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Lie–Bäcklund Tangent Transformations

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INTRODUCTION

The general theory of contact transformations which arose in the study of geometry and mechanics was developed principally by Sophus Lie. Lie's theory and various applications can be found in the fundamental works of Lie [1] and of Lie and Engel [2]. Lie's theory treats the case of one function of several variables in the context of continuous groups of first-order contact transformations and its main result is an infinitesimal characterization of these contact transformations. The direct generalizations of his formulation either to the treatment of more than one function or to the search for higher finite-order contact transformations yield in the former case only Lie point transformations, and in the latter case no new results beyond those of the theory of first-order contact transformations. The reasons for this situation are discussed in Section 1 of this work and these restrictions are shown to be fundamentally a consequence of the consideration of finite-order contact transformations.

One of the main results of the work presented here is that there exists for an arbitrary number of functions a generalization of Lie's formulation based on the notion of infinite-order contact transformations. Another one is that this generalization realizes and extends Bäcklund's original program [3] by studying his "surface transformations" in the language of point transformations of a necessarily infinite-dimensional space. Here we confine our attention to continuous groups of infinite-order contact transformations and in this way are able to give an infinitesimal characterization of these groups. We call them groups of Lie–Bäcklund tangent transformations. These results are discussed in Section 2.

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As an important application of Lie–Bäcklund tangent transformations, we obtain in Section 3 a group theoretical basis for the generalizations [4, 5] of the group theoretical analysis of differential equations [6].

The consideration of the group theoretic analysis of differential equations by means of Lie–Bäcklund tangent transformations leads us to the idea of considering the invariance of the tangent structure equations in conjunction with a given system of differential equations. This is also discussed in Section 3.

In order to bring out the algorithmic content of the notions presented here, we establish the possibility of operationally working with these notions in the language of infinitesimal operators. In particular, we systematically employ the calculation techniques developed by Ovsjannikov in his group theoretic analysis of differential equations [6]. Further, without loss of generality we treat only the case of one-parameter groups.

1. The Contact Transformations of Sophus Lie

Consider the group G of point transformations

$$G:\quad \begin{aligned} x'^i &= f^i(x, u, \underset{1}{u}; a), && i = 1,\dots, N,\\ u'^\alpha &= \phi^\alpha(x, u, \underset{1}{u}; a), && \alpha = 1,\dots, M,\\ u'^\alpha_i &= \psi_i^\alpha(x, u, \underset{1}{u}; a), && i = 1,\dots, N;\quad \alpha = 1,\dots, M, \end{aligned} \tag{1.1}$$

in the space of independent variables $(x, u, \underset{1}{u})$ where a is the group parameter, $x = (x^1,\dots, x^N) \in \mathbb{R}^N$, $u = (u^1,\dots, u^M) \in \mathbb{R}^M$ and $\underset{1}{u} = (u_1{}^1, u_2{}^1,\dots, u_N{}^M) \in \mathbb{R}^{NM}$ together with another group $\tilde{G}$ of point transformations in the space of independent variables $(x, u, \underset{1}{u}, dx, du, d\underset{1}{u})$. The group $\tilde{G}$ is obtained by the extension of the action of the group G to the differentials by means of the formulas

$$\begin{aligned} dx'^i &= \frac{\partial f^i}{\partial x^j}\,dx^j + \frac{\partial f^i}{\partial u^\beta}\,du^\beta + \frac{\partial f^i}{\partial u_j{}^\beta}\,du_j{}^\beta,\\ du'^\alpha &= \frac{\partial \phi^\alpha}{\partial x^j}\,dx^j + \frac{\partial \phi^\alpha}{\partial u^\beta}\,du^\beta + \frac{\partial \phi^\alpha}{\partial u_j{}^\beta}\,du_j{}^\beta,\\ du'^\alpha_i &= \frac{\partial \psi_i{}^\alpha}{\partial x^j}\,dx^j + \frac{\partial \psi_i{}^\alpha}{\partial u^\beta}\,du^\beta + \frac{\partial \psi_i{}^\alpha}{\partial u_j{}^\beta}\,du_j{}^\beta. \end{aligned} \tag{1.2}$$

Lie treated the case $M = 1$ and called the group G a group of contact transformations if the equation

$$du - u_i\,dx^i = 0 \tag{1.3}$$

is invariant with respect to the extended group $\tilde{G}$. Hereafter, we refer only to this particular group of transformations as the *group of contact transformations of S. Lie*.

A natural direction in which to attempt to generalize the notion of the contact transformations of S. Lie is to consider the case of arbitrary M. Indeed, we say that G is a group of contact transformations if the equations

$$du^\alpha - u_i^\alpha \, dx^i = 0, \qquad \alpha = 1,..., M, \tag{1.4}$$

are invariant with respect to the group $\tilde{G}$. One example of such a group of contact transformations is the group $\mathscr{P}$ of Lie point transformations in the (x, u) space

$$\mathscr{P}: \quad \begin{aligned} x'^i &= g^i(x, u; a), \qquad & i &= 1,..., N, \\ u'^\alpha &= h^i(x, u; a), \qquad & \alpha &= 1,..., M, \end{aligned} \tag{1.5}$$

extended in the usual way [1, 6, 7] to "derivatives" $\underset{1}{u}$. The following theorem states that such an extended group of point transformations is the only possible type of contact transformation for $M > 1$. The proof of this important result due to Ovsjannikov [8], not reproduced here because of space considerations, involves an infinitesimal technique which is employed throughout this paper.

THEOREM 1. *If $M > 1$, then every group G of contact transformations* (1.1) *is the extension to the derivatives $\underset{1}{u}$ of a group $\mathscr{P}$ of point transformations* (1.5).

For $M = 1$, this proof also yields Lie's infinitesimal characterization of his contact transformations, which can be expressed as the following proposition [2, 7].

THEOREM 2. *If $M = 1$, the transformations* (1.1) *form a group G of contact transformations, if there exists a function $W(x, u, \underset{1}{u})$ such that*

$$\xi^i = -\frac{\partial W}{\partial u_i}, \qquad \eta = W - u_i \frac{\partial W}{\partial u_i}, \qquad \zeta_i = \frac{\partial W}{\partial x^i} + u_i \frac{\partial W}{\partial u}, \tag{1.6}$$

where

$$\xi^i = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \ \eta = \left.\frac{\partial \phi}{\partial a}\right|_{a=0}, \ \zeta_i = \left.\frac{\partial \psi_i}{\partial a}\right|_{a=0}.$$

Summarizing the results thus far: Nontrivial (i.e., more general than extensions of point transformations $\mathscr{P}$) contact transformations exist only for the case of one function u $(M = 1)$.

Another suggested direction of generalization of Lie's definition is to introduce higher-order contact transformations. Therefore let us consider the group G_n of the point transformations

$$G_n: \quad \begin{aligned} x'^i &= f^i(x, u, \underset{1}{u},..., \underset{n}{u}; a), \qquad & i &= 1,..., N, \\ u'^\alpha &= \phi^\alpha(x, u, \underset{1}{u},..., \underset{n}{u}; a), \qquad & \alpha &= 1,..., M, \\ u_i'^\alpha &= \psi_i^\alpha(x, u, \underset{1}{u},..., \underset{n}{u}; a), \qquad & &\text{all } i, \alpha, \\ &\vdots & & \\ u'^\alpha_{i_1 \cdots i_n} &= \psi^\alpha_{i_1 \cdots i_n}(x, u, \underset{1}{u},..., \underset{n}{u}; a), \qquad & &\text{all } i_1,..., i_n, \alpha, \end{aligned} \tag{1.7}$$

in the space of variables $(x, u, \underset{1}{u}, ..., \underset{n}{u})$, where for every $s = 1, ..., n$; $\underset{s}{u} = \{u^\alpha_{i_1 \cdots i_s} \mid \alpha = 1, ..., M; i_1, ..., i_s = 1, ..., N\}$ (The quantities $u^\alpha_{i_1 \cdots i_s}$ are taken here to be symmetric in their lower indices. In all of our considerations and without loss of generality we consider all functions to depend only on the independent quantities.) In this case, we say that G_n is a group of contact transformations of nth-order if

$$du^\alpha - u_i^\alpha \, dx^i = 0, \qquad \alpha = 1, ..., M,$$
$$du^\alpha_{i_1 \cdots i_s} - u^\alpha_{i_1 \cdots i_s j} \, dx^j = 0, \qquad \alpha = 1, ..., M; \quad s = 1, ..., n-1, \tag{1.8}$$

are invariant with respect to the group $\tilde{G}_n$ obtained by the extension of the group G to the differentials dx^i, $du^\alpha, ..., du^\alpha_{i_1 \cdots i_n}$. The proof of Theorem 1 is not directly applicable to this case because the du_i^α are no longer independent variables. It is known that there do not exist nontrivial groups of higher-order contact transformations [3, 9, 10]. This result is fundamental to understanding the need for the definition of tangent transformations introduced in Section 2. Because of this and the fact that a proof of it in the language of infinitesimal operators is integral to the new proofs and results presented in this paper, we present such a proof here. Further, this extends the previous calculations of one of the authors [10] to the general case.

For the proof of this statement it is convenient to characterize the group G_n of contact transformations (1.7) in the language of the infinitesimal operator X of this group. The operator X is given in general by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i{}^\alpha \frac{\partial}{\partial u_i{}^\alpha} + \cdots + \zeta^\alpha_{i_1 \cdots i_n} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_n}}, \tag{1.9}$$

where

$$\xi^i = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \qquad \eta^\alpha = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}, ..., \zeta^\alpha_{i_1 \cdots i_s} = \left.\frac{\partial \psi^\alpha_{i_1 \cdots i_s}}{\partial a}\right|_{a=0}, \tag{1.10}$$

$s = 1, ..., n$. As in the preceding the infinitesimal operator of the group $\tilde{G}_n$ is given by

$$\tilde{X} = X + \tilde{\xi}^i \frac{\partial}{\partial (dx^i)} + \tilde{\eta}^\alpha \frac{\partial}{\partial (du^\alpha)} + \tilde{\zeta}_i^\alpha \frac{\partial}{\partial (du_i{}^\alpha)} + \cdots + \tilde{\zeta}^\alpha_{i_1 \cdots i_{n-1}} \frac{\partial}{\partial (du^\alpha_{i_1 \cdots i_{n-1}})} \tag{1.11}$$

(the operator of differentiation with respect to $du^\alpha_{i_1 \cdots i_n}$ is not needed for the following considerations and is omitted). From (1.7) and (1.10) it follows that

$$\tilde{\xi}^i = \frac{\partial \xi^i}{\partial x^j} dx^j + \frac{\partial \xi^i}{\partial u^\alpha} du^\alpha + \frac{\partial \xi^i}{\partial u_j^\alpha} du_j^\alpha + \cdots + \frac{\partial \xi^i}{\partial u^\alpha_{i_1 \cdots i_n}} du^\alpha_{i_1 \cdots i_n},$$

$$\tilde{\eta}^\alpha = \frac{\partial \eta^\alpha}{\partial x^j} dx^j + \frac{\partial \eta^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \eta^\alpha}{\partial u_j^\beta} du_j^\beta + \cdots + \frac{\partial \eta^\alpha}{\partial u^\beta_{i_1 \cdots i_n}} du^\beta_{i_1 \cdots i_n},$$

$$\tilde{\zeta}^\alpha_{i_1 \cdots i_s} = \frac{\partial \zeta^\alpha_{i_1 \cdots i_s}}{\partial x^j} dx^j + \frac{\partial \zeta^\alpha_{i_1 \cdots i_s}}{\partial u^\beta} du^\beta + \frac{\partial \zeta^\alpha_{i_1 \cdots i_s}}{\partial u_j^\beta} du_j^\beta + \cdots + \frac{\partial \zeta^\alpha_{i_1 \cdots i_s}}{\partial u^\beta_{j_1 \cdots j_n}} du^\beta_{j_1 \cdots j_n}, \tag{1.12}$$

where $s = 1,..., n$.

Now the criteria for the invariance of equations (1.8) are given by

$$\begin{aligned} \tilde{X}(du^\alpha - u_i^\alpha\, dx^i)|_{(1.8)} &= 0, \\ \tilde{X}(du^\alpha_{i_1 \cdots i_s} - u^\alpha_{i_1 \cdots i_s j}\, dx^j)|_{(1.8)} &= 0, \qquad s = 1,..., n-1, \end{aligned} \tag{1.13}$$

or from (1.11)

$$\begin{aligned} (\tilde{\eta}^\alpha - \zeta_i^\alpha\, dx^i - u_i^\alpha \tilde{\xi}^i)|_{(1.8)} &= 0, \\ (\tilde{\zeta}^\alpha_{i_1 \cdots i_s} - \zeta^\alpha_{i_1 \cdots i_s j}\, dx^j - u^\alpha_{i_1 \cdots i_s j} \tilde{\xi}^j)|_{(1.8)} &= 0, \qquad s = 1,..., n-1. \end{aligned} \tag{1.14}$$

It is now convenient to express our formulas in terms of the operators

$$\underset{s}{D_i} = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u^\alpha_{i i_1} \frac{\partial}{\partial u^\alpha_{i_1}} + \cdots + u^\alpha_{i i_1 \cdots i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \tag{1.15}$$

where s is a positive integer and $i = 1,..., N$. These operators naturally arise in group theoretic calculations.

Now by virtue of (1.8) it is possible to express the differentials $du^\alpha,..., du^\alpha_{i_1 \cdots i_{n-1}}$ in terms of the independent quantities $u_i^\alpha,..., u^\alpha_{i_1 \cdots i_n}$, dx^i. These expressions plus (1.12) when substituted into (1.14) yield after rearrangement, use of the operator $\underset{s}{D_i}$, and independence of dx^i and $du^\alpha_{i_1 \cdots i_n}$, the equivalent system of equations

$$\begin{aligned} \underset{n-1}{D_i}(\eta^\alpha) - u_j^\alpha \underset{n-1}{D_i}(\xi^j) - \zeta_i^\alpha &= 0, \\ \underset{n-1}{D_i}(\zeta^\alpha_{i_1 \cdots i_s}) - u^\alpha_{i_1 \cdots i_s j} \underset{n-1}{D_i}(\xi^j) - \zeta^\alpha_{i_1 \cdots i_s i} &= 0, \qquad s = 1,..., n-1, \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} \frac{\partial \eta^\alpha}{\partial u^\beta_{i_1 \cdots i_n}} - u_i^\alpha \frac{\partial \xi^i}{\partial u^\beta_{i_1 \cdots i_n}} &= 0, \\ \frac{\partial \zeta^\alpha_{i_1 \cdots i_s}}{\partial u^\beta_{j_1 \cdots j_n}} - u^\alpha_{i_1 \cdots i_s j} \frac{\partial \xi^j}{\partial u^\beta_{j_1 \cdots j_n}} &= 0, \qquad s = 1,..., n-1. \end{aligned} \tag{1.17}$$

Before attempting to solve these equations, we first rewrite (1.17) in terms of the new functions

$$W^\alpha = \eta^\alpha - \xi^i u_i^\alpha, \qquad\qquad (1.18)$$
$$W^\alpha_{i_1\cdots i_s} = \zeta^\alpha_{i_1\cdots i_s} - \xi^j u^\alpha_{i_1\cdots i_s j}, \qquad s = 1,\ldots, n-1,$$

and obtain the equivalent system

$$\frac{\partial W^\alpha}{\partial u^\beta_{j_1\cdots j_n}} = 0,$$
$$\frac{\partial W^\alpha_{i_1\cdots i_s}}{\partial u^\beta_{j_1\cdots j_n}} = 0, \qquad s = 1,\ldots, n-2, \qquad\qquad (1.19)$$

$$\frac{\partial W^\alpha_{i_1\cdots i_{n-1}}}{\partial u^\beta_{j_1\cdots j_n}} = -\xi^{j_n}\delta_\beta{}^\alpha\delta^{j_1\cdots j_{n-1}}_{i_1\cdots i_{n-1}}. \qquad\qquad (1.20)$$

Let us now consider the case $M > 1$; then (1.20) implies

$$\frac{\partial W^\alpha_{i_1\cdots i_{n-1}}}{\partial u^\beta_{j_1\cdots j_n}} = 0, \qquad \alpha \neq \beta, \qquad\qquad (1.21)$$

for all values of the indices α, $i_1,\ldots, i_{n-1}, j_1,\ldots, j_n$, and

$$\xi^j = -\frac{\partial W^\alpha_{i_1\cdots i_{n-1}}}{\partial u^\alpha_{i_1\cdots i_{n-1}j}} \qquad \text{(no summation on any of the indices).} \qquad\qquad (1.22)$$

From (1.22) we have immediately that ξ^j does not depend on $\underset{n}{u}$ and the general solution of (1.22) is

$$W^\alpha_{i_1\cdots i_{n-1}} = U^\alpha_{i_1\cdots i_{n-1}}(x, u, \underset{1}{u},\ldots, \underset{n-1}{u}) - \xi^j(x, u, \underset{1}{u},\ldots, \underset{n-1}{u})\, u^\alpha_{i_1\cdots i_{n-1}j},$$

with arbitrary functions $U^\alpha_{i_1\cdots i_{n-1}}$, $\alpha = 1,\ldots, M$. Further, from definitions (1.18) and Eq. (1.19), we obtain that the coordinates $\xi^i, \eta^\alpha, \zeta_i{}^\alpha,\ldots, \zeta^\alpha_{i_1\cdots i_{n-1}}$ of the operator (1.9) do not depend on $\underset{n}{u}$. Lie's theory of continuous groups gives us immediately that the transformation laws of the quantities $x, u, \underset{1}{u},\ldots, \underset{n-1}{u}$ in (1.7) do not depend upon $\underset{n}{u}$.

By induction and Theorem 1, we obtain that for $M > 1$, $\xi^i = \xi^i(x, u)$, $\eta^\alpha = \eta^\alpha(x, u)$, and the coordinates $\zeta^\alpha_{i_1\cdots i_s}$ ($s = 1,\ldots, n$) are given by (1.16). This means that G given by (1.7) is the nth-order extension of the group $\mathscr{P}$ of point transformations in the space of (x, u) only.

Now we consider the case $M = 1$. If $N > 1$, we obtain from (1.20)

$$\xi^j = -\frac{\partial W_{i_1\cdots i_{n-1}}}{\partial u_{i_1\cdots i_{n-1}j}} \text{ (no sum)}, \left.\frac{\partial W_{i_1\cdots i_{n-1}}}{\partial u_{j_1\cdots j_{n-1}j}}\right|_{(j_1,\ldots,j_{n-1})\neq(i_1,\ldots,i_{n-1})} = 0.$$

From this as in the preceding case, we have that the ξ^j's do not depend on $\underset{n}{u}$ and the general solution of the above equation is

$$W_{i_1\cdots i_{n-1}} = U_{i_1\cdots i_{n-1}}(x, u, \underset{1}{u},\ldots, \underset{n-1}{u}) - \xi^j(x, u, \underset{1}{u},\ldots, \underset{n-1}{u})\, u_{i_1\cdots i_{n-1}j}\,,$$

and again by induction, we prove that $\xi^i = \xi^i(x, u, \underset{1}{u})$, $\eta = \eta(x, u, \underset{1}{u})$, $\zeta_i = \zeta_i(x, u, \underset{1}{u})$, and the other coordinates $\zeta_{i_1\cdots i_s}$ $(s = 2,\ldots, n)$ are given by extension formulas. This means that the group G_n is the nth-order extension of the contact transformations (first order) of S. Lie.

The same conclusion is valid for $M = N = 1$ and can be proved in the following way. In this case Eqs. (1.18)–(1.20) yield

$$\begin{aligned} W &= \eta - \xi \underset{1}{u}, \\ W_1 &= \zeta_1 - \xi \underset{2}{u}, \\ &\vdots \\ W_{n-2} &= \underset{n-2}{\zeta} - \xi \underset{n-1}{u}\,, \\ W_{n-1} &= \underset{n-1}{\zeta} - \xi \underset{n}{u}, \end{aligned} \tag{1.23}$$

where

$$\underset{s}{\zeta} = \zeta_{\underbrace{1\cdots 1}_{s}}$$

and

$$\partial W/\partial \underset{n}{u} = 0,\ldots, \partial W_{n-2}/\partial \underset{n}{u} = 0, \tag{1.24}$$

$$\xi = -\partial W_{n-1}/\partial \underset{n}{u} \tag{1.25}$$

Using the last equation in (1.23) together with (1.16) we obtain

$$\begin{aligned} \underset{n-1}{D}(W_{n-2}) &= \underset{n-1}{D}(\underset{n-2}{\zeta}) - \underset{n-1}{u}\,\underset{n-1}{D}(\xi) - \underset{n}{u}\xi \\ &= \underset{n-1}{\zeta} - \xi \underset{n}{u} = W_{n-1}\,, \end{aligned}$$

or using the definition of $\underset{n-1}{D}$,

$$W_{n-1} = \frac{\partial W_{n-2}}{\partial x} + \cdots + \underset{n-1}{u}\frac{\partial W_{n-2}}{\partial \underset{n-2}{u}} + \underset{n}{u}\frac{\partial W_{n-2}}{\partial \underset{n-1}{u}}\,.$$

The latter equation together with (1.23) and (1.25) implies that

$$\xi = -\frac{\partial W_{n-2}}{\partial \underset{n-1}{u}},$$

$$\eta = W - \underset{1}{u}\frac{\partial W_{n-2}}{\partial \underset{n-1}{u}},$$

$$\underset{1}{\zeta} = W_1 - \underset{2}{u}\frac{\partial W_{n-2}}{\partial \underset{n-1}{u}}$$

$$\vdots$$

$$\underset{n-2}{\zeta} = W_{n-2} - \underset{n-1}{u}\frac{\partial W_{n-2}}{\partial \underset{n-1}{u}},$$

$$\underset{n-1}{\zeta} = \frac{\partial W_{n-2}}{\partial x} + \cdots + \underset{n-1}{u}\frac{\partial W_{n-2}}{\partial \underset{n-2}{u}}.$$

Equations (1.24) shows us that the coordinates $\xi, \eta,..., \underset{n-1}{\zeta}$ depend only on $x, u, \underset{1}{u},..., \underset{n-1}{u}$. Hence these considerations show that there do not exist any finite-order contact transformations beyond those considered by Lie. This statement can be formulated in the following way.

THEOREM 3. *Every group of* nth-*order contact transformations is either:*

(i) *an extended group of point transformations $\mathscr{P}$ in the case of more than one function, or*

(ii) *an extended group of contact transformations of S. Lie in the case of one function.*

2. GROUPS OF LIE–BÄCKLUND TANGENT TRANSFORMATIONS

Bäcklund [3] considered one function u of several variables x, and introduced transformations of x and u depending not only on these variables, but also upon a set of finite derivatives $\underset{1}{u},..., \underset{n}{u}$. He treated these transformations as surface transformations in the (x, u)-space under the assumption that they are infinite-order contact transformations of these surfaces. He showed that this implies that the transformation laws for successively higher derivatives must involve even higher derivatives.

The essence of our generalization is to wed Bäcklund's idea of infinite-order contact transformations with Lie's theory of continuous groups of transformations. In this way, we can consider the general case of an arbitrary number of "derivatives." We show here that by treating these transformations as point transformations in the infinite-dimensional $(x, u, \underset{1}{u}, ..., \underset{n}{u}, ...)$-space we can construct a generalization of Lie's theory of contact transformations. In fact, we show that this is the unique possibility within the context of Lie's original considerations. In order to distinguish this possibility from that of Lie, we shall call these infinite-order contact transformations, Lie–Bäcklund tangent transformations.

Let $x = (x^1, ..., x^N) \in \mathbb{R}^N$, $u = (u^1, ..., u^M) \in \mathbb{R}^M$, and for every $s = 1, 2, 3, ...$, $\underset{s}{u}$ is the set of quantities $u^\alpha_{i_1 \cdots i_s}$ $(\alpha = 1, ..., M;\ i_1, ..., i_s = 1, ..., N)$ symmetric in their lower indices. Let us consider a one-parameter group G of point transformations

$$G: \quad \begin{aligned} x'^i &= f^i(x, u, \underset{1}{u}, \underset{2}{u}, ...; a), \\ u'^\alpha &= \phi^\alpha(x, u, \underset{1}{u}, \underset{2}{u}, ...; a), \\ u_i'^\alpha &= \psi_i^\alpha(x, u, \underset{1}{u}, \underset{2}{u}, ...; a), \\ & \cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2.1}$$

in the infinite-dimensional $(x, u, \underset{1}{u}, \underset{2}{u}, ...)$-space. The number of arguments of each of the f^i and ϕ^α is a priori arbitrary. Together with the group G, we consider its extension $\tilde{G}$ to the differentials dx^i, du^α, $du_i^\alpha, ...$ by means of the formulas

$$\begin{aligned} dx'^i &= \frac{\partial f^i}{\partial x^j} dx^j + \frac{\partial f^i}{\partial u^\beta} du^\beta + \frac{\partial f^i}{\partial u_j^\beta} du_j^\beta + \cdots, \\ du'^\alpha &= \frac{\partial \phi^\alpha}{\partial x^j} dx^j + \frac{\partial \phi^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \phi^\alpha}{\partial u_j^\beta} du_j^\beta + \cdots, \\ du_i'^\alpha &= \frac{\partial \psi_i^\alpha}{\partial x^j} dx^j + \frac{\partial \psi_i^\alpha}{\partial u^\beta} du^\beta + \frac{\partial \psi_i^\alpha}{\partial u_j^\beta} du_j^\beta + \cdots, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2.2}$$

DEFINITION. A group G is called a group of *Lie–Bäcklund tangent transformations* if the infinite system of equations

$$\begin{aligned} du^\alpha - u_j^\alpha \, dx^j &= 0, \\ du_i^\alpha - u_{ij}^\alpha \, dx^j &= 0, \\ du_{i_1 i_2}^\alpha - u_{i_1 i_2 j}^\alpha \, dx^j &= 0, \\ & \cdots\cdots\cdots\cdots \end{aligned} \tag{2.3}$$

is invariant with respect to the group $\tilde{G}$.

Now we state the infinitesimal invariance criteria of (2.3) in a compact form employing the infinite version of the differential operators (1.15):

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ii_1}^\alpha \frac{\partial}{\partial u_{i_1}^\alpha} + u_{ii_1i_2}^\alpha \frac{\partial}{\partial u_{i_1i_2}^\alpha} + \cdots . \tag{2.4}$$

(We emphasize that this operator acts on functions of the infinite set of independent variables x^i, u^α, u_i^α, u_{ij}^α ,....) Let

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1\cdots i_s}^\alpha \frac{\partial}{\partial u_{i_1\cdots i_s}^\alpha} + \cdots \tag{2.5}$$

be the infinitesimal operator of the group G, where

$$\xi^i = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \qquad \eta^\alpha = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0},$$
$$\zeta_{i_1\cdots i_s}^\alpha = \frac{\partial \psi_{i_1\cdots i_s}^\alpha}{\partial a}\bigg|_{a=0}, \qquad s = 1, 2, 3,.... \tag{2.6}$$

In this case as in the finite-dimensional case, the infinitesimal operator (2.5) fully characterizes the group of transformations (2.1) if we ensure the existence and uniqueness of the solution of the Lie equation:

$$\frac{dF}{da} = \Xi(F), \qquad F\,|_{a=0} = z,$$

where

$$z = (x^i, u^\alpha, u_i^\alpha, \cdots),$$
$$F = (f^i, \phi^\alpha, \psi_i^\alpha, \cdots),$$
$$\Xi = (\xi^i, \eta^\alpha, \zeta_i^\alpha, \cdots).$$

This is possible, for example, if we consider the transformations (2.1) in a Banach space B of points z such that the map Ξ is a smooth map of the space B into itself. Another possibility is to consider the transformations (2.1) in the analytical scale and to take the map Ξ to be a quasi-differential one [11]. When the conditions of existence and uniqueness of the solution of the Lie equation are satisfied, the group property of transformations (2.1),

$$F(F(z, a), b) = F(z, a + b),$$

follows immediately from the uniqueness of the solution. The proof is the same as in the finite-dimensional case (e.g., [6, 11]).

The infinitesimal operator $\tilde{X}$ of the group $\tilde{G}$ will have the form

$$\tilde{X} = X + \tilde{\xi}^i \frac{\partial}{\partial(dx^i)} + \tilde{\eta}^\alpha \frac{\partial}{\partial(du^\alpha)} + \tilde{\zeta}_i^\alpha \frac{\partial}{\partial(du_i^\alpha)} + \cdots , \tag{2.7}$$

where

$$\tilde{\xi}^i = \frac{\partial(dx'^i)}{\partial a}\bigg|_{a=0}, \qquad \tilde{\eta}^\alpha = \frac{\partial(du'^\alpha)}{\partial a}\bigg|_{a=0}, \qquad \tilde{\zeta}_i^\alpha = \frac{\partial(du_i'^\alpha)}{\partial a}\bigg|_{a=0}, \qquad \ldots,$$

and from Eqs. (2.2) and (2.6), we have

$$\begin{aligned}
\tilde{\xi}^i &= \frac{\partial \xi^i}{\partial x^j}\, dx^j + \frac{\partial \xi^i}{\partial u^\beta}\, du^\beta + \frac{\partial \xi^i}{\partial u_j^\beta}\, du_j^\beta + \cdots, \\
\tilde{\eta}^\alpha &= \frac{\partial \eta^\alpha}{\partial x^j}\, dx^j + \frac{\partial \eta^\alpha}{\partial u^\beta}\, du^\beta + \frac{\partial \eta^\alpha}{\partial u_j^\beta}\, du_j^\beta + \cdots, \\
\tilde{\zeta}_i^\alpha &= \frac{\partial \zeta_i^\alpha}{\partial x^j}\, dx^j + \frac{\partial \zeta_i^\alpha}{\partial u^\beta}\, du^\beta + \frac{\partial \zeta_i^\alpha}{\partial u_j^\beta}\, du_j^\beta + \cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{2.8}$$

The infinitesimal conditions for the invariance of system (2.3) are given by

$$\begin{aligned}
\tilde{X}(du^\alpha - u_j^\alpha\, dx^j)|_{(2.3)} &= 0, \\
\tilde{X}(du^\alpha_{i_1\cdots i_s} - u^\alpha_{i_1\cdots i_s j}\, dx^j)|_{(2.3)} &= 0, \qquad s = 1, 2, \ldots.
\end{aligned} \tag{2.9}$$

Immediately, we obtain from (2.7) and (2.8) the infinite analog of (1.14), but in contrast to the case of finite-order transformations, we obtain only the infinite analog of Eqs. (1.16) as the equivalent form of Eqs. (2.9) because the only independent differentials now are dx^i ($i = 1,\ldots, N$). As a result we obtain the following theorem for the infinitesimal characterization of groups of Lie–Bäcklund tangent transformations.

THEOREM 4. *The group G of transformations* (2.1) *is a group of Lie–Bäcklund tangent transformations if and only if coordinates of the infinitesimal operator* (2.5) *satisfy the equations*

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\
\zeta^\alpha_{i_1 i_2} &= D_{i_2}(\zeta^\alpha_{i_1}) - u^\alpha_{i_1 j} D_{i_2}(\xi^j) \\
&\;\;\vdots \\
\zeta^\alpha_{i_1\cdots i_s} &= D_{i_s}(\zeta^\alpha_{i_1\cdots i_{s-1}}) - u^\alpha_{i_1\cdots i_{s-1} j} D_{i_s}(\xi^j), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{2.10}$$

Remark. If in the transformations (2.1), the transformed quantities x', and u' depend only upon x, u, and the group parameter a, then Theorem 4 shows us that the group G of Lie–Bäcklund transformations must be the infinite extension of a group $\mathscr{P}$ of point transformations (1.5). In this case Eqs. (2.10) become the well-known extension formulas.

Generalizing the considerations of Bäcklund [3], we now consider whether it is possible in this formulation of tangent transformations to find a finite-dimensional space of the variables x, u, u_1 ,..., u_n which transforms into itself under the transformations (2.1). Therefore we assume that transformations (2.1) take the form

$$\begin{aligned}
x'^i &= f^i(x, u, \underset{1}{u}, \ldots, \underset{n}{u}; a), \\
u'^\alpha &= \phi^\alpha(x, u, \underset{1}{u}, \ldots, \underset{n}{u}; a), \\
u_i'^\alpha &= \psi_i^\alpha(x, u, \underset{1}{u}, \ldots, \underset{n}{u}; a), \\
&\vdots \\
u'^\alpha_{i_1\cdots i_n} &= \psi^\alpha_{i_1\cdots i_n}(x, u, \underset{1}{u}, \ldots, \underset{n}{u}; a), \\
u'^\alpha_{i_1\cdots i_{n+1}} &= \psi^\alpha_{i_1\cdots i_{n+1}}(x, u, \underset{1}{u}, \ldots; a), \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{2.11}$$

In this case the coefficients ξ^i, η^α, $\zeta_{i_1\cdots i_s}$ $(s = 1,\ldots, n)$ depend only on x, u, $\underset{1}{u}$,..., $\underset{n}{u}$.

First, consideration of the case $n = 1$ reduces to the mathematical scope and hence results of Ovsjannikov's proof [8]. As a result we find that G is the extension of the group of contact transformations of S. Lie in the case of one function and G is the extension of the group $\mathscr{P}$ of point transformations (1.5) in the case of more than one function.

Now we turn to the case $n > 1$. In this case we consider the first n equations of (2.10).

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\
\zeta^\alpha_{i_1\cdots i_{s+1}} &= D_{i_{s+1}}(\zeta^\alpha_{i_1\cdots i_s}) - u^\alpha_{i_1\cdots i_s j} D_{i_{s+1}}(\xi^j) \qquad (s = 1,\ldots, n-1).
\end{aligned}$$

The right-hand sides of these equations can be rewritten by using the definition (1.15) of the operator D_{n-1} as

$$\begin{aligned}
&D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) \\
&\quad = \underset{n-1}{D_i}(\eta^\alpha) - u_j^\alpha \underset{n-1}{D_i}(\xi^j) + u^\beta_{i i_1\cdots i_n}\left(\frac{\partial \eta^\alpha}{\partial u^\beta_{i_1\cdots i_n}} - u_j^\alpha \frac{\partial \xi^j}{\partial u^\beta_{i_1\cdots i_n}}\right), \\
&D_{i_{s+1}}(\zeta^\alpha_{i_1\cdots i_s}) - u^\alpha_{i_1\cdots i_s j} D_{i_{s+1}}(\xi^j) \\
&\quad = \underset{n-1}{D_{i_{s+1}}}(\zeta^\alpha_{i_1\cdots i_s}) - u^\alpha_{i_1\cdots i_s j} \underset{n-1}{D_{i_{s+1}}}(\xi^j) \\
&\qquad + u^\beta_{i_{s+1} j_1\cdots j_n}\left(\frac{\partial \zeta^\alpha_{i_1\cdots i_n}}{\partial u^\beta_{j_1\cdots j_n}} - u^\alpha_{i_1\cdots i_s j} \frac{\partial \xi^j}{\partial u^\beta_{j_1\cdots j_n}}\right),
\end{aligned}$$

where the terms involving D_{n-1} are independent of the coordinates of $\underset{n+1}{u}$. Now because the left-hand sides $\zeta^{\alpha}_{i_1 \cdots i_s}$ $(s = 1,..., n)$ of the same equations depend only on $x, u, \underset{1}{u},..., \underset{n}{u}$, we immediately obtain Eqs. (1.16) and (1.17). Therefore, using the same argument as in Section 1, we obtain that in this case the coordinates of the operator (2.5) depend only on x, u.

So besides groups $\mathscr{P}$ of point transformations and the groups of contact transformations of S. Lie there exist no groups of Lie–Bäcklund tangent transformations which act invariantly on a finite-dimensional $(x, u, \underset{1}{u},..., \underset{n}{u})$-space for any $n \geqslant 1$. Hence the theory of groups of Lie–Bäcklund tangent transformations is essentially a transformation theory of an infinite-dimensional space.

3. Application to Differential Equations

In generalizations of the method of group theoretic analysis of differential equations, two different points of departure were taken from the Lie–Ovsjannikov theory. One was to generalize directly the infinitesimal operator [4, 5], and the other was to generalize the transformation law [5, 12]. In both cases the existent group structure was violated. One of the results of this work is the unification of these methods within a consistent group theoretic structure.

In particular, in the second generalization, the transformation laws

$$\begin{aligned} x'^i &= f^i(x, u(x), \underset{1}{u}(x),...; a), \\ u'^{\alpha}(x') &= \phi^{\alpha}(x, u(x), \underset{1}{u}(x),...; a), \end{aligned} \tag{3.1}$$

were interpreted as surface transformations $(x, u(x)) \to (x', u'(x'))$ depending on the derivatives $\underset{1}{u}(x), \underset{2}{u}(x),....$ Such an interpretation does not of itself yield a group theoretic interpretation of this transformation law. Now, we consider (3.1) as the first two transformation formulas of the point transformations (2.1) and the definition of the transformation laws for the higher derivatives in the previous work [12] as conditions (2.10) for Lie–Bäcklund tangent transformations.

In order to understand the group nature of the first generalization we consider the transformation properties of manifolds in (x, u)-space under the group G of point transformations (2.1). Suppose that

$$u^{\alpha} - \phi^{\alpha}(x) = 0$$

transforms under G into the manifold

$$u'^{\alpha} - \phi'^{\alpha}(x') = 0.$$

Using the transformation law (2.1), the last equation can be written in infinitesimal language in the form

$$u^{\alpha} - \phi'^{\alpha}(x) + [\eta^{\alpha}(x, u, \underset{1}{u},\ldots) - \xi^{i}(x, u, \underset{1}{u},\ldots)(\partial\phi'^{\alpha}(x)/\partial x^{i})]\, a + o(a) = 0.$$

This formula is valid for every point (x, u) of the original manifold so we can replace u^{α} and $\underset{1}{u},\ldots$ in this expression by $\phi^{\alpha}(x)$ and its derivatives. This shows that $\phi'^{\alpha}(x) = \phi^{\alpha}(x) + o(a)$. Using this fact and because the function $\phi^{\alpha}(x)$ is an arbitrary one, we obtain

$$u'^{\alpha}(x) - u^{\alpha}(x) = [\eta^{\alpha}(x, u, \underset{1}{u}(x),\ldots) - \xi^{i}(x, u, \underset{1}{u}(x),\ldots)(\partial u^{\alpha}(x)/\partial x^{i})]\, a + o(a).$$

We can write this formula in the form

$$u'(x) - u(x) = a\hat{Q}[u(x)] + o(a), \tag{3.2}$$

where

$$\hat{Q} = (\hat{Q}^{1},\ldots, \hat{Q}^{M}),$$

and

$$\begin{aligned}\hat{Q}^{\alpha}[u(x)] = {} & \eta^{\alpha}(x, u(x), (\partial u^{\beta}/\partial x^{j})(x),\ldots) \\ & - \xi^{i}(x, u(x), (\partial u^{\beta}(x)/\partial x^{j}),\ldots)(\partial u^{\alpha}(x)/\partial x^{i}).\end{aligned} \tag{3.3}$$

If in particular the functions ξ^{i} and η^{α} are given by the forms

$$\xi^{i} = \xi^{i}(x)$$

and

$$\eta^{\alpha} = \mu(x) + \mu_{\beta}{}^{\alpha}(x)\, u^{\beta}(x) + \sum_{s=1}^{\infty} \mu_{\beta}^{\alpha, j_1 \cdots j_s}(x)\, u^{\beta}_{j_1 \cdots j_s},$$

then $\hat{Q}$ is an infinite-order differential operator of the form

$$\hat{Q}^{\alpha}[u(x)] = q(x) + q_{\beta}{}^{\alpha}(x)\, u^{\beta}(x) + \sum_{s=1}^{\infty} q_{\beta}^{\alpha, i_1 \cdots i_s}(x) \frac{\partial^{s} u^{\beta}(x)}{\partial x^{1} \cdots x^{s}},$$

which appears in the examples illustrating a generalization of the infinitesimal operator [4, 5].

The existence of groups of Lie–Bäcklund tangent transformations provides us with the possibility of generalizing some of the results of group analysis of differential equations [13, 14, 15]. For example, we can pose the problem of the classification of differential equations by means of these groups. Computationally, the technique is closely patterned after that employed in the Lie–Ovsjannikov theory and uses the infinitesimal operator (2.5) together with the analog of Eqs. (2.10). In this case we can consider an arbitrary order differential equation without the need for considering extensions.

The notion of invariant or partially invariant solutions can be directly carried over from the classical theory to this case. But we immediately encounter technical difficulties which are connected with the theory of invariants in an infinite-dimensional space. In particular, the characterization of a functional basis for the invariants in such a theory is a principal difficulty and as a result we cannot simply translate the description of invariant manifolds in terms of invariants from the finite-dimensional case [6].

It would be interesting to study the possibility of generalizing the theory of Tresse [16], which establishes the existence of a finite basis for differential invariants for any group $\mathscr{P}$ of point transformations. Note that for groups of Lie–Bäcklund tangent transformations we have no intrinsic distinction between invariants and differential invariants. All the problems arising in these considerations need further development and it seems to be convenient in their investigation to use the results and ideas arising in the theory of infinite Lie groups [11, 17–19].

Finally, the consideration of the group theoretic analysis of differential equations by means of groups of Lie–Bäcklund tangent transformations leads us to the idea of considering the invariance of the tangent structure Eqs. (2.3) in conjunction with a given system of differential equations

$$\boldsymbol{\Omega}(x, u, \underset{1}{u},..., \underset{n}{u}) = 0$$

under the group of transformations (2.1). For this we take the manifold in the $(x, u, \underset{1}{u},...)$-space given by equations $\boldsymbol{\Omega} = 0$ and all their differential consequences:

$$\boldsymbol{\Omega} = 0, \qquad D_i\boldsymbol{\Omega} = 0, \qquad D_{i_1}(D_{i_2}\boldsymbol{\Omega}) = 0, \qquad \cdots, \tag{3.4}$$

and consider the simultaneous invariance of the systems (2.3) and (3.4) with respect to the extension $\tilde{G}$ of the group of transformations (2.1). Taking into account the fact that Eqs. (3.4) do not involve the differentials dx, du,..., the infinitesimal criteria for this invariance is

$$\begin{gathered}(\zeta_i^\alpha - D_i(\eta^\alpha) + u_j^\alpha D_i(\xi^j))|_{(3.4)} = 0,\\ (\zeta_{ij}^\alpha - D_j(\zeta_i^\alpha) + u_{ik}^\alpha D_j(\xi^k))|_{(3.4)} = 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{gathered} \tag{3.5}$$

and

$$X\boldsymbol{\Omega}\,|_{(3.4)} = 0, \qquad XD_i\boldsymbol{\Omega}\,|_{(3.4)} = 0, \qquad \cdots, \tag{3.6}$$

where X is given by (2.5).

We point out that instead of Eqs. (2.10) we obtain in this case the weaker conditions (3.5) for the coordinates of the operator X (2.5). As a result we obtain transformations (2.1) which are infinite-order contact transformations only for the solutions of the differential equations under consideration.

EXAMPLE. Consider the invariance of the time-independent Schrödinger equation for the bound states of the hydrogen atom.

$$-\tfrac{1}{2}\nabla\cdot\nabla\psi(x)+[K+U(x)]\,\psi(x)=0,$$

where in appropriate units

$$U(x)=1/[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2},$$
$$\nabla\cdot\nabla=(\partial/\partial x^1)^2+(\partial/\partial x^2)^2+(\partial/\partial x^3)^2,$$
$$-K=\text{energy}=\text{constant}<0,$$

and

$$\psi(x)\in\mathscr{L}_2(\mathbb{R}^3).$$

In this case we have for (3.4)

$$\Omega=\frac{1}{2}(u_{11}+u_{22}+u_{33})+\left(\frac{1}{r}-K\right)u=0,$$

$$D_i\Omega=\frac{1}{2}(u_{i11}+u_{i22}+u_{i33})+\left(\frac{1}{r}-K\right)u_i-\frac{x^i}{r^3}u=0,\qquad i=1,2,3,$$

$$D_jD_i\Omega=\frac{1}{2}(u_{ji11}+u_{ji22}+u_{ji33})-\frac{x^ju_i}{r^3}+\left(\frac{1}{r}-K\right)u_{ij}-\frac{\delta_j{}^iu}{r^3}$$
$$+\frac{3x^jx^iu}{r^5}-\frac{x^iu_j}{r^3}=0,$$

. . . .

where

$$r=[(x^1)^2+(x^2)^2+(x^2)^3]^{1/2}$$

and

$$u=u^1=\psi,$$
$$u_i=u_i{}^1,\quad\text{etc.}$$

Now the invariance conditions (3.6) are satisfied by the known 0(4) invariance algebra of Fock [20], but here are represented as Lie–Bäcklund tangent transformations. In particular, one can verify by direct substitution that the following six infinitesimal operators satisfy (3.6):

$$\xi^k_{(i)}=\epsilon_{ijk}x^j,\qquad\eta_{(i)}=0,\ i,j,k=1,2,3,$$

and

$$\xi^k_{(i)} = (2K)^{-1/2}\,\delta_{ik}\,,$$
$$\eta_{(i)} = (2K)^{-1/2}\,\{-\epsilon_{ijk}\epsilon_{jlm}x^l u_{mk} + (x^i u/r)\},\ i, j, k = 1, 2, 3,$$

where δ_{ik} is the Kronecker delta symbol, ϵ is a permutation symbol such that

$$\epsilon_{ijk} = 0 \text{ unless } i, j, k \text{ are all different,}$$
$$\begin{aligned}\epsilon_{ijk} &= +1 \quad \text{for even permutations of } 1, 2, 3,\\ &= -1 \quad \text{for odd permutations of } 1, 2, 3,\end{aligned}$$

and the associated ζ_i, $\zeta_{i_1 i_2}$,... are defined by (2.10). The first set of three operators corresponds to the set of components of the quantum mechanical angular momentum operator while the second set of three corresponds to the set of components of the quantum mechanical analog of the Runge–Lenz vector. The connection between the usual form of these operators used in quantum mechanics and the above form can be established by employing Eqs. (3.3) and its associated linear form.

The first set is completely accounted for in the theory of the contact transformations of S. Lie while the second is not. Further, this situation is not altered by the procedure of passing to any finite set of first-order partial differential equations equivalent to (3.4) and then reexpressing the infinitesimal operators of the second set in terms of the new u^α's and u_s^α's.

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