Combinatorics of Inductively Factored Arrangements

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A real arrangement of hyperplanes is a finite family $\mathcal{A}$ of hyperplanes through the origin in a finite-dimensional real vector space $V = \mathbb{R}^n$.

A real arrangement $\mathcal{A}$ of hyperplanes is said to be factored if there exists a partition $\Pi = (\Pi_1, \ldots, \Pi_l)$ of $\mathcal{A}$ into $l$ disjoint subsets such that the Orlik–Solomon algebra of $\mathcal{A}$ factors according to this partition. A real arrangement $\mathcal{A}$ of hyperplanes is called inductively factored if it is factored and there exists a hyperplane $H \in \mathcal{A}$ such that the arrangement obtained by removing $H$ from $\mathcal{A}$ and the arrangement on $H$ consisting of all intersections of elements of $\mathcal{A} - \{H\}$ with $H$ are both inductively factored.

A chamber of $\mathcal{A}$ is a connected component of the complement of $\mathcal{A}$. For a fixed base chamber, we may define a partial order on the set of chambers according to their combinatorial distances from the base chamber. Given an inductive factorization $\Pi = (\Pi_1, \ldots, \Pi_l)$ and a base chamber $C_0$, we define the counting map of $\mathcal{A}$ with respect to $C_0$ as a morphism from the poset of chambers to the poset $\{0, 1, \ldots, |\Pi_i|\} \times \cdots \times \{0, 1, \ldots, |\Pi_l|\}$. We prove that, for a suitable base chamber, the counting map is a bijection, the poset of chambers is a lattice, and its rank-generating function has a nice factorization.

We consider the dual decomposition of the sphere of $V$ induced by $\mathcal{A}$. We prove that, if $\mathcal{A}$ is inductively factored, then this cellular decomposition can be viewed as a decomposition of the boundary of the $l$-cube $[0, |\Pi_1|] \times \cdots \times [0, |\Pi_l|]$ by cubic cells.

1. Introduction

Let $K$ be a field and let $V$ be a vector space over $K$. An arrangement of hyperplanes in $V$ is a finite family $\mathcal{A}$ of hyperplanes of $V$ through the origin. An arrangement $\mathcal{A}$ of hyperplanes is said to be real (resp. complex) if $K = \mathbb{R}$ is the field of real numbers (resp. if $K = \mathbb{C}$ is the field of complex numbers).

With an arrangement $\mathcal{A}$ of hyperplanes, one can associate a graded torsion-free $\mathbb{Z}$-algebra $A(\mathcal{A})$, called the Orlik–Solomon algebra of $\mathcal{A}$. If $\mathcal{A}$ is a complex arrangement, then $A(\mathcal{A})$ is isomorphic to the cohomology algebra of the complement

$$M(\mathcal{A}) = V - \left( \bigcup_{H \in \mathcal{A}} H \right)$$

of $\mathcal{A}$ (see [9]). The Poincaré polynomial $\text{Poin}(\mathcal{A}, t)$ of $\mathcal{A}$ is the Poincaré polynomial of $A(\mathcal{A})$; namely,

$$\text{Poin}(\mathcal{A}, t) = \sum_{n=0}^{\infty} \dim(A^n(\mathcal{A}))t^n.$$  

We refer to [8] and [11] for good expositions on the theory of arrangements of hyperplanes and, more precisely, on Orlik–Solomon algebras.

Let $\mathcal{A}$ be a real arrangement of hyperplanes. A chamber of $\mathcal{A}$ is a connected component of $V - (\bigcup_{H \in \mathcal{A}} H)$. We denote by $\mathcal{C}(\mathcal{A})$ the set of chambers of $\mathcal{A}$. For $C, D \in \mathcal{C}(\mathcal{A})$, we denote by $\mathcal{S}(C, D)$ the set of hyperplanes of $\mathcal{A}$ which separate $C$ and $D$. For a fixed chamber $C_0 \in \mathcal{C}(\mathcal{A})$, we partially order $\mathcal{C}(\mathcal{A})$ by

$$C \leq D \quad \text{if} \quad \mathcal{S}(C_0, C) \subseteq \mathcal{S}(C_0, D).$$

$\mathcal{C}(\mathcal{A})$ provided with this order is denoted by $P(\mathcal{A}, C_0)$. It is a ranked bounded poset of finite rank, where $\text{rank}(C) = |\mathcal{S}(C_0, C)|$ for $C \in \mathcal{C}(\mathcal{A})$. Its smallest element is $C_0$ and its largest element is $\mathcal{A}$.

0195-6698/95/030267 + 26 $08.00/0 © 1995 Academic Press Limited
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its greatest one is the chamber \(-C_0\) opposite to \(C_0\). The rank-generating function of \(P(\mathcal{A}, C_0)\) is

\[
\zeta(P(\mathcal{A}, C_0), t) = \sum_{C \in \mathcal{C}(\mathcal{A})} t^{\text{rank}(C)}.
\]

The poset \(P(\mathcal{A}, C_0)\) has been introduced and investigated by Björner, Edelman and Ziegler [1, 2].

Let \(\mathcal{A}\) be a real arrangement of hyperplanes. If \(\mathcal{A}\) is either a supersolvable arrangement or a Coxeter arrangement, then there exist some integers \(b_1, \ldots, b_t\) and a chamber \(C_0 \in \mathcal{C}(\mathcal{A})\) such that the Poincaré polynomial of \(\mathcal{A}\) factors as

\[
Poin(\mathcal{A}, t) = \prod_{i=1}^{t} (1 + b_i t)
\]

(see [16] for supersolvable arrangements and [10] for Coxeter arrangements), the poset \(P(\mathcal{A}, C_0)\) is a lattice (see [1]), and the rank-generating function of \(P(\mathcal{A}, C_0)\) factors as

\[
\zeta(P(\mathcal{A}, C_0), t) = \prod_{i=1}^{t} (1 + t + \cdots + t^{b_i})
\]


Therefore, it is natural to ask whether there exists some relation between the Poincaré polynomial of a real arrangement \(\mathcal{A}\) and the poset \(P(\mathcal{A}, C_0)\) for some suitable chamber \(C_0 \in \mathcal{C}(\mathcal{A})\), and whether there exist other arrangements with such properties.

There is a class of arrangements of hyperplanes called free arrangements, introduced by Terao [17], and which contains supersolvable arrangements (see [6]) and Coxeter arrangements (see [18]). If \(\mathcal{A}\) is a free arrangement, then there exist some integers \(b_1, \ldots, b_t\) such that the Poincaré polynomial factors as (1) (see [17]). However, if \(\mathcal{A}\) is a free real arrangement of hyperplanes, then there does not necessarily exist a chamber \(C_0 \in \mathcal{C}(\mathcal{A})\) such that the rank-generating function of \(P(\mathcal{A}, C_0)\) factors as (2); Terao [19] has found that the arrangement \(A_4(17)\) from Grünbaum's list [4] is a counterexample.

Let \(\mathcal{A}\) be an arrangement of hyperplanes. The intersection lattice of \(\mathcal{A}\) is the geometric lattice

\[
\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\},
\]

ordered by reverse inclusion. \(V = \bigcap_{H \in \mathcal{A}} H\) is the smallest element of \(\mathcal{L}(\mathcal{A})\) and \(\bigcap_{H \in \mathcal{A}} H\) is its greatest one. For \(X \in \mathcal{L}(\mathcal{A})\), we set

\[
\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}, \quad \mathcal{A}^X = \{H \cap X \mid H \in \mathcal{A} - \mathcal{A}_X\}.
\]

A partition \(\Pi = (\Pi_1, \ldots, \Pi_l)\) of \(\mathcal{A}\) into \(l\) disjoint subsets is called independent if, for any choice of hyperplanes \(H_i \in \Pi_i (i = 1, \ldots, l)\), the rank of \(H_1 \cap \cdots \cap H_l\) is exactly \(l\). If \(X \in \mathcal{L}(\mathcal{A})\), then \(\Pi\) induces a partition \(\Pi_X\) of \(\mathcal{A}_X\) the blocks of which are the non-empty subsets \(\Pi_i \cap \mathcal{A}_X\). The partition \(\Pi\) is a factorization (or a nice partition) if

(1) \(\Pi\) is independent;
(2) if \(X \in \mathcal{L}(\mathcal{A}) - \{V\}\), then \(\Pi_X\) has at least a block which is a singleton. In particular, one of the blocks \(\Pi_i\) is a singleton. We say that \(\mathcal{A}\) is factored if \(\mathcal{A}\) has a factorization.

Factored arrangements have been introduced and investigated by Jambu, Falk and Terao [3, 21]. Supersolvable arrangements are factored (see [5, 20]).

The homogeneous component \(A^1(\mathcal{A})\) of degree 1 of the Orlik–Solomon algebra
A(\mathcal{A}) of an arrangement \mathcal{A} can be viewed as a free \(\mathbb{Z}\)-module spanned by the hyperplanes of \(\mathcal{A}\). For \(\mathcal{B} \subseteq \mathcal{A}\), we denote by \(B(\mathcal{B})\) the submodule of \(A^1(\mathcal{A})\) spanned by the hyperplanes included in \(\mathcal{B}\). The following theorem is due to Terao [21]:

**Theorem 1.1 (Terao [21]).** Let \(\mathcal{A}\) be an arrangement of hyperplanes. Let \(\Pi = (\Pi_1, \ldots, \Pi_l)\) be a partition of \(\mathcal{A}\). The Orlik-Solomon algebra of \(\mathcal{A}\), viewed as a graded \(\mathbb{Z}\)-module, factors as

\[ A(\mathcal{A}) = (\mathbb{Z} \oplus B(\Pi_1)) \otimes \cdots \otimes (\mathbb{Z} \oplus B(\Pi_l)) \]

iff \(\Pi\) is a factorization.

**Corollary 1.2 (Terao [21]).** Let \(\mathcal{A}\) be a factored arrangement of hyperplanes. Let \(\Pi = (\Pi_1, \ldots, \Pi_l)\) be a factorization of \(\mathcal{A}\). Then:

1. the Poincaré polynomial of \(\mathcal{A}\) factors as

\[ \text{Poin}(\mathcal{A}, t) = \prod_{i=1}^{l} (1 + |\Pi_i| t); \]

2. the multiset \(\{\Pi_1, \ldots, |\Pi_l|\}\) depends only on \(\mathcal{A}\);
3. for \(X \in \mathcal{L}(\mathcal{A})\),

\[ \text{rank}(X) = |\{i \mid \Pi_i \cap \mathcal{A}_X \neq \emptyset\}|. \]

In particular, \(l\) is the rank of \(\mathcal{A}\).

Let \(\mathcal{A}\) be a factored arrangement of hyperplanes and let \(\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_l)\) be a factorization of \(\mathcal{A}\). We say that a hyperplane \(H_0 \in \mathcal{A}\) is distinguished if \(\Pi\) induces a factorization \(\Pi'\) of \(\mathcal{A}' = \mathcal{A} - \{H_0\}\); namely, the non-empty subsets \(\Pi_i \cap \mathcal{A}'\) form a factorization of \(\mathcal{A}'\) (note that \(\Pi_i \cap \mathcal{A}' = \Pi_i \neq \emptyset\) if \(H_0 \notin \Pi_i\)). Given a distinguished hyperplane \(H_0 \in \Pi_1\), we write \(\Pi' = \{H \cap H_0 \mid H \in \Pi_1\}\) for \(i = 2, \ldots, l\). In this paper we prove the following result (see Proposition 2.1):

\[ \Pi'' = (\Pi_2', \ldots, \Pi_l') \text{ is a factorization of } \mathcal{A}'' = \mathcal{A}^{H_0}. \]

A factorization \(\Pi = (\Pi_1, \ldots, \Pi_l)\) of an arrangement \(\mathcal{A}\) of hyperplanes is said to be an inductive factorization if there exists a distinguished hyperplane \(H_0 \in \mathcal{A}\) such that \(\Pi'\) is an inductive factorization of \(\mathcal{A}' = \mathcal{A} - \{H_0\}\) and \(\Pi''\) is an inductive factorization of \(\mathcal{A}'' = \mathcal{A}^{H_0}\). We say that \(\mathcal{A}\) is inductively factored if \(\mathcal{A}\) has an inductive factorization.

A factored arrangement is not necessarily inductively factored; for example, the nine planes in \(\mathbb{C}^3\) which satisfy the equation \((x^3 - y^3)(x^3 - z^3)(y^3 - z^3) = 0\) form a factored arrangement which is not inductively factored. Nevertheless, we do not know any arrangement \(\mathcal{A}\) of hyperplanes in a real vector space which is factored and not inductively factored. Supersolvable arrangements are inductively factored.

In this paper we prove the following results:

1. Inductively factored arrangements are free (see Proposition 2.2).
2. Let \(\mathcal{A}\) be an inductively factored arrangement of hyperplanes in a real vector space. Let \(\Pi = (\Pi_1, \ldots, \Pi_l)\) be an inductive factorization of \(\mathcal{A}\). There exists a chamber \(C_0 \in \mathcal{C}(\mathcal{A})\) such that \(P(\mathcal{A}, C_0)\) is a lattice (see Theorem 5.1) and its rank-generating function factors as

\[ \zeta(P(\mathcal{A}, C_0), t) = \prod_{i=1}^{l} (1 + t + \cdots + t^{|\Pi_i|}) \] (see Proposition 3.4).
We do not know if (1) and (2) hold for any factored arrangement of hyperplanes.

An arrangement $\mathcal{A}$ of hyperplanes is essential if $\bigcap_{H \in \mathcal{A}} H = \{0\}$. Let $\mathcal{A}$ be an essential arrangement of hyperplanes in an $l$-dimensional real vector space $V = \mathbb{R}^l$. We provide $V$ with an arbitrary scalar product. Let $S^{l-1}$ denote the unit sphere. The arrangement $\mathcal{A}$ determines a cellular decomposition (defined in Section 4) of $S^{l-1}$, called the dual decomposition of $S^{l-1}$ induced by $\mathcal{A}$.

In this paper we prove the following result (see Theorem 4.2):

If $\mathcal{A}$ has an inductive factorization $\Pi = (\Pi_1, \ldots, \Pi_l)$, then $S^{l-1}$ can be viewed as the boundary $\partial \Omega$ of the $l$-cube

$$\Omega = [0, |\Pi_1|] \times \cdots \times [0, |\Pi_l|],$$

and each cell of the dual decomposition of $S^{l-1} = \partial \Omega$ has the form $I_1 \times \cdots \times I_l$, where for $i = 1, \ldots, l$, either $I_i = \{a_i\}$ ($a_i \in \mathbb{N}$) or $I_i = [a_i, b_i]$ ($a_i, b_i \in \mathbb{N}$).

In [13], Salvetti associated with a real arrangement $\mathcal{A}$ of hyperplanes a cellular complex $\text{Sal}(\mathcal{A})$ and proved that $\text{Sal}(\mathcal{A})$ has the same homotopy type as the complement $M(\mathcal{A}_c)$ of the complexification of $\mathcal{A}$. From the dual decomposition of $S^{l-1}$ induced by $\mathcal{A}$, we define a cellular decomposition of the (closed) unit disk $B^l$ by attaching an $l$-cell to $S^{l-1}$. For every chamber $C \in \mathcal{C}(\mathcal{A})$, we fix a copy $B(C)$ of $B^l$ provided with this decomposition. The complex $\text{Sal}(\mathcal{A})$ can be defined by

$$\text{Sal}(\mathcal{A}) = \left\{ \bigsqcup_{C \in \mathcal{C}(\mathcal{A})} B(C) \right\}/\sim,$$

where $\sim$ is an equivalence relation which, for each pair $(C, D)$ of chambers, identifies some cells of $B(C)$ with their corresponding cells of $B(D)$. Therefore, Theorem 4.2 may certainly be used to investigate the homotopy of the complement $M(\mathcal{A}_c)$ of the complexification of an inductively factored arrangement $\mathcal{A}$.

Let $\mathcal{A}$ be a real arrangement of hyperplanes. Assume that $\mathcal{A}$ has a factorization $\Pi = (\Pi_1, \ldots, \Pi_l)$. For $C, D \in \mathcal{C}(\mathcal{A})$ and $i \in \{1, \ldots, l\}$, we set $\mathcal{S}_i(C, D) = \mathcal{S}(C, D) \cap \Pi_i$. For $b \in \mathbb{N}$, we write $[b] = \{0, 1, \ldots, b\}$. The counting map of $\mathcal{A}$ with respect to a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ is the morphism $\phi(\Pi, C_0)$ of ranked posets defined by

$$\phi(\Pi, C_0): P(\mathcal{A}, C_0) \to [[|\Pi_1|]] \times \cdots \times [[|\Pi_l|]]$$

$$C \mapsto ([\mathcal{S}_1(C_0, C)], \ldots, [\mathcal{S}_l(C_0, C)]),$$

where $[[|\Pi_1|]] \times \cdots \times [[|\Pi_l|]]$ is partially ordered by

$$(a_1, \ldots, a_l) \leq (b_1, \ldots, b_l) \quad \text{if} \quad a_i \leq b_1, \ldots, a_l \leq b_l,$$

and, for $(a_1, \ldots, a_l) \in [[|\Pi_1|]] \times \cdots \times [[|\Pi_l|]],$

$$\text{rank}(a_1, \ldots, a_l) = a_1 + \cdots + a_l.$$

Note that, in general, the counting map $\phi(\Pi, C_0)$ is not an isomorphism of ranked posets, even if $\phi(\Pi, C_0)$ is a bijection; two chambers $C$ and $D$ may be not comparable even though their images are comparable.

Let $\Pi = (\Pi_1, \ldots, \Pi_l)$ be a factorization of $\mathcal{A}$. We say that $\Pi$ is a hyperfactorization if there exists a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)$ is a bijection. We say that $\mathcal{A}$ is hyperfactored if $\mathcal{A}$ has a hyperfactorization.

In this paper we prove the following result (see Theorem 6.1):

Inductively factored real arrangements are hyperfactored.
Most of the results on inductively factored arrangements mentioned above are actually proved for hyperfactored arrangements. We do not know any real arrangement \( \mathcal{A} \) which is hyperfactored and not inductively factored. In fact, we introduce this notion of 'hyperfactorization' for technical reasons: indeed, in order to prove that inductively factored arrangements are hyperfactored, we need several preliminary results on hyperfactored arrangements (Propositions 3.1–3.5, Proposition 4.1, Theorem 4.2 and Lemmas 6.2–6.5). So, all results stated in this paper for hyperfactored arrangements hold for inductively factored arrangements.

**Example.** Let \( \mathcal{A} \) be the arrangement in \( \mathbb{R}^3 \) shown projectively in Figure 1 (\( \mathcal{A} \) contains the 'line' at infinity). Set \( \Pi_1 = \{H_1, H_3, H_4\}, \Pi_2 = \{H_2, H_5, H_6\} \) and \( \Pi_3 = \{H_7\} \). Then \( \Pi = (\Pi_1, \Pi_2, \Pi_3) \) is a factorization of \( \mathcal{A} \). One can easily verify that \( H_1 \) is distinguished in \( \mathcal{A} \), \( H_2 \) is distinguished in \( \mathcal{A} - \{H_1\} \), \( H_3 \) is distinguished in \( \mathcal{A} - \{H_1, H_2\} \), and so on. Hence, \( \Pi \) is an inductive factorization.

The adjacency graph of \( \mathcal{A} \) is defined to be the graph the vertices of which are the chambers of \( \mathcal{A} \), and where two chambers are joined by an edge if they are adjacent (see Figure 2). From this graph, one can verify that \( \phi(\Pi, C_0) \) is a bijection from \( \mathcal{C}(\mathcal{A}) \) to \( [3] \times [3] \times [1] \), and thus \( \Pi \) is a hyperfactorization. Moreover, \( P(\mathcal{A}, C_0) \) is a lattice and

\[
\zeta(P(\mathcal{A}, C_0), t) = (1 + t + t^2 + t^3)^2(1 + t).
\]
We set $\Omega = ]0, 3[ \times ]0, 3[ \times ]0, 1[$. The dual decomposition of $S^2 = \partial \Omega$ induced by $\mathcal{A}$ is shown in Figure 3.

Our work is organized as follows.

In Section 2 we mainly prove that, if $\mathcal{A}$ is a factored arrangement of hyperplanes and $H_0 \in \mathcal{A}$ is distinguished, then $\mathcal{A}' = \mathcal{A}'_{H_0}$ is factored too (see Proposition 2.1). We also prove that inductively factored arrangements are free (see Proposition 2.2).

In Section 3 we prove several results on hyperfactored arrangements. In particular, we characterize the chambers $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)$ is a bijection (see Corollary 3.3), we prove there are $2^{\text{rank}(\mathcal{A})}$ of them (see Corollary 3.2), we prove that, for such a chamber $C_0$, the rank-generating function of $P(\mathcal{A}, C_0)$ factors as

$$\zeta(P(\mathcal{A}, C_0), t) = \prod_{i=1}^{l} (1 + t + \cdots + t^{\Pi_i})$$

(see Proposition 3.4), and, if $C$ is a chamber of a hyperfactored arrangement $\mathcal{A}$, then $C$ has at most $2^{\text{rank}(\mathcal{A})} - 1$ walls (see Corollary 3.6).

In Section 4 we prove that the dual decomposition of the sphere $S^{l-1}$ induced by a hyperfactored arrangement can be viewed as the decomposition of the boundary $\partial \Omega$ of the $l$-cube $\Omega$ described above (see Theorem 4.2).

In Section 5 we prove that, if $\mathcal{A}$ is a hyperfactored arrangement and $C_0 \in \mathcal{C}(\mathcal{A})$ is a suitable chamber, then the poset $P(\mathcal{A}, C_0)$ is a lattice (see Theorem 5.1).

In Section 6 we prove that inductively factored real arrangements are hyperfactored (see Theorem 6.1).

2. Inductively Factored Arrangements

Throughout this section, $\mathcal{A}$ is assumed to be an arrangement of hyperplanes in a vector space $V$ over an arbitrary field $K$.

Let $\Pi = (\Pi_1, \ldots, \Pi_l)$ be a factorization of $\mathcal{A}$. Let $H_0 \in \Pi_1$. Write $\mathcal{A}' = \mathcal{A} - \{H_0\}$. Recall that $H_0$ is distinguished if $\Pi$ induces a factorization of $\mathcal{A}'$; namely, the non-empty subsets $\Pi_i \cap \mathcal{A}'$ form a factorization of $\mathcal{A}'$.

Proposition 2.1. Let $\Pi = (\Pi_1, \ldots, \Pi_l)$ be a factorization of $\mathcal{A}$. Let $H_0 \in \Pi_1$ be a distinguished element. Write $\mathcal{B} = \Pi_2 \cup \Pi_3 \cup \cdots \cup \Pi_l$ and $\mathcal{A}' = \mathcal{A}'_{H_0}$. Then:

1. The map

$$R: \mathcal{B} \to \mathcal{A}'$$

$$H \mapsto H \cap H_0$$

is a bijection.
(2) Write $\Pi''_i = R(\Pi_i)$ for every $i = 2, \ldots, l$. Then $\Pi'' = (\Pi''_2, \ldots, \Pi''_l)$ is a factorization of $\mathcal{A}''$.

**Proof.** (1) Let $H \in \Pi_1 - \{H_0\}$. Let $L = H \cap H_0$. By Corollary 1.2, there is an $i \in \{2, \ldots, l\}$ such that $\Pi_i \cap \mathcal{A}_L \neq \emptyset$; thus, there is a hyperplane $H' \in \mathcal{B}$ such that $L = H' \cap H_0$. This shows that $R$ is a surjective map.

In order to prove that $R$ is a bijection, it suffices to show that

$$|\mathcal{A}''| = |\mathcal{B}| = \sum_{i=2}^l |\Pi_i|.$$ 

This equality holds since

$$Poin(\mathcal{A}'', t) = \frac{1}{t} \left( Poin(\mathcal{A}, t) - Poin(\mathcal{A}', t) \right) \quad \text{(by [22])}$$

$$= \frac{1}{t} \left( \prod_{i=1}^l \left( 1 + |\Pi_i| t \right) - \left( |\Pi_1| - 1 \right) t \prod_{i=2}^l \left( 1 + |\Pi_i| t \right) \right) \quad \text{(by Corollary 1.2)}$$

$$= \prod_{i=2}^l \left( 1 + |\Pi_i| t \right),$$

and

$$|\mathcal{A}''| = \frac{d}{dt} \left. Poin(\mathcal{A}'', t) \right|_{t=0} \quad \text{(by definition of Poin(\mathcal{A}'', t))}$$

$$= \sum_{i=2}^l |\Pi_i|.$$ 

(2) Choose an $L_i \in \Pi''_i$ for every $i = 2, \ldots, l$. Since $R$ is a bijection, there is a unique $H_i \in \Pi_i$ such that $L_i = H_i \cap H_0$. Then rank of $H_0 \cap H_2 \cap \cdots \cap H_l$ is $l$ in $\mathcal{L}(\mathcal{A})$ (since $\Pi$ is a factorization), thus the rank of $L_2 \cap \cdots \cap L_l = H_0 \cap H_2 \cap \cdots \cap H_l$ is exactly $l - 1$ in $\mathcal{L}(\mathcal{A}'')$.

Now, let $X \in \mathcal{L}(\mathcal{A}'')$. Assume that $\Pi'_i \cap \mathcal{A}'_X = R(\Pi_i \cap \mathcal{A}_X) \neq \emptyset$ for $i = 2, \ldots, r$ and $\Pi'_i \cap \mathcal{A}'_X = R(\Pi_i \cap \mathcal{A}_X) = \emptyset$ for $i = r + 1, \ldots, l$. We have $H_0 \in \Pi_i \cap \mathcal{A}_X$; thus $\Pi_i \cap \mathcal{A}_X \neq \emptyset$. Moreover, $\Pi_i \cap \mathcal{A}_X \neq \emptyset$ for $i = 2, \ldots, r$ and $\Pi_i \cap \mathcal{A}_X = \emptyset$ for $i = r + 1, \ldots, l$. It follows, by Corollary 1.2, that the rank of $X$ in $\mathcal{L}(\mathcal{A})$ is $r$ and, consequently, that the rank of $X$ in $\mathcal{L}(\mathcal{A}'')$ is $r - 1$. Set $\mathcal{B}_X = \mathcal{B} \cap \mathcal{A}_X$ and $Y = \cap_{H \in \mathcal{A}_X} H$. We clearly have:

(i) $\Pi_i \cap \mathcal{A}_Y = \Pi_i \cap \mathcal{A}_X \neq \emptyset$ for $i = 2, \ldots, r$;
(ii) $\Pi_i \cap \mathcal{A}_Y = \Pi_i \cap \mathcal{A}_X = \emptyset$ for $i = r + 1, \ldots, l$;
(iii) $X = Y \cap H_0$.

If $Y \neq X$, then $\text{rank}(Y) = \text{rank}(X) - 1 = r - 1$ (in $\mathcal{L}(\mathcal{A})$); thus $\Pi_i \cap \mathcal{A}_Y = \emptyset$ (by Corollary 1.2). So, since $\Pi$ is a factorization of $\mathcal{A}$, there exists an $i \in \{2, \ldots, r\}$ such that

$$|\Pi_i \cap \mathcal{A}_Y| = |\Pi_i \cap \mathcal{A}_X| = |\Pi'_i \cap \mathcal{A}_X| = 1.$$ 

Assume that $Y = X$. Then $Y = X \in \mathcal{L}(\mathcal{A}'')$ and the rank of $Y$ is $r$ in $\mathcal{L}(\mathcal{A}'')$ (since the rank of $X$ is $r$ in $\mathcal{L}(\mathcal{A})$). Moreover, $\Pi_i \cap \mathcal{A}_Y = \Pi_i \cap \mathcal{A}_X \neq \emptyset$ for $i = 2, \ldots, r$ and $\Pi_i \cap \mathcal{A}_Y = \Pi_i \cap \mathcal{A}_X = \emptyset$ for $i = r + 1, \ldots, l$. So, by Corollary 1.2, $(\Pi_1 - \{H_0\}) \cap \mathcal{A}_Y \neq \emptyset$ (since $\Pi' = (\Pi_1 - \{H_0\}, \Pi_2, \ldots, \Pi_l)$ is a factorization of $\mathcal{A}'$). It follows that
Let $\Pi = (\Pi_1, \ldots, \Pi_r)$ be a factorization of $\mathcal{A}$. Let $H_0 \in \Pi_1$ be a distinguished hyperplane. Write $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\mathcal{A}'' = \mathcal{A}^{H_0}$. Let $\Pi'$ denote the factorization of $\mathcal{A}'$ induced by $\mathcal{A}$. Let $\mathcal{B} = \Pi_2 \cup \cdots \cup \Pi_r$, let $R: \mathcal{B} \rightarrow \mathcal{A}''$ be the bijection of Proposition 2.1, and let $\Pi'' = (R(\Pi_2), \ldots, R(\Pi_r))$ denote the factorization of $\mathcal{A}''$ induced by $\mathcal{A}$. The class $\mathcal{I}$ of inductive factorization is the smallest class of partitions of arrangements of hyperplanes which satisfies the following two conditions:

1. If $\mathcal{A} = \{H\}$ is a singleton and $\Pi = (\mathcal{A})$, then $\Pi \in \mathcal{I}$.
2. If $\Pi = (\Pi_1, \ldots, \Pi_r)$ is a factorization of $\mathcal{A}$, the hyperplane $H_0 \in \Pi_1$ is distinguished, $\Pi' \in \mathcal{I}$, and $\Pi'' \in \mathcal{I}$, then $\Pi \in \mathcal{I}$.

We say that $\mathcal{A}$ is inductively factored if $\mathcal{A}$ has an inductive factorization.

**Proposition 2.2.** Let $\mathcal{A}$ be an inductively factored arrangement. Let $\Pi = (\Pi_1, \ldots, \Pi_r)$ be an inductive factorization of $\mathcal{A}$. Then $\mathcal{A}$ is free and $|\Pi_1|, \ldots, |\Pi_r|$ are the exponents of $\mathcal{A}$.

**Proof.** We prove Proposition 2.2 by induction on the cardinality of $\mathcal{A}$. If $\mathcal{A} = \{H\}$ is a singleton, then it is obvious. Now, assume that Proposition 2.2 holds for any inductively factored arrangement of cardinality $< |\mathcal{A}|$.

By definition of an inductive factorization, one may assume that there exists a distinguished hyperplane $H_0 \in \Pi_1$ such that $\Pi' = (\Pi_1 - \{H_0\}, \Pi_2, \ldots, \Pi_r)$ is an inductive factorization of $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\Pi'' = (R(\Pi_2), \ldots, R(\Pi_r))$ is an inductive factorization of $\mathcal{A}'' = \mathcal{A}^{H_0}$. By the inductive hypothesis, $\mathcal{A}'$ is free and $|\Pi_1| - 1$, $|\Pi_2|, \ldots, |\Pi_r|$ are the exponents of $\mathcal{A}'$, and $\mathcal{A}''$ is free and $|R(\Pi_2)| = |\Pi_2|, \ldots, |R(\Pi_r)| = |\Pi_r|$ are the exponents of $\mathcal{A}''$. It follows, by [17], that $\mathcal{A}$ is free and $|\Pi_1|, |\Pi_2|, \ldots, |\Pi_r|$ are the exponents of $\mathcal{A}$.

3. Hyperfactored Arrangements

Throughout this section, $\mathcal{A}$ is assumed to be an arrangement of hyperplanes in a real vector space $V$.

Let $\Pi = (\Pi_1, \ldots, \Pi_r)$ be a factorization of $\mathcal{A}$. Recall that the counting map of $\mathcal{A}$ with respect to a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ is the morphism $\phi(\Pi, C_0)$ of ranked posets defined by

$$\phi(\Pi, C_0): P(\mathcal{A}, C_0) \rightarrow \prod |\Pi_i|$$

where $\mathcal{C}(C_0, C)$ denotes the set of hyperplanes of $\Pi_i$ which separate $C_0$ and $C$. Recall that the factorization $\Pi$ is called a hyperfactorization if there exists a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)$ is a bijection.

Let $C \in \mathcal{C}(\mathcal{A})$. A wall of $C$ is a hyperplane $H \in \mathcal{A}$ such that $\text{codim}(H \cap \bar{C}) = 1$, where $\bar{C}$ is the closure of $C$. We denote by $\mathcal{W}(C)$ the set of walls of $C$. Two chambers $C$ and $D$ are adjacent if they have a common wall. We say that $C$ is coloured if $|\mathcal{W}(C) \cap \Pi_i| = 1$ for all $i = 1, \ldots, l$.

**Proposition 3.1.** Assume that $\mathcal{A}$ has a hyperfactorization $\Pi = (\Pi_1, \ldots, \Pi_r)$. Let $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)$ is a bijection. A chamber $C$ of $\mathcal{A}$ is coloured iff

$$\phi(\Pi, C_0)(C) \in \{0, |\Pi_1|\} \times \cdots \times \{0, |\Pi_r|\}$$.
Combining the inductively factored arrangements

Let $C \in \mathcal{C}$. Assume that

$$\phi(\Pi, C_0)(C) = (|\Pi_1|, \ldots, |\Pi_r|, 0, \ldots, 0).$$

Let $i \in \{1, \ldots, r\}$. Let $D_i \in \mathcal{C}$ such that

$$\phi(\Pi, C_0)(D_i) = (|\Pi_1|, \ldots, |\Pi_{i-1}|, |\Pi_i| - 1, |\Pi_{i+1}|, \ldots, |\Pi_r|, 0, \ldots, 0).$$

There exists only one hyperplane $H_i \in \mathcal{A}$ which separates $C$ and $D_i$, and this hyperplane is included in $\Pi_i$. So, $D_i$ is adjacent to $C$ and $H_i$ is a wall of $C$; thus $|\mathcal{W}(C) \cap \Pi_i| \geq 1$.

Similarly, $|\mathcal{W}(C) \cap \Pi_i| \geq 1$ for all $i = r + 1, \ldots, l$.

Suppose that $|\mathcal{W}(C) \cap \Pi_i| \geq 1$ for some $i \in \{1, \ldots, r\}$. Pick two distinct hyperplanes $H_i, H'_i \in \mathcal{W}(C) \cap \Pi_i$. Let $D_i$ be the chamber adjacent to $C$ and separated from $C$ by $H_i$, and let $D_i'$ be the chamber adjacent to $C$ and separated from $C$ by $H'_i$. We have

$$\phi(\Pi, C_0)(D_i) = \phi(\Pi, C_0)(D_i') = (|\Pi_1|, \ldots, |\Pi_{i-1}|, |\Pi_i| - 1, |\Pi_{i+1}|, \ldots, |\Pi_r|, 0, \ldots, 0);$$

thus $D_i = D_i'$ (since $\phi(\Pi, C_0)$ is a bijection) therefore $H_i = H'_i$. This is a contradiction. So, $|\mathcal{W}(C) \cap \Pi_i| = 1$ for all $i = 1, \ldots, r$.

Similarly, $|\mathcal{W}(C) \cap \Pi_i| = 1$ for all $i = r + 1, \ldots, l$.

Now, let $C$ be a coloured chamber. For $i = 1, \ldots, l$, let $K_i$ be the unique element of $\mathcal{W}(C) \cap \Pi_i$. Assume that $K_i$ separates $C_0$ and $C$ for $i = 1, \ldots, r$ and that $K_i$ does not separate $C_0$ and $C$ for $i = r + 1, \ldots, l$. Let $C_1 \in \mathcal{C}$ such that

$$\phi(\Pi, C_0)(C_1) = (|\Pi_1|, \ldots, |\Pi_l|, 0, \ldots, 0).$$

The hyperplane $K_i$ does not separate $C$ and $C_1$ for any $i = 1, \ldots, l$. If $C_1 \neq C$, then there exists an $H \in \mathcal{W}(C)$ which separates $C_1$ and $C$, but $H$ cannot be any $K_i$. So, $C_1 = C$.

**Corollary 3.2.** Assume that $\mathcal{A}$ has a hyperfactorization $\Pi = (\Pi_1, \ldots, \Pi_l)$. Then there are exactly $2^l$ coloured chambers in $\mathcal{C}(\mathcal{A})$.

**Corollary 3.3.** Assume that $\mathcal{A}$ has a hyperfactorization $\Pi = (\Pi_1, \ldots, \Pi_l)$. Let $C_0 \in \mathcal{C}(\mathcal{A})$ be a coloured chamber. Then $\phi(\Pi, C_0)$ is a bijection iff $C_0$ is coloured.

**Proof.** If $\phi(\Pi, C_0)$ is a bijection, then $C_0$ is coloured (since $\phi(\Pi, C_0)(C_0) = (0, \ldots, 0)$).

Assume that $C_0$ is coloured. Let $C_0$ be a chamber of $\mathcal{A}$ such that $\phi(\Pi, C_0)$ is a bijection. By Proposition 3.1, one can assume that

$$\phi(\Pi, C_0)(C_0) = (|\Pi_1|, \ldots, |\Pi_l|, 0, \ldots, 0).$$

Then $\phi(\Pi, C_0)$ is a bijection since, for any chamber $C \in \mathcal{C}(\mathcal{A})$,

$$\phi(\Pi, C_0)(C) = (|\Pi_1| - a_1, \ldots, |\Pi_l| - a_r, a_{r+1}, \ldots, a_l)$$

if

$$\phi(\Pi, C_0)(C) = (a_1, \ldots, a_r, a_{r+1}, \ldots, a_l).$$

**Proposition 3.4.** Assume that $\mathcal{A}$ has a hyperfactorization $\Pi = (\Pi_1, \ldots, \Pi_l)$. Let $C_0 \in \mathcal{C}(\mathcal{A})$ be a coloured chamber. Then

$$\xi(P(\mathcal{A}, C_0), t) = \prod_{i=1}^l (1 + t + \cdots + t^{|\Pi_i|}).$$
PROOF.

\[ \zeta(P(\mathcal{A}, C_0), t) = \sum_{C \in \mathcal{C}(\mathcal{A})} t^{\text{rank}(C)} \]

\[ = \sum_{\sigma \in \Pi_1} t^{\text{rank}(\sigma)} \]

\[ = \prod_{i=1}^{l} \left( 1 + t + \cdots + t^{\text{rank}(\sigma)} \right). \]

\[ \square \]

PROPOSITION 3.5. Assume that \( \mathcal{A} \) has a hyperfactorization \( \Pi = (\Pi_1, \ldots, \Pi_l) \). Let \( C_0 \in \mathcal{C}(\mathcal{A}) \) be a coloured chamber. Let \( C \in \mathcal{C}(\mathcal{A}) \). Write \( \phi(\Pi, C_0)(C) = (a_1, \ldots, a_i) \). Then:

1) if either \( a_i = 0 \) or \( a_i = |\Pi_i| \), then \( |\mathcal{W}(C) \cap \Pi_i| \leq 1; \)

2) if \( a_i \neq 0 \) and \( a_i \neq |\Pi_i| \), then \( |\mathcal{W}(C) \cap \Pi_i| \leq 2. \)

PROOF. Let \( H_i \in \mathcal{W}(C) \cap \Pi_i \). Let \( D_i \) be the chamber adjacent to \( C \) and separated from \( C \) by \( H_i \). Either

\[ \phi(\Pi, C_0)(D_i) = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_l) \]

or

\[ \phi(\Pi, C_0)(D_i) = (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_l). \]

Since \( \phi(\Pi, C_0) \) is a bijection, there exists at most one such \( D_i \) if either \( a_i = 0 \) or \( a_i = |\Pi_i| \), and there exist at most two such \( D_i \) if \( a_i \neq 0 \) and \( a_i \neq |\Pi_i| \).

\[ \square \]

COROLLARY 3.6. Let \( \mathcal{A} \) be a hyperfactored arrangement of rank \( l \). Let \( C \in \mathcal{C}(\mathcal{A}) \). Then

\[ |\mathcal{W}(C)| \leq 2l - 1. \]

REMARK. Corollary 3.6 does not give the optimal inequality; in fact, the best one is \( |\mathcal{W}(C)| \leq 2l - 1 \), as shown in Section 4 (see Corollary 4.3).

4. DUAL DECOMPOSITION OF THE SPHERE INDUCED BY A HYPERFACTORED ARRANGEMENT

Throughout this section, \( \mathcal{A} \) is assumed to be an essential arrangement of hyperplanes in an \( l \)-dimensional real vector space \( V = \mathbb{R}^l \).

The hyperplanes of \( \mathcal{A} \) subdivide \( V \) into facets. We denote by \( \mathcal{F}(\mathcal{A}) \) the set of all facets. The support \( |F| \in \mathcal{L}(\mathcal{A}) \) of a facet \( F \) is the vector space spanned by \( F \). Every facet is open in its support. We denote by \( \bar{F} \) the closure of \( F \) in \( V \). There is a partial order in \( \mathcal{F}(\mathcal{A}) \) defined by \( F \leq G \) if \( F \subseteq \bar{G} \). Note that a chamber is a 0-codimensional facet, a face is a 1-codimensional facet, and two chambers \( C \) and \( D \) are adjacent iff they have a common face.

We provide \( V \) with an arbitrary scalar product. Let \( S^{l-1} \) denote the unit sphere. \( \mathcal{A} \) determines a cellular decomposition of \( S^{l-1} \) called the canonical decomposition of \( S^{l-1} \) induced by \( \mathcal{A} \). To a facet \( F \neq \{0\} \) corresponds the (open) cell \( \Delta(F) = F \cap S^{l-1} \). Every cell of this decomposition has that form.

From the canonical decomposition of \( S^{l-1} \) we define a simplicial decomposition of \( S^{l-1} \) called the barycentric subdivision of \( S^{l-1} \) induced by \( \mathcal{A} \). For every facet \( F \neq \{0\} \), we fix a point \( \omega(F) \in \Delta(F) \). A chain \( \{0\} \neq F_0 < F_1 < \cdots < F_r \) of facets determines an
(open and convex) simplex \( S = \omega(F_0) \vee \omega(F_1) \vee \cdots \vee \omega(F_t) \). Every simplex of this decomposition has that form.

From the barycentric subdivision of \( S^{t-1} \) we define a new cellular decomposition of \( S^{t-1} \) called the dual decomposition of \( S^{t-1} \) induced by \( \mathcal{A} \). To a facet \( F \neq \{0\} \), corresponds the (open) cell \( A^*(F) \) which is the union of all the simplexes \( S = \omega(F_0) \vee \omega(F_1) \vee \cdots \vee \omega(F_t) \) such that \( \{0\} \neq F_0 < F_1 < \cdots < F_t \) and \( F_0 = F \). Every cell of this decomposition has that form.

If \( F \in \mathcal{F}(\mathcal{A}) - \{0\} \), then \( \dim(A(F)) = \dim(F) - 1 \) and \( \dim(A^*(F)) = \text{codim}(F) \). If \( C \in \mathcal{C}(\mathcal{A}) \), then \( A^*(C) = \omega(C) \) is a point.

We denote by \( S(\mathcal{A}) \) the sphere \( S^{t-1} \) provided with its dual decomposition induced by \( \mathcal{A} \).

Fix \( F \in \mathcal{F}(\mathcal{A}) \). Let \( X = |F| \) be the support of \( F \). Let \( \mathcal{C}_F(\mathcal{A}) = \{ C \in \mathcal{C}(\mathcal{A}) \mid C \supset F \} \). Let \( p_X: \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}_X) \) denote the map which associates to every \( C \in \mathcal{C}(\mathcal{A}) \) the unique chamber \( p_X(C) \in \mathcal{C}(\mathcal{A}_X) \) which contains \( C \). Let \( \iota_F: \mathcal{C}(\mathcal{A}_X) \to \mathcal{C}(\mathcal{A}) \) denote the map which associates to every \( D \in \mathcal{C}(\mathcal{A}_X) \) the unique chamber \( \iota_F(D) \in \mathcal{C}(\mathcal{A}) \) contained in \( D \). We clearly have \( p_X \circ \iota_F = \text{Id}_{\mathcal{C}(\mathcal{A}_X)} \) and \( \iota_F \circ p_X(C) = C \) for every \( C \in \mathcal{C}(\mathcal{A}) \).

**Proposition 4.1.** Assume that \( \mathcal{A} \) has a hyperfactorization \( \Pi = (\Pi_1, \ldots, \Pi_l) \). Let \( C_0 \in \mathcal{C}(\mathcal{A}) \) be a coloured chamber, let \( F \in \mathcal{F}(\mathcal{A}) \), and let \( X = |F| \) be the support of \( F \). Write \( \Pi_{X,i} = \Pi_i \cap \mathcal{A}_X \) for every \( i = 1, \ldots, l \). Assume that \( \Pi_{X,i} \neq \emptyset \) for \( i = 1, \ldots, r \) and \( \Pi_{X,i} = \emptyset \) for \( i = r + 1, \ldots, l \). Write \( \Pi_X = (\Pi_{X,1}, \ldots, \Pi_{X,r}) \) and \( b_x^i = |\Pi_{X,i}| \) for \( i = 1, \ldots, r \). Then:

1. The map
   \[
   \phi(\Pi_X, p_X(C_0)): \mathcal{C}(\mathcal{A}_X) \to [b_x^1] \times \cdots \times [b_x^r]
   \]
   is a bijection. In particular, \( \Pi_X \) is a hyperfactorization of \( \mathcal{A}_X \) and \( p_X(C_0) \) is a coloured chamber.

2. Write
   \[
   (a_1^r, \ldots, a_t^r) = \phi(\Pi, C_0)(\iota_F \circ p_X(C_0)).
   \]
   Then
   \[
   \phi(\Pi, C_0): \mathcal{C}(\mathcal{A}) \to (a_1^r, \ldots, a_t^r) + ([b_x^1] \times \cdots \times [b_x^r] \times \{0\} \times \cdots \times \{0\})
   \]
   is a bijection.

**Proof.** Let \( C \in \mathcal{C}(\mathcal{A}) \) and let \( D \in \mathcal{C}(\mathcal{A}_X) \). Then
\[
\mathcal{S}(C, \iota_F(D)) = \mathcal{S}(C, \iota_F \circ p_X(C)) \cup \mathcal{S}(\iota_F \circ p_X(C), \iota_F(D)).
\]
This union is disjoint,
\[
\mathcal{S}(C, \iota_F \circ p_X(C)) \subseteq \mathcal{A} - \mathcal{A}_X,
\]
and
\[
\mathcal{S}(\iota_F \circ p_X(C), \iota_F(D)) = \mathcal{S}(p_X(C), D) \subseteq \mathcal{A}_X.
\]
So, if \( D \in \mathcal{C}(\mathcal{A}_X) \), then
\[
\phi(\Pi, C_0)(\iota_F(D)) = \phi(\Pi, C_0)(\iota_F \circ p_X(C_0)) + (\phi(\Pi_X, p_X(C_0))(D), 0, \ldots, 0).
\]
(*)
This equality shows that \( \phi(\Pi_X, p_X(C_0)) \) is injective (since \( \iota_F \) and \( \phi(\Pi, C_0) \) are injective). Thus, \( \phi(\Pi_X, p_X(C_0)) \) is a bijection since
\[
|\mathcal{C}(\mathcal{A}_X)| = \text{Poin}(\mathcal{A}_X, 1) = \prod_{i=1}^l (1 + b_x^i)
\]

Proposition 4.1.2 is an obvious consequence of Proposition 4.1.1 and of (*)

Now, with the assumptions and notations of Proposition 4.1, set
\[ \Omega_F = (a_1^F, \ldots, a_n^F) + (0, b_1^F \times \cdots \times 0, b_2^F \times \{0\} \times \cdots \times \{0\}). \]
If \( F = C \) is a chamber, then \( \Omega_C = \{ \phi(P, C_0)(C) \} \). If \( F = \{0\} \), then
\[ \Omega_0 = \Omega = [0, |\Pi_1| \times \cdots \times 0, |\Pi_l|]. \]
Note that, by Proposition 4.1, \( \tilde{\Omega}_F \) is the convex hull of \( \{ \phi(P, C_0)(C) \mid C \in \mathcal{C}_F(\mathcal{A}) \} \), where \( \tilde{\Omega}_F \) denotes the closure of \( \Omega_F \) in \( \mathbb{R}^l \). Furthermore, \( \phi(P, C_0)(C) \in \tilde{\Omega}_F \) iff \( C \in \mathcal{C}_F(\mathcal{A}) \).

The main goal of this section is to prove the following theorem.

**Theorem 4.2.** There exists a homeomorphism \( \rho: S(\mathcal{A}) \to \partial \Omega \) which sends homeomorphically \( \Delta^*(F) \) onto \( \Omega_F \) for every \( F \in \mathcal{F}(\mathcal{A}) - \{0\} \).

**Corollary 4.3.** Let \( \mathcal{A} \) be a hyperfactored arrangement. Let \( C \in \mathcal{C}(\mathcal{A}) \). Then
\[ |W(C)| \leq 2(l - 1). \]

**Proof of Corollary 4.3.** Let \( \Pi = (\Pi_1, \ldots, \Pi_l) \) be a hyperfactorization of \( \mathcal{A} \). Let \( C_0 \) be a coloured chamber. Assume that \( |\Pi_l| = 1 \). Suppose there exists a chamber \( C \in \mathcal{C}(\mathcal{A}) \) such that \( |W(C)| > 2(l - 1) \). Then, by Proposition 3.5, \( |W(C)| = 2l - 1 \), \( |W(C) \cap \Pi_i| = 2 \) for all \( i = 1, \ldots, l - 1 \), \( |W(C) \cap \Pi_l| = 1 \), and
\[ \phi(\Pi, C_0)(C) \in [0, |\Pi_1| \times \cdots \times 0, |\Pi_l| \times \{0, 1\}). \]

Let \( H_i \) be the unique element of \( \Pi_i \). The hyperplane \( H_i \) is a wall of \( C \). Let \( D \) be the chamber adjacent to \( C \) and separated from \( C \) by \( H_i \). Let \( F \in \mathcal{F}(\mathcal{A}) \) be the face common to \( C \) and \( D \). Obviously, \( \mathcal{C}_F(\mathcal{A}) = \{C, D\} \). One can assume that
\[ \phi(\Pi, C_0)(C) = [0, |\Pi_1| \times \cdots \times 0, |\Pi_l| \times \{0\}], \]
\[ \phi(\Pi, C_0)(D) = [0, |\Pi_1| \times \cdots \times 0, |\Pi_l| \times \{1\}]. \]
So,
\[ \rho(\Delta^*(F)) = \Omega_F \subseteq [0, |\Pi_1| \times \cdots \times 0, |\Pi_l| \times |0, 1|[ = \Omega \]
This contradicts the fact that \( \rho(\Delta^*(F)) \subseteq \partial \Omega \) (Theorem 4.2).

We prove Theorem 4.2 by induction on the rank of \( \mathcal{A} \). The case \( \operatorname{rank}(\mathcal{A}) = 1 \) is obvious. So, from now on, we assume that Theorem 4.2 holds for any hyperfactored arrangement of rank \( < l \).

We assume that \( \mathcal{A} \) is an (essential) hyperfactored arrangement of rank \( l \), that \( \Pi = (\Pi_1, \ldots, \Pi_l) \) is a hyperfactorization of \( \mathcal{A} \), and that \( C_0 \in \mathcal{C}_F(\mathcal{A}) \) is a coloured chamber.

The following Lemmas 4.4–4.8 are preliminary results to the proof of Theorem 4.2.

**Lemma 4.4.** There exists a continuous map \( \rho: S(\mathcal{A}) \to \mathbb{R}^l \) which sends homeomorphically \( \Delta^*(F) \) onto \( \Omega_F \) for every \( F \in \mathcal{F}(\mathcal{A}) - \{0\} \).

**Proof.** Let \( \Delta^*_r \) be the \( r \)-skeleton of \( S(\mathcal{A}) \). We define the restriction \( \rho_r: \Delta^*_r \to \mathbb{R}^l \) of \( \rho \) to \( \Delta^*_r \) by induction on \( r \).
We set \( p_0(\omega(C)) = \phi(\Pi, C_0)(C) \) for all \( C \in \mathcal{C}(\mathcal{A}) \).

Assume \( \rho_{r-1}; \Delta^*_r \to \mathbb{R}^l \) to be defined. Recall that a \( r \)-cell of \( S(\mathcal{A}) \) has the form \( \Delta^*_r(F) \), where \( F \in \mathcal{F}(\mathcal{A}) - \{0\} \) and \( \text{codim}(F) = r \). Fix such a facet \( F \). Let \( X \) be its support. We assume that \( \Pi_i \cap \mathcal{A}_X = \Pi_{x,i} \neq \emptyset \) for \( i = 1, \ldots, r \) and \( \Pi_i \cap \mathcal{A}_X = \emptyset \) for \( i = r + 1, \ldots, l \). Write

\[
\mathcal{A}_X = \{ H/X \mid H \in \mathcal{A}_X \},
\]

\[
\tilde{\Pi}_X = \{ H/X \mid H \in \Pi_{x,i} \} \quad \text{for every } i = 1, \ldots, r,
\]

\[
\tilde{\Pi}_X = (\tilde{\Pi}_{X, 1}, \ldots, \tilde{\Pi}_{X, r}),
\]

\[
\tilde{C}_0 = p_X(C_0)/X,
\]

\[
\tilde{\Omega}_X = \{ 0, b^*_1 \times \cdots \times 0, b^*_r \subseteq \mathbb{R}^r \}.
\]

By Proposition 4.1, \( \tilde{\Pi}_X \) is a hyperfactorization of \( \mathcal{A}_X \) and \( \tilde{C}_0 \) is a coloured chamber. By the inductive hypothesis, there exists a homeomorphism \( \tilde{\rho}_X : S(\mathcal{A}_X) \to \partial \tilde{\Omega}_X \) which sends homeomorphically \( \Delta^*(\tilde{G}) \) onto its corresponding subset \( \Omega_{\tilde{G}} \) of \( \mathbb{R}^r \) for every \( \tilde{G} \in \mathcal{F}(\mathcal{A}_X) - \{0\} \). In particular, the set \( \{ \Omega_{\tilde{G}} \mid \tilde{G} \in \mathcal{F}(\mathcal{A}_X) - \{0\} \} \) forms a cellular decomposition of \( \partial \tilde{\Omega}_X \).

Consider the map \( g_F : \partial \tilde{\Omega}_X \to \partial \Omega_F \) defined by

\[
g_F(x_1, \ldots, x_r) = (a^*_1 + x_1, \ldots, a^*_r + x_r, a^*_{r+1}, \ldots, a^*_r).
\]

Obviously, \( g_F \) is a homeomorphism. Thus, the set \( \{ g_F(\Omega_{\tilde{G}}) \mid \tilde{G} \in \mathcal{F}(\mathcal{A}_X) - \{0\} \} \) forms a cellular decomposition of \( \partial \Omega_F \).

Let \( \mathcal{F}_r(\mathcal{A}) \) denote the set of facets \( G \in \mathcal{F}(\mathcal{A}) \) such that \( G > F \). Clearly, \( \{ \Delta^*(G) \mid G \in \mathcal{F}_r(\mathcal{A}) - \{F\} \} \) forms a cellular decomposition of \( \partial \Delta^*(F) \).

There is an isomorphism \( q : \mathcal{F}^*(\mathcal{A}) \to \mathcal{F}^*(\mathcal{A}) \) of posets defined as follows. Pick a \( \tilde{G} \in \mathcal{F}(\mathcal{A}_X) \). Let \( G_1 \in \mathcal{F}(\mathcal{A}_X) \) such that \( \tilde{G} = G_1/X \). Then \( q(\tilde{G}) \) is the unique facet of \( \mathcal{F}_r(\mathcal{A}) \) contained in \( F_1 \). Fix a \( \tilde{G} \in \mathcal{F}(\mathcal{A}_X) - \{0\} \). Write \( G = q(\tilde{G}) \). Let \( \mathcal{C} \in \mathcal{C}(\mathcal{A}_X) \) and let \( C = q(\tilde{C}) \). Clearly, \( \mathcal{C} = p_X(C)/X \); thus, as in the proof of Proposition 4.1,

\[
\phi(\Pi, C_0)(C) = (a^*_1, \ldots, a^*_r) + (\phi(\Pi_X, p_X(C_0))(p_X(C)), 0, \ldots, 0)
\]

\[
= (a^*_1, \ldots, a^*_r) + (\phi(\tilde{\Pi}_X, \tilde{C}_0)(\tilde{C}), 0, \ldots, 0)
\]

\[
= g_F(\phi(\tilde{\Pi}_X, \tilde{C}_0)(\tilde{C})).
\]

This shows that \( g_F \) sends bijectively \( \{ \phi(\tilde{\Pi}_X, \tilde{C}_0)(\tilde{C}) \mid \tilde{C} \in \mathcal{C}(\mathcal{A}_X) \} \) onto \( \{ \phi(\Pi, C_0)(C) \mid C \in \mathcal{C}(\mathcal{A}) \} \). The set \( \Omega_{\tilde{G}} \) is the convex hull of \( \{ \phi(\tilde{\Pi}_X, \tilde{C}_0)(\tilde{C}) \mid \tilde{C} \in \mathcal{C}(\mathcal{A}_X) \} \), the set \( \tilde{\Omega}_G \) is the convex hull of \( \{ \phi(\Pi, C_0)(C) \mid C \in \mathcal{C}(\mathcal{A}) \} \), and \( g_F \) is an affine function; thus \( g(\tilde{\Omega}_G) = \tilde{\Omega}_G \), and therefore \( g_F(\Omega_{\tilde{G}}) = \Omega_G \). So, the set \( \{ \Omega_G \mid G \in \mathcal{F}_r(\mathcal{A}) - \{F\} \} \) is a cellular decomposition of \( \partial \Omega_F \).

Now, the restriction of \( \rho_{r-1} \) to \( \partial \Delta^*(F) \) is a continuous map \( \rho_{r-1}; \Delta^*(F) \to \partial \Omega_F \) which sends homeomorphically \( \Delta^*(G) \) onto \( \Omega_G \) for all \( G \in \mathcal{F}_r(\mathcal{A}) - \{F\} \); thus it is a regular cellular function and a bijection, and therefore \( \rho_{r-1}; \Delta^*(F) \to \partial \Omega_F \) is a homeomorphism (see [7, Ch. I, Sec. 4]). So, \( \rho_{r-1} \) can be extended to \( \Delta^*(F) \) sending homeomorphically \( \Delta^*(F) \) onto \( \Omega_F \).

□

**Lemma 4.5.** The map \( \rho: S(\mathcal{A}) \to \mathbb{R}^l \) is injective.

**Proof.** It suffices to show that, if \( F_1, F_2 \in \mathcal{F}(\mathcal{A}) - \{0\} \) are such that \( \Omega_{F_1} \cap \Omega_{F_2} \neq \emptyset \), then \( F_1 = F_2 \).
Let 

\[ x = (x_1, \ldots, x_l) \in \Omega_{F_1} \cap \Omega_{F_2}. \]

Assume that \( x_i \in \mathbb{Z} \) for \( i = 1, \ldots, r \) and \( x_i \notin \mathbb{Z} \) for \( i = r + 1, \ldots, l \). For \( t \in \mathbb{R} \), we denote by \([t]\) the integer part of \( t \). Let \( \mathcal{H} \) be the set of chambers \( C \in \mathcal{C}(\mathcal{A}) \) such that

\[ \phi(\Pi, C_0)(C) \in ([x_1], [x_1] + 1) \times \cdots \times ([x_r], [x_r] + 1) \times \{x_{r+1}\} \times \cdots \times \{x_l\}. \]

Let \( G \) be the greatest facet of \( \mathcal{A} \) such that \( C \not\supseteq G \) for all \( C \in \mathcal{H} \) (\( G \) may be \( \{0\} \) a priori). The region \( \Omega_{F_1} \) has the form

\[ \Omega_{F_1} = (a^1_1, \ldots, a^r_1) + (l^1_1 \times \cdots \times l^r_1), \]

where either \( l^i_1 = 0, b^r_1 \) or \( l^i_1 = \{0\} \). Since \( x \in \Omega_{F_1} \), we have \( \phi(\Pi, C_0)(C) \in \tilde{\Omega}_{F_1} \) for all \( C \in \mathcal{H} \). Recall that

\[ \mathcal{C}(\mathcal{A}) = \{ C \in \mathcal{C}(\mathcal{A}) \mid \phi(\Pi, C_0)(C) \in \tilde{\Omega}_{F_1} \}. \]

Thus, \( C \not\supseteq F_1 \) for all \( C \in \mathcal{H} \) and, consequently, \( G \not\supseteq F_1 \). In particular, \( G \neq \{0\} \). Let \( A \) be the convex hull of \( \{ \phi(\Pi, C_0)(C) \mid C \in \mathcal{H} \} \). Clearly, \( x \in A \subseteq \tilde{\Omega}_G \). If \( G \neq F_1 \), then \( x \in \tilde{\Omega}_G \cap \Omega_{F_1} \) and \( \tilde{\Omega}_G \supseteq \partial \Omega_{F_1} \) (since \( G \not\supseteq F_1 \)). This is a contradiction, So, \( G = F_1 \). Similarly, \( G = F_2 \).

\[ \square \]

**Corollary 4.6.** The map \( \rho: \mathcal{S}(\mathcal{A}) \rightarrow \mathbb{R}^l \) is a homeomorphism from \( \mathcal{S}(\mathcal{A}) \) onto its image.

**Lemma 4.7.** Let \( \mathbb{E}^n \) be an euclidean space. Let \( \mathbb{S}^{n-1} \) be the unit sphere, and let \( \mathbb{B}^n \) be the (closed) unit disk in \( \mathbb{E}^n \). Let \( X \) be a contractible subspace of \( \mathbb{E}^n \) such that \( \mathbb{S}^{n-1} \subseteq X \subseteq \mathbb{B}^n \). Then \( X = \mathbb{B}^n \).

**Proof.** Suppose that there exists an \( x \in \mathbb{B}^n - \mathbb{S}^{n-1} \) such that \( x \notin X \). The radial projection \( \mathbb{B}^n - \{x\} \rightarrow \mathbb{S}^{n-1} \) induces a retraction \( r: X \rightarrow \mathbb{S}^{n-1} \). So, there exists an epimorphism \( r_*: H_{n-1}(X, \mathbb{Z}) = \{0\} \rightarrow H_{n-1}(S^{n-1}, \mathbb{Z}) = \mathbb{Z} \). This is a contradiction. \( \square \)

Note that \( \partial \Omega \) is a parallelopide, and thus one may consider the facets of \( \partial \Omega \).

**Lemma 4.8.** Let \( \Sigma \) be a (closed) facet of \( \partial \Omega \). Then \( \Sigma \subseteq \rho(\mathcal{S}(\mathcal{A})) \).

**Proof.** We prove Lemma 4.8 by induction on the dimension of \( \Sigma \).

If \( \dim \Sigma = 0 \), then \( \Sigma \) is a point included in \( \{0, |\Pi_1|\} \times \cdots \times \{0, |\Pi_l|\} \), and thus \( \Sigma = \rho(\omega(C)) = \phi(\Pi, C_0)(C) \) for some coloured chamber \( C \) (Proposition 3.1).

Now, assume that \( \dim \Sigma = r > 0 \) and that all facets of \( \Omega \) of dimension \( <r \) are contained in \( \rho(\mathcal{S}(\mathcal{A})) \). Assume that

\[ \Sigma = [0, |\Pi_1|] \times \cdots \times [0, |\Pi_l|] \times \{0\} \times \cdots \times \{0\}. \]

Write

\[ \mathcal{B} = \Pi_{r+1} \cup \cdots \cup \Pi_l. \]

Let \( D_0 \in \mathcal{C}(\mathcal{B}) \) be the chamber which contains \( C_0 \). Note that \( D_0 \) is a union of facets of \( \mathcal{A} \) and is open.

**Assertion 1.** Let \( F, G \in \mathcal{F}(\mathcal{A}) \) such that \( F < G \). If \( F \subseteq D_0 \), then \( G \subseteq D_0 \).

**Proof of Assertion 1.** Let \( H \in \mathcal{B} \). Then \( F \) and \( G \) cannot be contained into different
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connected components of \( V - H \) (since \( F \subseteq G \)). Moreover, if \( G \subseteq H \), then \( F \subseteq H \) (since \( F < G \)), and this is not the case. So, \( G \subseteq D_0 \).

**Assertion 2.** Let \( F, G \in \mathcal{F}(\mathcal{A}) \) such that \( F < G \). If \( G \subseteq D_0 \), then \( F \subseteq D_0 \).

**Proof of Assertion 2.** Let \( H \in \mathcal{B} \). then \( F \) and \( G \) cannot be contained into different connected components of \( V - H \) (since \( F \subseteq G \)). So, \( F \subseteq D_0 \).

Now, consider the barycentric subdivision of \( S^{l-1} \) induced by \( \mathcal{A} \). Fix a simplex \( S = \omega(F_0) \vee \omega(F_1) \vee \cdots \vee \omega(F_s) \) (where \( \{0\} \neq F_0 < F_1 < \cdots < F_s \)).

**Assertion 3.** \( S \subseteq D_0 \cap S^{l-1} \) iff \( F_i \subseteq D_0 \).

**Proof of Assertion 3.** If \( S \subseteq D_0 \cap S^{l-1} \), then \( F_i \cap D_0 \neq \emptyset \) (since \( S \subseteq F_i \)); thus \( F_i \subseteq D_0 \). If \( F_i \subseteq D_0 \), then \( S \subseteq D_0 \) since \( S \subseteq F_i \).

**Assertion 4.** \( \rho(S) \subseteq \rho(S(\mathcal{A})) \cap \Sigma \) iff \( F_0 \subseteq D_0 \).

**Proof of Assertion 4.** Assume that \( \rho(S) \subseteq \rho(S(\mathcal{A})) \cap \Sigma \). Then \( \rho(\omega(F_0)) \in \Sigma \), and thus \( \rho(\omega(F_0)) \) has the form

\[
\rho(\omega(F_0)) = (x_1, \ldots, x_r, 0, \ldots, 0).
\]

Recall that \( \Omega_{F_0} \) has the form

\[
\Omega_{F_0} = (a_1^{F_0}, \ldots, a_t^{F_0}) + (0^r \times \cdots \times 0^r),
\]

where either \( I_i^{F_0} = [0, b_i^{F_0}] \) or \( I_i^{F_0} = (0) \). From the inclusion \( \rho(\omega(F_0)) \in \rho(\Delta^*(F_0)) = \Omega_{F_0} \), we deduce that \( 0 \in a_i^{F_0} + I_i^{F_0} \) for \( i = r + 1, \ldots, l \); thus \( a_i^{F_0} = 0 \) and \( I_i^{F_0} = (0) \) if \( i = r + 1, \ldots, l \). This shows that \( \Omega_{F_0} \subseteq \Sigma \). So, if \( C \in \mathcal{C}_{F_0}(\mathcal{A}) \), then \( \phi(\Pi, C_0)(C) \in \bar{\Omega}_{F_0} \subseteq \Sigma \) (Proposition 4.1); thus no hyperplane of \( \mathcal{B} \) separates \( C_0 \) and \( C \), and therefore \( C \subseteq D_0 \). If follows that \( F_0 \subseteq D_0 \) (since \( D_0 \) is a convex set).

Now, assume that \( F_0 \subseteq D_0 \). Then \( C \subseteq D_0 \) for all \( C \in \mathcal{C}_{F_0}(\mathcal{A}) \) (Assertion 1); thus \( \phi(\Pi, C_0)(C) \in \Sigma \) for all \( C \in \mathcal{C}_{F_0}(\mathcal{A}) \), and therefore \( \bar{\Omega}_{F_0} \subseteq \Sigma \) (since \( \bar{\Omega}_{F_0} \) is the convex hull of \( \{\phi(\Pi, C_0)(C) \mid C \in \mathcal{C}_{F_0}(\mathcal{A})\} \)). Furthermore, by definition of \( \Delta^*(F_0) \), we have \( S \subseteq \Delta^*(F_0) \). So, \( \rho(S) \subseteq \rho(\Delta^*(F_0)) = \Omega_{F_0} \subseteq \Sigma \).

**End of the proof of Lemma 4.8.** Assertions 2 and 3 show that \( D_0 \cap S^{l-1} \) is the union of all simplices \( S = \omega(F_0) \vee \omega(F_1) \vee \cdots \vee \omega(F_s) \) such that at least one \( F_i \) is contained in \( D_0 \). Assertions 1 and 4 show that \( \rho^{-1}(\rho(S(\mathcal{A})) \cap \Sigma) \) is the union of all simplices \( S = \omega(F_0) \vee \omega(F_1) \vee \cdots \vee \omega(F_s) \) such that \( F_i \) is contained in \( D_0 \) for all \( i = 0, 1, \ldots, s \). This implies that \( D_0 \cap S^{l-1} \) is the star of \( \rho^{-1}(\rho(S(\mathcal{A})) \cap \Sigma) \) in \( S^{l-1} \); thus \( \rho^{-1}(\rho(S(\mathcal{A})) \cap \Sigma) \) is a strong deformation retract of \( D_0 \cap S^{l-1} \) (see [15, Cor. 11, Ch. 3, Sec. 3]), and therefore \( \rho(S(\mathcal{A})) \cap \Sigma \) is contractible (since \( D_0 \cap S^{l-1} \) is contractible). On the other hand, by the inductive hypothesis, \( \partial \Sigma \subseteq \rho(S(\mathcal{A})) \cap \Sigma \). So, by Lemma 4.7, \( \rho(S(\mathcal{A})) \cap \Sigma = \Sigma \).

**Corollary 4.9.** \( \partial \Omega \subseteq \rho(S(\mathcal{A})) \).

**Proof of Theorem 4.2.** Let \( \rho : S(\mathcal{A}) \to \mathbb{R}^l \) be the map defined in Lemma 4.4. \( \rho \) is a homeomorphism from \( S(\mathcal{A}) \) onto its image (Corollary 4.6) and sends homeomorphically \( \Delta^*(F) \) onto \( \Omega_F \) for all \( F \in \mathcal{F}(\mathcal{A}) - \{0\} \) (Lemma 4.4). Furthermore, \( \rho(S(\mathcal{A})) \subseteq \bar{\Omega} \) (where \( \bar{\Omega} \) is the closure of \( \Omega \)) and \( \partial \Omega \subseteq \rho(S(\mathcal{A})) \) (Corollary 4.9). Suppose that
Pick an \( x \in S(\mathcal{A}) \) such that \( \rho(x) \notin \partial \Omega \). Then \( \rho(S(\mathcal{A}) - \{x\}) \) is contractible and \( \partial \Omega \subseteq \rho(S(\mathcal{A}) - \{x\}) \subseteq \Omega \); thus \( \rho(S(\mathcal{A}) - \{x\}) = \bar{\Omega} \) (Lemma 4.7). This cannot happen; otherwise, \( x \) would not have any image by \( \rho \). Thus, \( \rho(S(\mathcal{A})) = \partial \Omega \)

5. Poset of Chambers of a Hyperfactored Arrangement

Throughout this section, \( \mathcal{A} \) is assumed to be an (essential) arrangement of hyperplanes in an \( l \)-dimensional real vector space \( V = \mathbb{R}^l \).

The main goal of this section is to prove the following theorem.

**Theorem 5.1.** Assume that \( \mathcal{A} \) has a hyperfactorization \( \Pi = (\Pi_1, \ldots, \Pi_l) \). Let \( C_0 \in \mathcal{C}(\mathcal{A}) \) be a coloured chamber. Then \( P(\mathcal{A}, C_0) \) is a lattice.

The following Lemma 5.2 is a preliminary result to the proof of Theorem 5.1. Its proof is inspired by the proof of [1, Lemma 3.1].

Let \( P \) be a poset. Let \( x, y \in P \). We say that \( x \) covers \( y \) if \( x > y \) and \( x > z \implies z = y \). In \( P(\mathcal{A}, C_0) \), a chamber \( C \) covers a chamber \( D \) iff \( C \) and \( D \) are adjacent and \( D \) is in the same side of \( H \) as \( C_0 \), where \( H \) is the unique hyperplane of \( \mathcal{A} \) which separates \( C \) and \( D \).

**Lemma 5.2.** Let \( P \) be a bounded ranked poset of finite rank. Let \( a, b \in P \) such that \( a < b \). Assume that, for any two elements \( x, y \in [a, b] \), if \( x \) and \( y \) both cover an element \( z \in P \), then the join \( x \vee y \) exists in \( P \). Then, for any two elements \( x, y \in [a, b] \), the join \( x \vee y \) exists in \( P \).

**Proof.** We prove Lemma 5.2 by induction on \( \text{rank}(b) - \text{rank}(a) \). If \( \text{rank}(b) - \text{rank}(a) = 1 \), then Lemma 5.2 is obvious. Assume that \( \text{rank}(b) - \text{rank}(a) > 1 \).

Let \( x, y \in [a, b] \). If \( x = a \), then \( x \vee y = y \). If \( y = a \), then \( x \vee y = x \). So, we may assume that \( x \neq a \) and \( y \neq a \). Let \( c_1, c_2 \in [a, b] \) such that \( c_1 \) and \( c_2 \) both cover \( a \), \( x \geq c_1 > a \), and \( y \geq c_2 > a \). If \( c_1 = c_2 = c \), then \( x, y \in [c, b] \subseteq [a, b] \); thus, by the inductive hypothesis, \( x \vee y \) exists in \( P \).

Assume that \( c_1 \neq c_2 \). Since \( c_1 \) and \( c_2 \) both cover \( a \), the join \( c_1 \vee c_2 \) exists in \( P \). We have \( x, c_1 \vee c_2 \in [c_1, b] \subseteq [a, b] \); thus, by the inductive hypothesis, \( x \vee (c_1 \vee c_2) \) exists in \( P \). We have \( y, x \vee (c_1 \vee c_2) \in [c_2, b] \subseteq [a, b] \); thus, by the inductive hypothesis, \( d = y \vee x \vee (c_1 \vee c_2) \) exists in \( P \), and \( d = x \vee y \).

**Proof of Theorem 5.1.** We prove Theorem 5.1 by induction on the rank of \( \mathcal{A} \). Theorem 5.1 is obvious if \( \text{rank}(\mathcal{A}) = 1 \). We assume that \( \text{rank}(\mathcal{A}) > 1 \) and that Theorem 5.1 holds for any hyperfactored arrangement of rank \( < l = \text{rank}(\mathcal{A}) \). Recall that

\[
\Omega = [0, |\Pi_1|] \times \cdots \times [0, |\Pi_l|].
\]

For a (closed) facet \( \Sigma \) of \( \partial \Omega \), we set

\[
\mathcal{C}_\Sigma = \{ C \in \mathcal{C}(\mathcal{A}) | \phi(\Pi, C_0)(C) \in \Sigma \}.
\]

**Assertion 1.** Let \( \Sigma \) be a facet of \( \partial \Omega \). Let \( C_1, C_2 \in \mathcal{C}_\Sigma \). If \( C_1 \) and \( C_2 \) both cover a chamber \( D \in \mathcal{C}(\mathcal{A}) \), then the join \( C_1 \vee C_2 \) exists in \( P(\mathcal{A}, C_0) \).

**Proof of Assertion 1.** Write

\[
\phi(\Pi, C_0)(D) = (a_1, a_2, a_3, \ldots, a_l).
\]
Since $C_1$ and $C_2$ both cover $D$, we may assume that
\[ \phi(\Pi, C_0)(C_1) = (a_1 + 1, a_2, a_3, \ldots, a_i), \quad \phi(\Pi, C_0)(C_2) = (a_1, a_2 + 1, a_3, \ldots, a_i). \]
Let
\[ Q = [a_1, a_1 + 1] \times [a_2, a_2 + 1] \times \cdots \times [a_i]. \]
$\phi(\Pi, C_0)(C_1)$ and $\phi(\Pi, C_0)(C_2)$ are both included in $\Sigma$; thus $Q \cap \Sigma \neq \emptyset$, and therefore $Q \subseteq \Sigma$. So, by Theorem 4.2, there exists a facet $F \in \mathcal{F}(\mathcal{A}) - \{0\}$ such that $Q \subseteq \Omega_F \subseteq \Sigma$. Let $X$ be the support of $F$. Note that $\phi(\Pi, C_0)(C_1)$ and $\phi(\Pi, C_0)(C_2)$ are both included in $\tilde{Q} \subseteq \tilde{\Omega}_F$, thus $C_1, C_2 \in \mathcal{C}_F(\mathcal{A})$ (Proposition 4.1). By the inductive hypothesis, $P(\mathcal{A}_X, p_X(C_0))$ is a lattice, thus $p_X(C_1) \lor p_X(C_2)$ exists in $P(\mathcal{A}_X, p_X(C_0))$. Let $E = \cup_F(p_X(C_1) \lor p_X(C_2))$.

This shows that $E \geq C_1$. Similarly, $E \geq C_2$. Let $E' \in \mathcal{C}(\mathcal{A})$ such that $E' \geq C_1$ and $E' \geq C_2$:

\[ \mathcal{F}(C_0, E') \supseteq \mathcal{F}(C_0, C_1) \quad (\text{since } C_1 \leq E') \]
\[ \supseteq \mathcal{F}(C_0, \iota_F \circ p_X(C_0)) \quad (\text{since } C_1 \in \mathcal{C}_F(\mathcal{A})). \]

For $j = 1, 2$,

\[ \mathcal{F}(p_X(C_0), p_X(E')) = \mathcal{F}(C_0, E') \cap \mathcal{A}_X \]
\[ \supseteq \mathcal{F}(C_0, C_j) \cap \mathcal{A}_X \quad (\text{since } E' \geq C_j) \]
\[ = \mathcal{F}(p_X(C_0), p_X(C_j)). \]

This shows that $p_X(E') \geq p_X(C_1)$ and $p_X(E') \geq p_X(C_2)$; thus $p_X(E') \geq p_X(E)$, and therefore

\[ \mathcal{F}(C_0, E') \supseteq \mathcal{F}(p_X(C_0), p_X(E')) \supseteq \mathcal{F}(p_X(C_0), p_X(E)). \]

It follows that

\[ \mathcal{F}(C_0, E') \supseteq \mathcal{F}(C_0, \iota_F \circ p_X(C_0)) \cup \mathcal{F}(p_X(C_0), p_X(E)) \]
\[ = \mathcal{F}(C_0, E) \quad (\text{since } E \in \mathcal{C}_F(\mathcal{A})), \]
and thus $E' \geq E$. So, $E = C_1 \lor C_2$.

**Assertion 2.** Let $\Sigma$ be a facet of $\partial \Omega$ Let $C_1, C_2 \in \mathcal{C}_\Sigma$. Then the join $C_1 \lor C_2$ exists in $P(\mathcal{A}, C_0)$.

**Proof of Assertion 2.** We may assume that
\[ \Sigma = \{b_1\} \times \cdots \times \{b_r\} \times [0, |I_{i+1}|] \times \cdots \times [0, |I_0|], \]
where, for $i = 1, \ldots, r$, either $b_i = 0$ or $b_i = |I_i|$. Let $A \in \mathcal{C}(\mathcal{A})$ be the chamber such that $\phi(\Pi, C_0)(A) = (b_1, \ldots, b_r, 0, \ldots, 0)$, and let $B \in \mathcal{C}(\mathcal{B})$ be the chamber of $\mathcal{B}$ such that $\phi(\Pi, C_0)(B) = (b_1, \ldots, b_r, |I_{i+1}|, \ldots, |I_0|)$. The set $\mathcal{C}_\Sigma$ is equal to the interval $[A, B]$ of $P(\mathcal{A}, C_0)$. So, Assertion 2 is a direct consequence of Assertion 1 and Lemma 5.2.

**Assertion 3.** Let $\Sigma$ be a facet of $\partial \Omega$ Let $C_1, C_2 \in \mathcal{C}_\Sigma$. Then the meet $C_1 \land C_2$ exists in $P(\mathcal{A}, C_0)$.

**Proof of Assertion 3.** $P(\mathcal{A}, -C_0)$ is the dual poset of $P(\mathcal{A}, C_0)$; thus the join of $C_1$...
and \( C_2 \) in \( P(\mathcal{A}, -C_0) \) (which exists by Assertion 2) is the same chamber as the meet \( C_1 \wedge C_2 \) in \( P(\mathcal{A}, C_0) \).

**Assertion 4.** Let \( C_1, C_2 \in \mathcal{C}(\mathcal{A}) \). Then the join \( C_1 \vee C_2 \) exists in \( P(\mathcal{A}, C_0) \).

**Proof of Assertion 4.** One may assume that \(|\Pi_1| = 1\). Write
\[
\phi(\Pi, C_0)(C_1) = (a_1, a_2, \ldots, a_i), \quad \phi(\Pi, C_0)(C_2) = (a_1', a_2', \ldots, a_i').
\]
First, assume that \( a_1 = a_1' = 0 \). Let
\[
\Sigma = \{0\} \times [0, |\Pi_2|] \times \cdots \times [0, |\Pi_i|].
\]
We have \( C_1, C_2 \in \mathcal{C}_\Sigma \); thus, by Assertion 2, \( C_1 \vee C_2 \) exists in \( P(\mathcal{A}, C_0) \).

Now, assume that \( a_1 = 1 \). Let
\[
\Sigma = \{1\} \times [0, |\Pi_2|] \times \cdots \times [0, |\Pi_i|].
\]
Let \( A \) be the chamber of \( \mathcal{A} \) such that \( \phi(\Pi, C_0)(A) = (1, 0, \ldots, 0) \), and let \( B \) be the chamber of \( \mathcal{A} \) such that \( \phi(\Pi, C_0)(B) = (1, |\Pi_2|, \ldots, |\Pi_i|) \) (i.e. \( B = -C_0 \)). Clearly, \( \mathcal{C}_\Sigma = [A, B] \). Let \( \mathcal{M}(C_1, C_2) \) be the set of chambers \( C \in \mathcal{C}_\Sigma \) such that \( C \supseteq C_1 \) and \( C \supseteq C_2 \). The chamber \( B \) is included in \( \mathcal{M}(C_1, C_2) \); hence, \( \mathcal{M}(C_1, C_2) \neq \emptyset \). If \( D_1, D_2 \in \mathcal{M}(C_1, C_2) \), then \( D_1 \wedge D_2 \) exists in \( P(\mathcal{A}, C_0) \) (Assertion 3), \( D_1 \wedge D_2 \supseteq C_1 \) and \( D_1 \wedge D_2 \supseteq C_2 \), and \( D_1 \wedge D_2 \in [A, B] = \mathcal{C}_\Sigma \); thus \( D_1 \wedge D_2 \in \mathcal{M}(C_1, C_2) \). It follows that \( \mathcal{M}(C_1, C_2) \) has a smallest element
\[
E = \bigwedge_{C \in \mathcal{M}(C_1, C_2)} C.
\]
By definition, \( E \supseteq C_1 \) and \( E \supseteq C_2 \). Let \( E' \in \mathcal{C}(\mathcal{A}) \) such that \( E' \supseteq C_1 \) and \( E' \supseteq C_2 \). We have \( A \subseteq C_1 \subseteq E' \subseteq B \), thus \( E' \in [A, B] = \mathcal{C}_\Sigma \); therefore \( E' \in \mathcal{M}(C_1, C_2) \), and hence \( E' \supseteq E \). So, \( E = C_1 \wedge C_2 \). 

**Assertion 5.** Let \( C_1, C_2 \in \mathcal{C}(\mathcal{A}) \). Then the meet \( C_1 \wedge C_2 \) exists in \( P(\mathcal{A}, C_0) \).

**Proof of Assertion 5.** \( P(\mathcal{A}, -C_0) \) is the dual poset of \( P(\mathcal{A}, C_0) \); thus the join of \( C_1 \) and \( C_2 \) in \( P(\mathcal{A}, -C_0) \) (which exists by Assertion 4) is the same chamber as the meet \( C_1 \wedge C_2 \) in \( P(\mathcal{A}, C_0) \). \( \square \)

### 6. Inductively Factored Arrangements are Hyperfactored

Throughout this section, \( \mathcal{A} \) is assumed to be an (essential) arrangement of hyperplanes in an \( l \)-dimensional real vector space \( V = \mathbb{R}^l \).

The main goal of this section is to prove the following theorem.

**Theorem 6.1.** If \( \mathcal{A} \) is an inductively factored arrangement of hyperplanes and \( \Pi = (\Pi_1, \ldots, \Pi_i) \) is an inductive factorization of \( \mathcal{A} \), then \( \Pi \) is a hyperfactorization.

The following Lemmas 6.2–6.10 are preliminary results to the proof of Theorem 6.1.

Throughout Lemmas 6.2–6.4, \( \Pi = (\Pi_1, \ldots, \Pi_i) \) is assumed to be a hyperfactorization of \( \mathcal{A} \) and \( C_0 \in \mathcal{C}(\mathcal{A}) \) a fixed coloured chamber.

**Lemma 6.2.** Let \( C \in \mathcal{C}(\mathcal{A}) \) such that
\[
\phi(\Pi, C_0)(C) \in [0, |\Pi_1|] \times \{0\} \times \cdots \times \{0\}.
\]
Then \(|\mathcal{W}(C) \cap \Pi_1| = 2\).
PROOF. Let $D_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)(D_0) = (|\Pi_1|, 0, \ldots, 0)$. The set of hyperplanes of $\mathcal{A}$ which separate $C_0$ and $D_0$ is $\Pi_1$. Choose a minimal gallery $(C_0, C_1, \ldots, C_{|\Pi_1|} = D_0)$ from $C_0$ to $D_0$. Obviously, $\phi(\Pi, C_0)(C_j) = (j, 0, \ldots, 0)$ for every $j = 0, 1, \ldots, |\Pi_1|$. So, since $\phi(\Pi, C_0)$ is a bijection, $C = C_a$ for some $a \in \{1, \ldots, |\Pi_1| - 1\}$. Let $H_1$ be the hyperplane of $\mathcal{A}$ which separates $C_{a-1}$ and $C_a$, and let $H_2$ be the hyperplane of $\mathcal{A}$ which separates $C_{a+1}$ and $C_a$. We have $H_1 \neq H_2$ and $H_1, H_2 \in \mathcal{W}(C) \cap \Pi_1$; thus $|\mathcal{W}(C) \cap \Pi_1| \geq 2$. On the other hand, by Proposition 3.5, $|\mathcal{W}(C) \cap \Pi_1| \leq 2$. \[\square\]

LEMMA 6.3. Let $C \in \mathcal{C}(\mathcal{A})$ such that

$$\phi(\Pi, C_0)(C) \in \emptyset, |\Pi_1| \times \{0\} \times \cdots \times \{0\}.$$ Write $\mathcal{W}(C) \cap \Pi_1 = \{H_1, H_2\}$. Let $F_1, F_2 \in \mathcal{F}(\mathcal{A})$ be the faces (1-codimensional facets) of $C$ having respectively $H_1$ and $H_2$ as support. Let $G \in \mathcal{F}(\mathcal{A})$ be a 1-dimensional facet. If $G < C$, then either $G < F_1$ or $G < F_2$.

PROOF. Write $\phi(\Pi, C_0)(C) = (a, 0, \ldots, 0)$. Let $C_1$ be the chamber of $\mathcal{A}$ adjacent to $C$ and separated from $C$ by $H_1$, and let $C_2$ be the chamber of $\mathcal{A}$ adjacent to $C$ and separated from $C$ by $H_2$. One may assume that $\phi(\Pi, C_0)(C_1) = (a - 1, 0, \ldots, 0)$ and $\phi(\Pi, C_0)(C_2) = (a + 1, 0, \ldots, 0)$. The cell $\Omega_C$ has the form

$$\Omega_C = (a_1^G, a_2^G, \ldots, a_l^G) + (I_1^G \times I_2^G \times \cdots \times I_l^G),$$

where, for $i = 1, 2, \ldots, l$, either $I_i^G = \emptyset$ or $I_i^G = \{0\}, b_i^G[0]$. Moreover, since $G$ is a 1-dimensional facet, there is only one $i \in \{1, 2, \ldots, l\}$ such that $I_i^G = \emptyset$. Assume that $G < C$. Then $\phi(\Pi, C_0)(C) = (a, 0, \ldots, 0) \in \Omega_C$ (Proposition 4.1), thus $a_i^G = \cdots = a_i^G = 0$. If $I_i^G = \emptyset$, then $a_i^G = a$ and

$$\Omega_G = (a, 0, \ldots, 0) + ([0] \times [0], b_1^G[0] \times \cdots \times [0], b_l^G[0]) \in \Omega.$$ This contradicts Theorem 4.2. So, $I_i^G = \emptyset, b_i^G[0]$. This implies, since $(a, 0, \ldots, 0) \in \Omega_C$, that either $\phi(\Pi, C_0)(C_1) = (a - 1, 0, \ldots, 0) \in \Omega_C$ or $\phi(\Pi, C_0)(C_2) = (a + 1, 0, \ldots, 0) \in \Omega_C$; thus either $G < C_1$ or $G < C_2$ (Proposition 4.1), and therefore either $G < F_1$ or $G < F_2$. \[\square\]

LEMMA 6.4. Let $C, D \in \mathcal{C}(\mathcal{A})$ such that

$$\phi(\Pi, C_0)(C) = (a_1, a_2, \ldots, a_l), \quad \phi(\Pi, C_0)(D) = (a_1 + 1, a_2, \ldots, a_l).$$ Let $F$ be the greatest facet of $\mathcal{A}$ such that $F < C$ and $F < D$. Let $X$ be the support of $F$. Then $|\Pi_1 \cap \mathcal{A}_X| = 1$, and, for $i = 2, \ldots, l$, either $|\Pi_1 \cap \mathcal{A}_X| \geq 2$ or $\Pi_1 \cap \mathcal{A}_X = \emptyset$.

PROOF. We prove Lemma 6.4 by induction on the rank of $\mathcal{A}$.

First, suppose that $F \neq \emptyset$. Assume that $\Pi_1 \cap \mathcal{A}_X \neq \emptyset$ for $i = 2, \ldots, r$ and $\Pi_1 \cap \mathcal{A}_X = \emptyset$ for $i = r + 1, \ldots, l$. There exists at least one $H \in \Pi_1$ which separates $C$ and $D$, and such an $H$ has to contain $X$; thus $\Pi_1 \cap \mathcal{A}_X \neq \emptyset$. Write $\phi(\Pi_X, p_X(C_0))(p_X(C)) = (x_1, x_2, \ldots, x_r)$. We have $C, D \in \mathcal{C}_p(\mathcal{A})$; thus, as in the proof of Proposition 4.1,

$$\phi(\Pi, C_0)(C) = (a_1^f, a_2^f, \ldots, a_l^f) + (\phi(\Pi_X, p_X(C_0))(p_X(C)), 0, \ldots, 0),$$

$$\phi(\Pi, C_0)(D) = (a_1^f, a_2^f, \ldots, a_l^f) + (\phi(\Pi_X, p_X(C_0))(p_X(D)), 0, \ldots, 0);$$

therefore $\phi(\Pi_X, p_X(C_0))(p_X(D)) = (x_1 + 1, x_2, \ldots, x_r)$. Moreover, the greatest facet $G \in \mathcal{F}(\mathcal{A}_X)$ such that $G < p_X(C)$ and $G < p_X(D)$ is $G = X$. So, by the inductive
hypothesis applied to \( \mathcal{A}_X, p_X(C) \) and \( p_X(D) \), we have \( |\Pi_i \cap \mathcal{A}_X| = 1, |\Pi_i \cap \mathcal{A}_X| \geq 2 \) for all \( i = 2, \ldots, r \), and \( \Pi_i \cap \mathcal{A}_X = \emptyset \) for all \( i = r + 1, \ldots, l \).

Now, assume that \( F = \{0\} \). In that case, \( X = \{0\} \) and \( \mathcal{A}_X = \mathcal{A} \). Write

\[
\{C, D[ = |a_1, a_1 + 1 [ \times \{a_2\} \times \cdots \times \{a_l\}.
\]

Suppose that \( \{C, D[ \subseteq \mathcal{A}_0 \). Then there exists a facet \( G \in \mathcal{F}(\mathcal{A}) - \{0\} \) such that \( \{C, D[ \subseteq \mathcal{O}_G \) (Theorem 4.2). It follows that \( \phi(\Pi, C_0)(C) \in \mathcal{O}_G \) and \( \phi(\Pi, C_0)(D) \in \mathcal{O}_G \); thus \( G < C \) and \( G < D \) (Proposition 4.1), and therefore \( \{0\} \neq G < F \). This contradicts the hypothesis \( F = \{0\} \). So, \( \{C, D[ \notin \mathcal{A}_0 \). Thus, \( |\Pi_i| = 1, a_i = 0 \), and, for \( i = 2, \ldots, l, a_i \in \{0, |\Pi_i| \}. \) This shows that \( |\Pi_i| = |\Pi_i \cap \mathcal{A}_X| = 1 \) and, for \( i = 2, \ldots, l, |\Pi_i| = |\Pi_i \cap \mathcal{A}_X| \geq 2 \).

Throughout Lemmas 6.5–6.10, we fix the following assumptions and notations:

1. \( \Pi = (\Pi_1, \Pi_2, \ldots, \Pi_l) \) is an inductive factorization of \( \mathcal{A} \).
2. \( H_0 \in \Pi_1 \) is a distinguished hyperplane.
3. \( \Pi' = (\Pi_1 - \{H_0\}, \Pi_2, \ldots, \Pi_l) \) is an inductive factorization of \( \mathcal{A}' = \mathcal{A} - \{H_0\} \). In particular, we assume that \( \Pi_1 - \{H_0\} \neq \emptyset \).
4. \( \mathcal{A}' = \mathcal{A}'^{H_0} \).
5. \( \mathcal{B} = \Pi_2 \cup \cdots \cup \Pi_l \), and \( R: \mathcal{B} \rightarrow \mathcal{A}' \) (\( \rightarrow H \cap H_0 \)) is the bijection of Proposition 2.1.
6. \( \Pi'' = (\Pi'_2, \ldots, \Pi'_l) = (R(\Pi_2), \ldots, R(\Pi_l)) \) is an inductive factorization of \( \mathcal{A}'' \).
7. Theorem 6.1 holds for \( \mathcal{A}' \) and \( \mathcal{A}'' \), namely, \( \Pi' \) and \( \Pi'' \) are both hyperfactorizations.
8. \( \mathcal{A}' \) is a fixed coloured chamber of \( \mathcal{A}' \).

Let \( C' \) be a coloured chamber of \( \mathcal{A}' \). Let \( C'' \in \mathcal{C}(\mathcal{A}'') \). Let \( L_1, \ldots, L_r \) be the walls of \( C'' \). For \( j = 1, \ldots, r \), let \( K_j \) be the unique hyperplane of \( \mathcal{B} \) such that \( L_j = K_j \cap H_0 \). We say that \( C'' \) is minimal with respect to \( C' \) if \( K_j \) does not separate \( C' \) and \( C'' \) for any \( j = 1, \ldots, r \).

**Lemma 6.5.** Let \( C'' \) be a coloured chamber of \( \mathcal{A}'' \). Then \( C'' \) is minimal with respect to exactly two coloured chambers \( C_1 \) and \( C_2 \) of \( \mathcal{A}' \). Moreover, the set of hyperplanes of \( \mathcal{A}' \) which separate \( C_1 \) and \( C_2 \) is \( \Pi_1 - \{H_0\} \).

**Proof.** Let \( L_i \) be the unique element of \( \mathcal{W}(C'') \cap \Pi_i \) \( (i = 2, \ldots, l) \). Let \( K_i \) be the unique hyperplane of \( \Pi_i \) such that \( L_i = K_i \cap H_0 \) \( (i = 2, \ldots, l) \). Assume that \( K_i \) separates \( C'_0 \) and \( C'' \) for \( i = 2, \ldots, r \) and \( K_i \) does not separate \( C'_0 \) and \( C'' \) for \( i = r + 1, \ldots, l \). Let \( C'_1, C'_2 \in \mathcal{C}(\mathcal{A}') \) such that

\[
\phi(\Pi', C'_0)(C'_1) = (0, |\Pi_2|, \ldots, |\Pi_l|, 0, \ldots, 0),
\]

\[
\phi(\Pi', C'_0)(C'_2) = (|\Pi_1| - 1, |\Pi_2|, \ldots, |\Pi_l|, 0, \ldots, 0).
\]

Then \( C'_1 \) and \( C'_2 \) are both coloured (Proposition 3.1), \( C'' \) is minimal with respect to \( C'_1 \) and to \( C'_2 \), the chamber \( C'' \) is not minimal with respect to another coloured chamber of \( \mathcal{A}' \), and the set of hyperplanes of \( \mathcal{A}' \) which separate \( C'_1 \) and \( C'_2 \) is \( \Pi_1 - \{H_0\} \). \( \square \)

**Lemma 6.6.** Let \( C'' \in \mathcal{C}(\mathcal{A}'') \). If there exists a coloured chamber \( C' \in \mathcal{C}(\mathcal{A}') \) such that \( C'' \) is minimal with respect to \( C' \), then \( C'' \) is coloured.

**Proof.** We assume that \( C' = C'_0 \). In order to prove Lemma 6.6, it suffices to show that \( |\mathcal{W}(C'') \cap \Pi_i| \leq 1 \) for all \( i = 2, \ldots, l \); indeed, \( C'' \) has at least \( \text{rank}(\mathcal{A}'') \) walls and \( \text{rank}(\mathcal{A}') = l - 1 \).
Let $i \in \{2, \ldots, l\}$. Let $L \in \mathcal{W}(C'') \cap \Pi_i$. Let $K$ be the unique hyperplane of $\Pi_i$ such that $L = K \cap H_0$. Let $D'$ be the chamber of $\mathcal{A}'$ such that $C'' = D' \cap H_0$. Write
\[
\phi(\Pi', C_0')(D') = (a_1, a_2, \ldots, a_l).
\]
Let $E'$ be the chamber of $\mathcal{A}'$ such that
\[
\phi(\Pi', C_0')(E') = (a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_l).
\]
We prove that $K$ separates $D'$ and $E'$ and there is no other hyperplane in $\mathcal{B}$ which separates $D'$ and $E'$. This shows that $K$ is unique if it exists and, consequently, that $|\mathcal{W}(C'') \cap \Pi_i| \leq 1$.

Let $F$ be the face of $C''$ having $L$ as support. Let $\mathcal{A}_L' = \{H \in \mathcal{A}' \mid H \supseteq L\}$.

If $\mathcal{A}_L' = \{K\}$, then $D'$ and $E'$ are adjacent chambers and $K$ is the unique hyperplane of $\mathcal{A}'$ which separates $D'$ and $E'$ (see Figure 4).

Assume that $\mathcal{A}_L' \neq \{K\}$. Then $F$ is a 2-codimensional facet of $\mathcal{A}'$. Write $\mathcal{A}_L' = \{K, H_1, \ldots, H_{m-1}\}$, and
\[
\mathcal{C}_F(\mathcal{A}') = \{A' \in \mathcal{C}(\mathcal{A}') \mid A' \succ F\} = \{A_1', \ldots, A_{2m}'\},
\]
placed as in Figure 5 and such that $D' = A_j'$ for some $j_0 \in \{1, \ldots, m\}$. For $j = 1, \ldots, m - 1$, we have $H_j \cap H_0 = K \cap H_0 = L$; thus $H_j \in \Pi_i - \{H_0\}$ (since $R : \mathcal{B} \rightarrow \mathcal{A}''$ is a bijection). Let $k \in \{1, \ldots, 2m\}$ such that
\[
|\mathcal{I}(C_0, A_k')| = \min_{1 \leq j \leq 2m} |\mathcal{I}(C_0, A_j')|.
\]
(Recall that, if \( A', B' \in \mathcal{C}(\mathcal{A}) \), then \( \mathcal{S}(A', B') \) denotes the set of hyperplanes of \( \mathcal{A} \) which separate \( A' \) and \( B' \).) Since \( D' = A_j \) is in the same side of \( K \) as \( C_\ell \in \{1, \ldots, m\} \), we have \( k \in \{1, \ldots, m\} \). If \( k \neq 1 \) and \( k \neq m \), then \( A_k \) is in the same side of \( H_{k-1} \) as \( C_0 \), and \( A_k \) is in the same side of \( H_k \) as \( C_\ell \); thus

\[
\phi(\Pi', C_0)(A_k) = \phi(\Pi', C_0)(A_{k+1}) = \phi(\Pi', C_0)(A_j) + (1, 0, \ldots, 0).
\]

This is a contradiction. So, either \( k = 1 \) or \( k = m \) (say, \( k = 1 \)). We have \( E' = A_{2m-j_0+1} \):

indeed,

\[
\phi(\Pi', C_0)(A_{2m-j_0+1}) = \phi(\Pi', C_0)(A_j) + (j_0 - 1, 0, \ldots, 0, 1, 0, \ldots, 0)
\]

\[
= \phi(\Pi', C_0)(A_j) + (0, 0, \ldots, 0, 1, 0, \ldots, 0)
\]

\[
= (a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_t).
\]

So, if \( H \in \mathcal{A} \) separates \( D' \) and \( E' \), then \( H \in \mathcal{A} \) = \{\( K, H_1, \ldots, H_{m-1}\)\}. It follows that \( K \) is the unique hyperplane of \( \mathcal{B} \) which separates \( D' \) and \( E' \).

**Lemma 6.7.** Let \( C' \) be a coloured chamber of \( \mathcal{A} \). Then there exists a unique coloured chamber \( C'' \) of \( \mathcal{A} \) which is minimal with respect to \( C' \).

**Proof.** Let \( \text{Coul}(\mathcal{A}) \) be the set of coloured chambers of \( \mathcal{A} \), and let \( \text{Coul}(\mathcal{A}') \) be the set of coloured chambers of \( \mathcal{A}' \). Let \( \sim \) be the equivalence relation on \( \text{Coul}(\mathcal{A}') \) defined by \( C_1 \sim C_2 \) if the set of hyperplanes of \( \mathcal{A}' \) which separate \( C_1 \) and \( C_2 \) is \( \Pi_1 \{H_0\} \). Let \( f: \text{Coul}(\mathcal{A}') \to \text{Coul}(\mathcal{A}')/\sim \) be the map which associates to each \( C'' \in \text{Coul}(\mathcal{A}') \) the class formed by the two chambers \( C_1, C_2 \in \text{Coul}(\mathcal{A}') \) such that \( C'' \) is minimal with respect to \( C_1 \) and \( C_2 \) (Lemma 6.5). Since

\[
|\text{Coul}(\mathcal{A}')| = |\text{Coul}(\mathcal{A}')/\sim| = 2^{l-1}
\]

(Corollary 3.2), in order to prove Lemma 6.7, it suffices to show that \( f \) is a surjective map.

Let \( C' \in \text{Coul}(\mathcal{A}') \) (say, \( C' = C_0 \)). For \( C'' \in \mathcal{C}(\mathcal{A}') \) and \( i \in \{2, \ldots, l\} \), we denote by \( \mathcal{S}_i(C_0, C'') \) the set of hyperplanes of \( \Pi_i \) which separate \( C_0 \) and \( C'' \). Let

\[
\phi': \mathcal{C}(\mathcal{A}') \to [\vert \Pi_2 \vert] \times \cdots \times [\vert \Pi_l \vert]
\]

\[
C'' \mapsto (\mathcal{S}_2(C_0, C''), \ldots, \mathcal{S}_l(C_0, C'')).
\]

Recall that \( [\vert \Pi_2 \vert] \times \cdots \times [\vert \Pi_l \vert] \) is partially ordered by

\[
(a_2, \ldots, a_t) \leq (b_2, \ldots, b_t) \quad \text{if} \quad a_2 \leq b_2, \ldots, a_t \leq b_t.
\]

Pick a \( C'' \in \mathcal{C}(\mathcal{A}') \) such that \( \phi''(C') \) is minimal. Write \( \phi''(C'') = (a_2, \ldots, a_t) \). Let \( L \) be a wall of \( C'' \). Let \( i \in \{2, \ldots, l\} \) such that \( L \in \Pi_i \). Let \( K \) be the unique hyperplane of \( \Pi_i \) such that \( L = K \cap H_0 \). Let \( D'' \) be the chamber of \( \mathcal{A}'' \) adjacent to \( C'' \) and separated from \( C'' \) by \( L \). If \( K \) separates \( C_0 \) and \( C'' \), then

\[
\phi''(D'') = (a_2, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_t).
\]

This contradicts the minimality of \( \phi''(C'') \). So, \( C'' \) is minimal with respect to \( C_0 \) and is coloured (Lemma 6.6): thus \( f(C'') \) is the class in \( \text{Coul}(\mathcal{A}')/\sim \) which contains \( C' = C_0 \).

**Lemma 6.8.** Let \( C_0 \) be the unique coloured chamber of \( \mathcal{A}'' \) which is minimal with respect to \( C_0 \). There is no hyperplane in \( \mathcal{B} \) which separates \( C_0 \) and \( C_0 \).
Proof. Let $K \in \mathcal{A}$. Let $K^+$ be the half-space of $V$ bordered by $K$ and containing $C_0'$. Let $L = K \cap H_0$ and let $L^+ = K^+ \cap H_0$. Let

$$
\phi^*: \mathcal{C}(\mathcal{A'}) \to [[\Pi_2]] \times \cdots \times [[\Pi_l]]
$$

$$
\mathcal{C}'' \to (|\mathcal{F}(C_0', C'')|, \ldots, |\mathcal{F}(C_0', C''')|)
$$

be the map defined in the proof of Lemma 6.7. Pick a $C'' \in \mathcal{C}(\mathcal{A'})$ contained in $L^+$ and such that $\phi^*(C'')$ is minimal among the chambers of $\mathcal{A'}$ contained in $L^+$. Write $\phi^*(C'') = (a_2, \ldots, a_l)$. Let $L'$ be a wall of $C''$. Let $i \in \{2, \ldots, l\}$ such that $L' = \Pi_i$. Let $K'$ be the hyperplane of $\Pi_i$ such that $L' = K' \cap H_0$. Let $D''$ be the chamber of $\mathcal{A''}$ adjacent to $C''$ and separated from $C''$ by $L'$. If $K'$ separates $C_0'$ and $C''$, then

$$
\phi^*(D'') = (a_2, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_l).
$$

Moreover, $K$ does not separate $C_0'$ and $C''$; thus $K \neq K'$, and therefore $D'' \subseteq L^+$. This contradicts the minimality of $\phi^*(C'')$. So, $C''$ is minimal with respect to $C_0'$ and, consequently, $C'' = C_0''$ is the unique coloured chamber of $\mathcal{A''}$ which is minimal with respect to $C_0'$ (Lemma 6.7). This shows that $K$ does not separate $C_0'$ and $C_0''$. 

Lemma 6.9. Let $(a_2, \ldots, a_l) \in [[\Pi_2]] \times \cdots \times [[\Pi_l]]$. There exists a unique chamber $C' \in \mathcal{C}(\mathcal{A'})$ such that $C' \cap H_0 \neq \emptyset$ and

$$
\phi(\Pi', C_0')(C') \in [[\Pi_1] - 1] \times \{a_2\} \times \cdots \times \{a_l\}.
$$

Proof. Let $C_0'$ be the unique coloured chamber of $\mathcal{A'}$ which is minimal with respect to $C_0'$. Let $C'' \in \mathcal{C}(\mathcal{A''})$ such that

$$
\phi(\Pi'', C_0')(C'') = (a_2, \ldots, a_l).
$$

Let $C' \in \mathcal{C}(\mathcal{A'})$ such that $C'' = C' \cap H_0$. Then, by Lemma 6.8,

$$
\phi(\Pi', C_0')(C') \in [[\Pi_1] - 1] \times \{a_2\} \times \cdots \times \{a_l\}.
$$

Now, let $C_1'$ and $C_2'$ be two chambers of $\mathcal{A'}$ such that $C_j' \cap H_0 \neq \emptyset$ ($j = 1, 2$) and

$$
\phi(\Pi', C_0')(C_j') \in [[\Pi_1] - 1] \times \{a_2\} \times \cdots \times \{a_l\} \quad (j = 1, 2).
$$

Write $C_1'' = C_1' \cap H_0$ and $C_2'' = C_2' \cap H_0$. Then, by Lemma 6.8,

$$
\phi(\Pi'', C_0')(C_1'') = \phi(\Pi'', C_0')(C_2'') = (a_2, \ldots, a_l);
$$

thus $C_1'' = C_2''$, and therefore $C_1' = C_2'$.

Let $D_0 \in \mathcal{C}(\mathcal{A'})$ such that $\phi(\Pi', C_0')(D_0) = (|\Pi_1| - 1, 0, \ldots, 0)$. By Lemma 6.9, replacing $C_0'$ by $D_0$ if necessary, we may assume that $C_0' = C_0$ is a coloured chamber of $\mathcal{A}$. If $D_0' \cap H_0 = \emptyset$, then $D_0 = D_0'$ is a chamber of $\mathcal{A}$. If $D_0' \cap H_0 \neq \emptyset$, then we denote by $D_0$ the chamber of $\mathcal{A}$ contained in $D_0'$ and separated from $C_0$ by $H_0$.

Lemma 6.10. $\mathcal{F}(C_0, D_0) = \Pi_1$.

Proof. If $D_0 \cap H_0 \neq \emptyset$, then, by definition of $D_0$, the hyperplane $H_0$ separates $C_0$ and $D_0$; thus,

$$
\mathcal{F}(C_0, D_0) = \mathcal{F}(C_0, D_0') \cup \{H_0\} = \Pi_1.
$$
Now, assume that $D_0 \cap H_0 = \emptyset$. By Lemma 6.9, there exists a unique chamber $C' \in \mathcal{C}(\mathcal{A}')$ such that $C' \cap H_0 \neq \emptyset$ and
\[
\phi(\Pi', C'_0)(C') \in [[[\Pi_1]] - 1] \times \{0\} \times \cdots \times \{0\}.
\]
Write $\phi(\Pi', C'_0)(C') = (a, 0, \ldots, 0)$. By the starting hypothesis, we have $a \neq 0$ and $a \neq |\Pi_1| - 1$ (since $C' \neq C_0$ and $C' \neq D_0$). Let $(C_0, C_1, \ldots, C_{|\Pi_1| - 1} = D_0)$ be a minimal gallery of $\mathcal{A}'$ from $C_0$ to $D_0$. We have $\phi(\Pi', C'_0)(C'_j) = (j, 0, \ldots, 0)$ for $j = 0, 1, \ldots, |\Pi_1| - 1$ and $C' = C_j$. Let $H_1$ be the hyperplane of $\Pi_1 - \{H_0\}$ which separates $C'_{j-1}$ and $C'_j$, and let $H_2$ be the hyperplane of $\Pi_1 - \{H_0\}$ which separates $C'_{j+1}$ and $C'_j$. Let $F_1$ and $F_2$ be the faces of $C'$ having, respectively, $H_1$ and $H_2$ as support. There are exactly two chambers $B_1$ and $B_2$ of $\mathcal{A}$ contained in $C'$. Clearly, $B_1$ and $B_2$ are adjacent and separated by $H_0$.

We have $H_0 \cap F_i = \emptyset$. If not, there exists a point $x \in H_0 \cap F_i \subseteq H_1 \cap H_0$. Since $\Pi$ is a factorization, there exists a $K \in \mathcal{B}$ such that $K \cap H_1 = H_0 \cap H_1$. It follows that $x \in K \cap H_1$ and, consequently, that $F_i \subseteq K \cup H_1$. This contradicts the fact that $H_1$ is the support of $F_i$.

Therefore, either $F_1 \subseteq B_1$ or $F_1 \subseteq B_2$. Similarly, either $F_2 \subseteq B_1$ or $F_2 \subseteq B_2$.

Let $F$ be the face of $\mathcal{A}$ common to $B_1$ and $B_2$. Pick a 1-dimensional facet $G$ of $\mathcal{A}$ such that $G \cap F = \emptyset$ and $G < B_1$. Since $G$ is a half-line and the support of $G$ is included in $\mathcal{L}(\mathcal{A}')$ (since $G \notin H_0$), $G$ is also a facet of $\mathcal{A}'$. Furthermore, $G \subseteq \overline{B}_1 \subseteq \overline{C}'$; thus $G < C'$ (in $\mathcal{P}(\mathcal{A}')$). By Lemma 6.3, either $G < F_1$ or $G < F_2$ (say, $G < F_1$). We cannot have $F_1 \subseteq \overline{B}_2$; otherwise, $G \subseteq \overline{F}_1 \subseteq \overline{B}_2$ and $G \subseteq \overline{B}_1$ imply that $G \subseteq \overline{B}_1 \cap \overline{B}_2 = \overline{F}$, and this contradicts our hypothesis that $G \cap \overline{F} = \emptyset$. It follows that $F_1 \subseteq B_1$.

Therefore, we may assume that $F_1 \subseteq B_1$ and $F_2 \subseteq B_2$. By Lemma 6.9, $C'_j = C_j$ is a chamber of $\mathcal{A}$ if $j \neq a$. The face $F_1 \in \mathcal{F}(\mathcal{A}')$ is also a face of $\mathcal{A}$ (since $F_1 \cap H_0 = \emptyset$) and is common to $C_{a-1}$ and $B_1$. The face $F_2 \in \mathcal{F}(\mathcal{A}')$ is also a face of $\mathcal{A}$ (since $F_2 \cap H_0 = \emptyset$) and is common to $C_{a+1}$ and $B_2$. The sequence $(C_0, C_1, \ldots, C_{a-1}, B_1, B_2, C_{a+1}, \ldots, C_{|\Pi_1| - 1} = D_0)$ is a gallery which crosses once each element of $\Pi_1$ (since $H_0$ included); thus it is a minimal gallery of $\mathcal{A}$ from $C_0$ to $D_0$ and $\mathcal{P}(C_0, D_0) = \Pi_1$.

**Proof of Theorem 6.1.** We prove Theorem 6.1 by induction on the cardinality of $\mathcal{A}$. We assume that any inductive factorization of an arrangement of cardinality $<|\mathcal{A}|$ is a hyperfactorization. By definition of an inductive factorization, we may assume that there exists a distinguished hyperplane $H_0 \in \Pi_1$ such that $\Pi$ induces an inductive factorization of $\mathcal{A} = \mathcal{A} - \{H_0\}$ and $\Pi$ induces an inductive factorization of $\mathcal{A}'' = \mathcal{A}''_0$. The case $\Pi_1 - \{H_0\} = \emptyset$ can be proved in a similar way as the case $\Pi_1 - \{H_0\} \neq \emptyset$; thus we may fix the assumptions and notations of Lemmas 6.5-6.10.

As in Lemma 6.10, we assume that $C_0 = C_0$ is a coloured chamber of $\mathcal{A}$. We prove that $\phi(\Pi, C_0): \mathcal{C}(\mathcal{A}) \to [[[\Pi_1]]] \times \cdots \times [[[\Pi_1]]]$ is a bijection. It suffices to show that $\phi(\Pi, C_0)$ is injective: indeed, by Corollary 1.2 and by [22],
\[
|\mathcal{C}(\mathcal{A})| = \text{Poin}(\mathcal{A}, 1) = \prod_{i=1}^{l} (1 + |\Pi_i|).
\]

Let $C, D \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)(C) = \phi(\Pi, C_0)(D) = (a_1, a_2, \ldots, a_l)$. Let $C'$ be the chamber of $\mathcal{A}'$ containing $C$, and let $D'$ be the chamber of $\mathcal{A}'$ containing $D$. Either $\phi(\Pi', C'_0)(C') = (a_1 - 1, a_2, \ldots, a_l)$ or $\phi(\Pi', C'_0)(C') = (a_1, a_2, \ldots, a_l)$, and either $\phi(\Pi', C'_0)(D') = (a_1 - 1, a_2, \ldots, a_l)$ or $\phi(\Pi', C'_0)(D') = (a_1, a_2, \ldots, a_l)$.

**Case 1:** $\phi(\Pi', C'_0)(C') = \phi(\Pi', C'_0)(D') = (a_1 - 1, a_2, \ldots, a_l)$.

Then $C' = D'$ (since $\phi(\Pi', C'_0)$ is a bijection). If $C' \cap H_0 = \emptyset$, then $C'$ is a chamber of $\mathcal{A}$; thus $C = D = C' = D'$. If $C' \cap H_0 \neq \emptyset$, then $C'$ contains exactly two chambers
Case 2: \( \phi(\Pi', C_0)(C') = (a_1 - 1, a_2, \ldots, a_l) \) and \( \phi(\Pi, C_0)(D') = (a_1, a_2, \ldots, a_l) \).

There exists a hyperplane \( H_1 \in \Pi_1 - \{H_0\} \) such that \( H_1 \) does not separate \( C_0 \) and \( C' \) and \( H_1 \) separates \( C_0 \) and \( D' \). On the other hand, \( H_0 \) separates \( C_0 \) and \( C \) (since \( \phi(\Pi, C_0)(C) = \phi(\Pi', C_0')(C') + (1, 0, \ldots, 0) \)) and \( H_0 \) does not separate \( C_0 \) and \( D \) (since \( \phi(\Pi, C_0)(D) = \phi(\Pi', C_0')(D') \)). Let \( D_0 \in \mathcal{E}(\mathcal{A}) \), as defined in Lemma 6.10. Both \( H_0 \) and \( H_1 \) separate \( C_0 \) and \( D_0 \) by Lemma 6.10 (see Figure 6).

Pick a point \( p_0 \in C_0 \) and a point \( q_0 \in D_0 \). Let \( [p_0, q_0] \) be the segment joining \( p_0 \) to \( q_0 \). Since \( \Pi \) is a factorization and \( H_0, H_1 \in \Pi_1 \), there exists a \( K \in \mathcal{B} \) such that \( K \supseteq H_0 \cap H_1 \). Let \( T_0 \) be the chamber of \( \mathcal{B} \) containing \( C_0 \). We have \( C_0, D_0 \subseteq T_0 \); thus \( [p_0, q_0] \subseteq T_0 \) (since \( T_0 \) is a convex set), and therefore \( K \cap [p_0, q_0] = \emptyset \). It follows that \( K \) separates \( C \) and \( D \) (see Figure 6).

Let \( F \) be the greatest facet of \( \mathcal{A}' \) common to \( C' \) and \( D' \). Let \( X \) be the support of \( F \). By Lemma 6.4, we may assume that \( |\Pi_1 \cap \mathcal{A}'_X| = 1 \), that \( |\Pi_i \cap \mathcal{A}'_X| \geq 2 \) for \( i = 2, \ldots, r \), and that \( \Pi_i \cap \mathcal{A}'_X = \emptyset \) for \( i = r + 1, \ldots, l \). Both \( H_1 \) and \( K \) separate \( C' \) and \( D' \) (since they separate \( C \) and \( D \)); thus \( F \subseteq H_1 \cap K \), and therefore \( X \subseteq H_1 \cap K \subseteq H_0 \). This shows that \( H_0 \in \mathcal{A}_X \). So,

\[
|\Pi_1 \cap \mathcal{A}_X| = |(\Pi_1 \cap \mathcal{A}_X) \cup \{H_0\}| = 2,
\]

\[
|\Pi_i \cap \mathcal{A}_X| = |\Pi_i \cap \mathcal{A}_X| \geq 2 \quad \text{if} \quad i = 2, \ldots, r,
\]

\[
|\Pi_i \cap \mathcal{A}_X| - \Pi_i \cap \mathcal{A}_X = \emptyset \quad \text{if} \quad i = r + 1, \ldots, l.
\]

This contradicts the fact that \( \Pi \) is a factorization. So, Case 2 cannot hold.

Case 3: \( \phi(\Pi', C_0)(C') = \phi(\Pi', C_0)(D') = (a_1, a_2, \ldots, a_l) \).

Then \( C' = D' \) (since \( \phi(\Pi', C_0) \) is a bijection). If \( C' \cap H_0 = \emptyset \), then \( C' \) is a chamber of \( \mathcal{A} \); thus \( C = D = C' = D' \). If \( C' \cap H_0 \neq \emptyset \), then \( C' \) contains exactly two chambers \( C_1 \) and \( C_2 \) of \( \mathcal{A} \), and \( \phi(\Pi, C_0)(C_1) = (a_1, a_2, \ldots, a_l) \) and \( \phi(\Pi, C_0)(C_2) = (a_1 + 1, a_2, \ldots, a_l) \); thus \( C = D = C_1 \).

REFERENCES


Received 26 April 1993 and accepted 5 April 1994

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