Comments on “Non-Linear Piezoelectric Vibration Energy Harvesting From a Vertical Cantilever Beam with Tip Mass”

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Abstract

This paper first points out the retaining higher order terms in the main equation of previous study (PS) of Friswell et al. (Journal of Intelligent Material Systems and Structures 23(13): 1505–1521, 2012). The equations of motion for the system are derived following energy (Lagrange) based approaches and the correct equation with the extra higher order terms is provided and simulations are redone in the current study (CS).

Keywords : Energy harvester, base excitation, piezoelectric, control, non-linear dynamics

1. Introduction

The paper “Non-linear piezoelectric vibration energy harvesting from a vertical cantilever beam with tip mass”, published in Journal of Intelligent Material Systems and Structures (Friswell, M.I., Ali, S.F., Bilgen, O., Adhikari, S., Lees, A.W., & Litak, G., 2012), presented an equation of motion of a kinematically excited elastic beam - tip mass system by using the Lagrange’s equation. The equation of motion of the beam-mass system has been expressed as (see Eq. (15) in (Friswell, M.I., Ali, S.F., Bilgen, O., Adhikari, S., Lees, A.W., & Litak, G., 2012)):

\[
\left[p\alpha N_1 + M_1 + I_t N_5^2 + \left(p\alpha N_3 + M_2 N_4^2 + I_t N_5^4\right)^2\right]\ddot{v} + \left[p\alpha N_3 + M_2 N_4^2 + I_t N_5^4\right]\dot{v}^2 \nu + [EIN_6 - p\alpha g N_6 - M_6 g N_4 + 2EIN_7\nu^2] \nu = -(p\alpha N_2 + M_2)\ddot{\nu}
\]  

(1)

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However, there is a mistake in the Eq. (1). To our knowledge the equation of motion of the beam-mass system has not been correctly derived by Michael I Friswell and his colleagues (2012, 2014). We state that a significant difference occurs when Lagrange’s equations is derived. On the other hand, the interested reader can easily verify that Eq.(1) does not involve information for the value of N_b.

2. Derivation of Equation of Motion

In order to obtain the equations of motion of the system, The Lagrangian is defined as

\[ L = T - V, \]

and if all forces are conservative, Lagrange’s equation becomes

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{w}} \right) - \frac{\partial L}{\partial w} = 0, \]

where T and V denote the kinetic and potential energies, respectively. The kinetic energy of this beam-mass system was expressed by Friswell and his colleagues as

\[ T = \frac{1}{2} \rho A[ N_1 \dot{v}^2 + 2 N_2 \dot{z} \ddot{v} + \dot{z}^2 L + N_3 (\ddot{v}^2)] + \frac{1}{2} M_t \left[ (\dot{v} + \dot{z})^2 + N_4 (\dot{v} \ddot{v})^2 \right] + \frac{1}{2} I_t \left[ N_5 \dot{v} + \frac{1}{2} N_5^3 v^2 \dot{v} \right]^2 \]

and the potential energy of this system was written by Friswell and his colleagues as

\[ V = \frac{1}{2} EI \left[ N_6 v^2 + N_7 v^4 + \frac{1}{4} N_8 v^6 \right] - \frac{1}{2} N_9 \rho Ag v^2 - \frac{1}{2} N_4 M_t g v^2 \]

Substitution of equations (4) and (5) into (2) leads to

\[ L = T - V = \frac{1}{2} \rho A \left[ N_1 \dot{v}^2 + 2 N_2 \dot{z} \ddot{v} + N_3 (\ddot{v}^2) \right] + \frac{1}{2} M_t \left[ (\dot{v} + \dot{z})^2 + N_4 (\dot{v} \ddot{v})^2 \right] + \frac{1}{2} I_t \left[ N_5 \dot{v} + \frac{1}{2} N_5^3 v^2 \dot{v} \right]^2 - \frac{1}{2} EI \left[ N_6 v^2 + N_7 v^4 + \frac{1}{4} N_8 v^6 \right] - \frac{1}{2} N_9 \rho Ag v^2 - \frac{1}{2} N_4 M_t g v^2 \]

Then,

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}} \right) = \frac{1}{2} \rho A [2 N_1 \ddot{v} + 2 N_2 \dddot{z} + 2 N_3 (v' v'' + 2 v' \dddot{v}^2)] + \frac{1}{2} M_t \left[ 2 \ddot{v} + 2 \dddot{z} + 2 N_4 (v'' \ddot{v} + 2 v'' \dddot{v}^2) \right] + \frac{1}{2} I_t \left[ 2 \dddot{v} + 2 \dddot{z} + 2 N_5 \dddot{v}^2 \right] \]

\[ + \frac{1}{2} I_t \left[ 2 N_5^2 \dddot{v} + 2 N_5 \dddot{v}^2 + 2 \dddot{v} \dddot{v}^2 \right] + \frac{1}{2} N_5^6 (v'' \dddot{v} + 4 v'' \dddot{v}^2) \]

(7)

and

\[ \frac{\partial L}{\partial \dot{v}} = \frac{1}{2} \rho A [2 N_3 v' \dddot{v}^2] + \frac{1}{2} M_t [2 N_4 \dddot{v}^2 + 2 N_5 \dddot{v}^3 \dddot{v}^2] + \frac{1}{2} I_t [2 N_5^4 \dddot{v}^2 + 2 N_5^6 \dddot{v}^3 \dddot{v}^2] - \frac{1}{2} EI \left[ 2 N_6 v + 4 N_7 v^3 + \frac{3}{2} N_8 v^5 \right] \]

\[ + N_9 \rho Ag v + N_4 M_t g v \]

(8)

Substitution of equations (7) and (8) into (3), the equation of motion of the beam-mass system is derived in terms of the displacement of the tip mass using Lagrange’s equations as

\[ \left[ \rho AN_1 + M_t + I_t N_5^2 + \left( \rho AN_3 + M_t N_4^2 + I_t N_5^4 \right) v'^2 + \frac{1}{2} I_t N_5^6 v^4 \right] \ddot{v} + \left[ \rho AN_3 + M_t N_4^2 + I_t N_5^4 + \frac{1}{2} I_t N_5^6 v^2 \right] \dddot{v}^2 v + \left[ EIN_6 - \rho Ag N_9 - M_t g N_4 + 2 EIN_7 v^2 + \frac{1}{4} EIN_8 v^4 \right] v = - (\rho AN_2 + M_t) \dddot{v} \]
2.1 Equilibrium positions

The equilibrium positions with no forcing are obtained by setting the velocity and acceleration terms to zero in equation (9) to give

$$\left[ EIN_6 - \rho AgN_9 - M_tN_4 + 2EIN_7v^2 + \frac{3}{4}EIN_8v^4 \right] v = 0$$

(10)

Then the non-zero equilibrium positions are given by

$$v_{ob} = \pm \frac{-4EIN_7 \pm 2 \sqrt{E[4EIN_7 - 3N_6(EIN_6 - \rho AgN_9 - M_tN_4)]}}{3EIN_8}$$

(11)

Hence, the natural frequencies about both buckled equilibrium positions are

$$\omega_{nb}^2 = \frac{4EIN_7v_{ob}^2 + \frac{3}{2}EIN_8v_{ob}^4}{\rho AN_1 + M_t + I_tN_5^2 + (\rho AN_3 + M_tN_4^2 + I_tN_5^4)v_{ob}^2 + \frac{1}{4}I_tN_5^6v_{ob}^4}$$

(12)

3. Results and Discussions

In order to study the chaotic motion evolution process of strongly non-linear piezoelectric vibration energy harvesting vibration system, different kinds of numerical methods are applied such as bifurcation diagram, phase trajectory and Poincar’e map. These methods are all very useful tools for examining chaotic properties and exploring chaotic attractors. In the following calculations, a rectangle-cross steel beam is considered with length $L=300$ mm, width $b=16$ mm, thickness $h=0.254$ mm, density $\rho=7850$ kg/m$^3$, and Young’s modulus along the axial direction $E=2.1x10^{11}$ N/m$^2$. Equations may be resolved by using a Matlab software tool that involves the fourth-order Runge-Kutta method. The parameters of the numerical example are given in Table 1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Numerical values</th>
<th>Description</th>
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<tbody>
<tr>
<td>$E$</td>
<td>$210 \times 10^9$ N/m$^2$</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>$h$</td>
<td>0.254 mm</td>
<td>beam thickness</td>
</tr>
<tr>
<td>$b$</td>
<td>16 mm</td>
<td>beam width</td>
</tr>
<tr>
<td>$L_{c}$</td>
<td>300 mm</td>
<td>length of the beam</td>
</tr>
<tr>
<td>$\rho$</td>
<td>7850 kg/m$^3$</td>
<td>density of the beam mass</td>
</tr>
<tr>
<td>$M_t$</td>
<td>10.0 g.</td>
<td>the tip mass about to buckle</td>
</tr>
<tr>
<td>$I_t/M_t$</td>
<td>40.87 mm$^2$</td>
<td>ratio of mass moment of inertia</td>
</tr>
<tr>
<td>$I$</td>
<td>$2.185 \times 10^{14}$ m$^4$</td>
<td>geometrical moment of inertia</td>
</tr>
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<td>$L_c$</td>
<td>52.5 mm</td>
<td>active length of piezoelectric layers</td>
</tr>
<tr>
<td>$\gamma_c$</td>
<td>$-4.00 \times 10^4$ Nm/V</td>
<td>constant of piezoelectric device</td>
</tr>
<tr>
<td>$C_p$</td>
<td>51.4 nF</td>
<td>capacitance of the piezoelectric patch.</td>
</tr>
<tr>
<td>$R_l$</td>
<td>$10^9$ Ω</td>
<td>load resistor</td>
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Table 2: Effective parameters of the equation of motion of the beam-mass system in SI units

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<td>N4</td>
<td>N5</td>
<td>N6</td>
<td>N7</td>
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</table>

3.1 The model without coupling to the electric circuit

Using the analysis described in section ‘Equilibrium positions’, Figure 1 shows the equilibrium position of the tip mass and the corresponding natural frequency of the linearised system as a function of the tip mass amplitude $M_t$; due to the same values of previous study (PS) and current study (CS) for small vibrations before buckling ($M_t < M_{tb}$), only post-buckled responses are given. The equilibrium positions and the natural frequencies are different after buckling ($M_t > M_{tb}$). The value of the tip mass is now swept from 10 to 40 g. The equilibrium position of CS changes with the nonlinearity term $\frac{3}{4}EIN_b v^4$, which provides that the post-buckled response of CS has lower absolute values of equilibrium positions for $M_t > 10$ g. In Figure 1a. In addition to this, Figure 1b shows that the post-buckled response of CS has higher natural frequencies, which is the result of additional nonlinear terms $\left[\frac{1}{4}I_t N_5^2 v^4\right]$ and $\left[\frac{3}{4}EIN_b v^4\right]$, respectively.

Fig. 1. The effect of the tip mass of PS (dashed) and CS (solid) on (a) the equilibrium position, (b) the corresponding natural frequencies for the stable equilibrium positions.

3.2 Coupled electromechanical model

Then, the equation of motion of the coupled electromechanical model, which becomes

\[
\rho AN_1 + M_t + I_t N_5^2 + \left(\rho AN_3 + M_t N_4^2 + I_t N_5^4\right)v^2 + \frac{1}{4}I_t N_5^6 v^4 \ddot{v} + \\
\rho AN_3 + M_t N_4^2 + I_t N_5^4 + \frac{1}{2}I_t N_5^6 v^2 \dot{v}^2 v + \\
EIN_6 - \rho AgN_9 - M_6 g N_4 + 2EIN_7 v^2 + \frac{3}{2}EIN_6 v^3 \left(\dot{V} - \Theta_1 V - \Theta_2 v^2 V = -(\rho AN_2 + M_t)\ddot{z} \right)
\]

(13)

Table 3: Effective parameters of electrical equation in SI units

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<tbody>
<tr>
<td>$\Theta_1$</td>
<td>-0.00020918</td>
</tr>
<tr>
<td>$\Theta_2$</td>
<td>-0.002860</td>
</tr>
</tbody>
</table>
Fourth-order Runge-Kutta method is employed to numerical study of Non-linear piezoelectric vibration energy harvesting system. The fourth-order Runge-Kutta routine is used for numerical integration. In order to identify the dynamical behavior, the time response is simulated, with zero initial displacement and velocity. The time step is $T/100$, and the total time is $8000T$. Because the beginning of the calculation yields unstable responses, we discard the response of the first 7900 periods and use only the response of the final 100$T$, where $T$ is the period of the system vibration. Only the stable limit sets are plotted. For the last 100 cycles, the Poincare’ points are sampled. Bifurcation under the variation of the tip mass $M_t$ is studied at a base excitation of $z=16$mm, frequency 0.5 Hz, and for a load resistance $R_l=100$ k$\Omega$. The tip mass is increased from 0 to 20 g with increment step 0.05 g. In general, the results show that the tip mass has a significant effect upon the dynamic displacement of the beam mass system.

![Fig. 2](image1.png)

(a)

![Fig. 2](image2.png)

(b)

Fig. 2. The effect of the variation of the tip mass for a base excitation of $z=16$mm at frequency 0.5 Hz and for a load resistance $R_l=100$ k$\Omega$: bifurcation diagram of (a) PS and (b) CS. The results were obtained using zero initial conditions for the tip mass displacement and velocity.

The points corresponding to the displacement response of the tip mass are shown in Fig. 2 as a bifurcation diagram (Nayfeh, A., & Mook, D., 1979). Fig. 2 shows the global bifurcation diagram against the tip mass, in which (a) of PS and (b) of CS. From these figures, one can find chaotic regions alternate with the period regions with the increase of the tip mass. From Fig. 2, we can see that periodic and chaotic motion interval occur with the increase of tip mass. When tip mass is small, system response is period-1 motion. With the increase of tip mass, system jumps into chaotic motion. With further increase of mass, system finally enters chaotic state after period-doubling bifurcation. The bifurcation diagram of PS shows us that there was periodicity from 0 to 10 of tip mass. As can be seen in the bifurcation plots, there is slightly more disturbance for the plot PS than the CS. In the Fig 2a, one can defiantly see large chaotic clouds of points arranged around $M_t=10$-14g. However, CS (Fig.2b) represents close to period doubling behavior, which can be clearly seen in the theoretical Poincar’e and phase plots (Fig. 6). Period
doubling behavior typically occurs when a system is ready to transition from periodicity to chaos. In this situation the data PS does not resemble the CS data after the tip mass of 10g. Moreover, the general trend of chaotic motion is found in the plot PS: the dots are not focused at one point but rather spread throughout the diagram, showing that at equal time increments the motion is not same. From Fig. 2b, we can see that periodic and chaotic motion interval occur with the increase of tip mass. The motion of the system is a period-1 before \( M_t = 10 \) g which is the same as that in Fig 2a, and then the system steps into a period-doubling process at \( M_t = 13 \) g. In this situation the data PS does not resemble the CS data again. In Fig. 2b, a narrow zone of chaotic motion comes into being, which is the result of the damping effect of additional nonlinear terms

\[
\left[ \frac{1}{4} I_1 N_5 v^4 \right] \ddot{v}, \left[ \frac{1}{2} I_1 N_5^2 v^2 \right] \dot{v}^2 v \quad \text{and} \quad \left[ \frac{3}{4} E I N_6 v^4 \right] v,
\]

respectively, which can be clearly seen in the theoretical Poincaré and phase plots of \( nT \)-periodic motion in Fig. 4, 6 and 8.

To test for the existence of a chaotic behavior under the variation of the tip mass \( M_t \), Figures 3-8 illustrate the phase diagrams, power spectra, and Poincaré maps for PS and CS in the horizontal direction of the beam-mass system at a fixed base excitation of \( \omega = 16 \) mm, frequency of 0.5 Hz, and load resistance of \( R_l = 100 \) kΩ for the tip masses of 10.5, 13.0 and 18.6 g, respectively. In Fig. 3 and 4, chaotic and periodic motion is clearly visible, respectively. The motion of PS, which is different from that in CS, in which chaos appears in the region \( M_t = 10.5-14.0 \) g. When tip mass is \( M_t = 10.5 \) g, system response is period-1 motion for CS; however, for PS of \( M_t = 10.5 \) g (Fig. 3), the system jumps into chaotic domain. The case of PS at \( M_t = 13 \) g presents a chaotic attractor in Fig. 5, which differs from the mapping points of the Poincaré map of period doubling of CS. There are seven Poincaré mapping points in Fig. 6d, in which there is shown obviously period-7 motion. It can also be seen that the orbit repeats after rotating seven cycles. As a result, in the case of PS, the phase diagrams are highly disordered and the power spectra reveal numerous excitation frequencies. In other words, the results presented in these figures all indicate that the extra non-linear parameters

\[
\left[ \frac{1}{4} I_1 N_5 v^4 \right] \ddot{v}, \left[ \frac{1}{2} I_1 N_5^2 v^2 \right] \dot{v}^2 v \quad \text{and} \quad \left[ \frac{3}{4} E I N_6 v^4 \right] v
\]

of CS would be able to suppress chaotic motions to be \( nT \)-periodic motions and even escape the undesired motions.
Fig. 3 (a) Phase plane, (b) Time history, (c) Fourier spectra, and (d) Poincaré section of PS for $M_t = 10.5 \text{ g}$.

Fig. 4 (a) Phase plane, (b) Time history, (c) Fourier spectra, and (d) Poincaré section of CS for $M_t = 10.5 \text{ g}$. 
Fig. 5 (a) Phase plane, (b) Time history, (c) Fourier spectra, and (d) Poincaré section of PS for $M_t = 13$ g.

Fig. 6 (a) Phase plane, (b) Time history, (c) Fourier spectra, and (d) Poincaré section of CS for $M_t = 13$ g.
Fig. 7 (a) Phase plane, (b) Time history, (c) Fourier spectra, and (d) Poincaré section of PS for $M_t = 18.6$ g.

Fig. 8 (a) Phase plane, (b) Time history, (c) Fourier spectra, and (d) Poincaré section of CS for $M_t = 18.6$ g.
4. Conclusions

In this paper, a set of governing equations for the beam-mass system in terms of the displacement of the tip mass is presented. The typical nonlinear behaviors of this system are investigated. Complicated global bifurcation diagrams are illustrated, under the variation of tip mass and excitation amplitude. The work allows one to see a complete solution structure with the variation of the tip mass condition and obtain the parameter regions of chaos. CS has presented a numerical analysis of the nonlinear dynamic response of a beam-mass system subject to the nonlinear damping effects of the parameters \( \frac{1}{4} I_1 N_5 v^4 \), \( \frac{1}{2} I_2 N_5 v^2 \), \( \frac{3}{4} E I N_5 v^4 \) and \( \frac{3}{4} E I N_5 v^4 \), respectively. The dynamics of the system have been analyzed by reference to its dynamic trajectories, power spectra, Poincaré maps and bifurcation diagrams. The analysis has investigated the dynamic response of the beam-mass system as a function of PS and the CS.

The results have shown that CS exhibits an extensive range of periodic, and sub-harmonic. For the case of PS without a drop-out terms effect \( \frac{1}{4} I_1 N_5 v^4 \), \( \frac{1}{2} I_2 N_5 v^2 \) and \( \frac{3}{4} E I N_5 v^4 \), it has been shown that a chaotic behavior takes place at a broad band values of the tip mass, i.e., \( M_t = 10.5-14.0 \) g. In PS without a drop-out terms effect, synchronous 1T-periodic behavior is observed at very low values of the tip mass, i.e., \( M_t < 10.5 \) g. However, at values of the tip mass in the range 10.5–14.0, the beam-mass system exhibits a chaotic response. However, for values of CS in the same range (10.5–14.0 g), the system returns to a periodic or period-doubling behavior. As the tip mass is increased from \( M_t = 18.5 \) to \( 18.6 \) g, the case of PS reverts once again to a period doubling response whereas that with a drop-out terms effect of CS exhibits a periodic motion. Comparing the dynamic behaviors of PS and CS over tip mass range associated with typical practical applications, i.e., \( M_t = 10.5-14.0 \) g, it has been shown that CS performs persistent nT-periodic motion. Thus, the results confirm the importance of taking these drop-out terms into account when predicting the dynamic response of beam-mass systems.

I do agree absolutely that this paper is a much more refined study than the previous one performed by the author and co-workers (PS). On the other hand the effects of these drop-out terms are not negligible as compared to the incomplete equation of motion PS, as has also been shown via numerical simulations. From the bifurcation diagrams one can concludes: (1) In CS, the equilibrium or periodic occurs with the effects of these drop-out terms in a wide range of tip mass; (2) chaos or quasi-periodic motion is likely to appear with using the equation motion of PS in a wide range of tip mass.

Acknowledgments

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References