge such that $u \in N(x)$. If $u \neq y, z$ then let e_8 be a hyperedge containing u and x. Now, the hyperedges e_7, e_2, e_1, e_6, e_8 constitute a pentagon on the vertices u, v_3, v_2, y, x, a contradiction. If $u = y$ then the hyperedges e_3, e_4, e_5, e_6, e_7 constitute a pentagon on the vertices v_3, v_4, v_5, z, y, a contradiction.

So, we may assume that $u = z$ and $e_7 = v_3v_7z$. Since G_4 has minimum degree at least 3, there is an edge v_4v_8 in G_4 such that $v_8 \neq v_3, v_5$ and there is a hyperedge $e_8 = v_4v_8v$ defining this edge such that $v \in N(x)$. If $v \neq y, z$ then let e_9 be a hyperedge containing v and x. Now, the hyperedges e_4, e_5, e_6, e_9, e_8 constitute a pentagon on the vertices v_4, v_5, z, x, v, a contradiction. If $v = z$ then the hyperedges e_1, e_2, e_3, e_8, e_6 constitute a pentagon on the vertices y, v_2, v_3, v_4, z, a contradiction. Finally, if $v = y$ then the hyperedges e_8, e_3, e_7, e_5, e_4 constitute a pentagon on the vertices v_4, y, v_3, z, v_5, a contradiction. Since G_4 has minimum degree at least 3, there is an edge v_3v_7 in G_4 such that $v_7 \neq v_2, v_4$ and there is a hyperedge $e_7 = v_3v_7u$ defining this edge such that $u \in N(x)$. If $u \neq y, z$ then let e_8 be a hyperedge containing u and x. Since G_4 has minimum degree at least 3, there is an edge v_3v_7 in G_4 such that $v_7 \neq v_2, v_4$ and there is a hyperedge $e_7 = v_3v_7u$ defining this edge such that $u \in N(x)$. If $u \neq y, z$ then let e_8 be a hyperedge containing u and x. Since G_4 has minimum degree at least 3, there is an edge v_3v_7 in G_4 such that $v_7 \neq v_2, v_4$ and there is a hyperedge $e_7 = v_3v_7u$ defining this edge such that $u \in N(x)$. If $u \neq y, z$ then let e_8 be a hyperedge containing u and x. □

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