A fourth-order compact finite difference method for nonlinear higher-order multi-point boundary value problems

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A B S T R A C T

A fourth-order compact finite difference method is proposed for a class of nonlinear 2nth-order multi-point boundary value problems. The multi-point boundary condition under consideration includes various commonly discussed boundary conditions, such as the three- or four-point boundary condition, (n + 2)-point boundary condition and 2(n − m)-point boundary condition. The existence and uniqueness of the finite difference solution are investigated by the method of upper and lower solutions, without any monotone requirement on the nonlinear term. The convergence and the fourth-order accuracy of the method are proved. An efficient monotone iterative algorithm is developed for solving the resulting nonlinear finite difference systems. Various sufficient conditions for the construction of upper and lower solutions are obtained. Some applications and numerical results are given to demonstrate the high efficiency and advantages of this new approach.

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1. Introduction

Multi-point boundary value problems for differential equations arise in various physical processes. A great deal of work has been devoted to these problems, and most of the discussions are for the existence and multiplicity of solutions (see e.g., [1–13]). In this paper, we seek an efficient numerical method for the following 2nth-order nonlinear multi-point boundary value problem:

\[
\begin{align*}
(-1)^n u^{(2n)}(x) & = f(x, u(x), -u^{(2)}(x), \ldots, (-1)^{i-1}u^{(2i-2)}(x), \ldots, (-1)^{n-1}u^{(2(n-1))}(x)), \\
0 < x < 1,
\end{align*}
\]

\[
u^{(2i)}(0) = \sum_{j=1}^{p} \alpha_{i,j} u^{(2j)}(\xi_{i,j}), \quad u^{(2j)}(1) = \sum_{j=1}^{p} \beta_{i,j} u^{(2j)}(\eta_{i,j}), \quad 0 \leq i \leq n - 1,
\]

where \(u^{(k)} \equiv d^k u/dx^k, n \geq 1, f \) is a continuous function of its arguments, and for each \(i \) and \(j \), \(\alpha_{i,j}, \beta_{i,j} \in [0, +\infty)\) and \(\xi_{i,j}, \eta_{i,j} \in (0, 1)\). The above problem has appeared in many fields of applied science. For example, the problem for \(n = 1\) arises from the design of a large size bridge with multi-point supports (see e.g., [14]), while the case \(n = 2\) of the problem describes static deflection of an elastic bending beam. We refer the readers to the works of Zill and Cullen [15] and Timoshenko [16] for a brief and easily accessible discussion and the physical interpretation for some of the boundary conditions associated...
with the beam equations. It is allowed in (1.1) that \( \alpha_{i,j} = 0 \) or \( \beta_{i,j} = 0 \) for some or all \( i \) and \( j \). Various combinations of \( \alpha_{i,j} = 0 \) and \( \beta_{i,j} = 0 \) for each \( i \) and \( j \) lead to different boundary conditions. This implies that the boundary condition in (1.1) includes various commonly discussed multi-point boundary conditions. In particular, the boundary condition in (1.1) is reduced to

\[
\begin{align*}
  u^{(2)}(0) &= 0, & u^{(2)}(1) &= \sum_{j=1}^{p} \beta_{i,j} u^{(2)}(\eta_{ij}), & 0 \leq i \leq n - 1, \\
\end{align*}
\]  

(1.1a)

if \( \alpha_{i,j} = 0 \) for all \( i \) and \( j \) (see [17–20]); to the form

\[
\begin{align*}
  u^{(2)}(0) &= \sum_{j=1}^{p} \alpha_{i,j} u^{(2)}(\xi_{ij}), & u^{(2)}(1) &= 0, & 0 \leq i \leq n - 1, \\
\end{align*}
\]  

(1.1b)

if \( \beta_{i,j} = 0 \) for all \( i \) and \( j \) (see [8]); to the (2\( m \))-point boundary condition

\[
\begin{align*}
  \begin{cases}
  u^{(2)}(0) = u^{(2)}(1) = 0, & 0 \leq i \leq m, \ m \leq n - 2, \\
  u^{(2)}(0) = au^{(2)}(\xi_{1}), & u^{(2)}(1) = bu^{(2)}(\eta_{1}), \ m + 1 \leq i \leq n - 1, 
  \end{cases} \\
\end{align*}
\]  

(1.1c)

if \( p = 1 \) and \( \alpha_{i,1} = \beta_{i,1} = 0 \) for all \( 0 \leq i \leq m \) (see [5,21]); to the four-point boundary condition

\[
\begin{align*}
  u^{(2)}(0) &= a_{i} u^{(2)}(\xi_{1}), & u^{(2)}(1) &= b_{i} u^{(2)}(\eta_{1}), & 0 \leq i \leq n - 1, \\
\end{align*}
\]  

(1.1d)

if \( p = 1 \) and \( \xi_{i,1} = \xi_{1}, \eta_{i,1} = \eta_{1} \) for all \( 0 \leq i \leq n - 1 \) (see [22]); and to the two-point Lidstone boundary condition

\[
\begin{align*}
  u^{(2)}(0) &= u^{(2)}(1) = 0, & 0 \leq i \leq n - 1, \\
\end{align*}
\]  

(1.1e)

if \( \alpha_{i,j} = 0 \) and \( \beta_{i,j} = 0 \) for all \( i \) and \( j \) (see [23–29]). Conditions (1.1a) and (1.1b) with \( p = 1 \) become, respectively, the \( (n + 2) \)-point boundary conditions (see [10,30])

\[
\begin{align*}
  u^{(2)}(0) &= 0, & u^{(2)}(1) &= \beta_{i,1} u^{(2)}(\eta_{1}), & 0 \leq i \leq n - 1, \\
\end{align*}
\]  

(1.1f)

and

\[
\begin{align*}
  u^{(2)}(0) &= \alpha_{i,1} u^{(2)}(\xi_{1}), & u^{(2)}(1) &= 0, & 0 \leq i \leq n - 1. \\
\end{align*}
\]  

(1.1g)

Condition (1.1g) includes the three-point boundary condition when \( \xi = \eta \) (see [10,13]).

Literature dealing with the multi-point boundary value problem in the form (1.1) is extensive. Most of the investigations are concerned with the existence and multiplicity of solutions by using different methods such as the fixed point index theorem in cones and the method of upper and lower solutions. Using the fixed point theorem in cones, the works in [8,12,17–20,23–29] showed the existence of one or more solutions to either the problem (1.1)–(1.1a) or the problem (1.1)–(1.1b) or the problem (1.1)–(1.1c). The fourth-order equations under the fully multi-point boundary condition in (1.1) was treated recently in [6] with \( \alpha_{i,j} = \alpha_{j}^{*}, \beta_{i,j} = \beta_{j}^{*}, \xi_{i,j} = \xi_{j}^{*}, \eta_{i,j} = \eta_{j}^{*} \) and with a special function \( f(x,u(x),-u^{(2)}(x)) = f_{0}(x,u(x)) = b_{0} u^{(2)}(x) + c_{0} u(x) \) where \( b_{0} \) and \( c_{0} \) are two constants subjecting certain restrictions, and by the fixed point index theorem in cones, the author obtained some sufficient conditions so that the problem has a positive solution. In [11,31], the method of upper and lower solutions was applied to a fourth-order nonlinear four-point boundary value problem with or without a bending term and an existence result was obtained. The same method was used in [8,30] for the fourth-order equation in (1.1) with both the boundary condition (1.1f) or the boundary condition (1.1g). By constructing upper and lower solutions, the work in [24,32,33] gave the existence of positive solutions to the 2n-th-order equation in (1.1) with the two-point Lidstone boundary condition (1.1f). More recently, the application of the method of upper and lower solutions was extended in [21] to the 2n-th-order (2\( n \)-point) boundary value problem (1.1)–(1.1f), and in [34,35] to the 2n-th-order multi-point boundary value problem (1.1) with the more general multi-point boundary conditions. Using the upper and lower solutions as initial iterations, the authors of these works constructed two monotone iterative sequences which converge to the extremal solutions or a unique solution of the problem.

On the other hand, there are also some works that are devoted to numerical methods for the solutions of multi-point boundary value problems. The work in [36] made use of the Chebyshev series for approximating solutions of nonlinear first-order multi-point boundary value problems, and the work in [37] showed how an adaptive finite difference technique can be developed to produce efficient approximations to the solutions of nonlinear multi-point boundary value problems for first-order systems of equations. Another method for computing the solutions of nonlinear first-order multi-point boundary value problems was described in [38], where a multiple shooting technique was developed. For some other works that deal with computational methods for first-order multi-point boundary value problems, see e.g., [39–41]. In [42–44] the authors gave several constructive methods for the solutions of multi-point discrete boundary value problems, including the method of adjoints, invariant embedding method and shooting type method. In the case of 2n-th-order multi-point boundary value problems, there are only a few computational algorithms in the literature. Based upon the shooting technique, a numerical method was developed in [14] for approximating solutions and fold bifurcation solutions of a class of second-order multi-point boundary value problems. The paper [45] set up a reproducing kernel Hilbert space method for the solution of a
second-order three-point boundary value problem. This method was latterly extended in [46] to a fourth-order four-point boundary value problem.

In the context of finite difference discretizations, as we know, a reasonable approach is to develop a higher-order compact finite difference method, which not only provides accurate numerical results and saves computational work, but also is easier to treat boundary conditions. Recently, a fourth-order compact finite difference method was developed in [47] for the 2nth-order differential equation in (1.1) with the two-point Lidstone boundary condition (1.1p). To the best of our knowledge, no higher-order compact finite difference methods are available in the literature to the 2nth-order differential equation in (1.1) with fully multi-point boundary conditions. The study presented in this paper is aimed at extending the fourth-order compact finite difference method in [47] to the 2nth-order multi-point boundary value problem (1.1) with the more general boundary conditions, including the multi–point boundary conditions (1.1a), (1.1b), (1.1c) and (1.1d). It is not difficult to give a compact finite difference approximation to (1.1) in the same manner as in [47]. However, a lack of explicit information about the boundary value of the solution in the multi-point boundary conditions prevents us from using the standard analysis process in [47] and so we here develop a different approach for the analysis of the compact finite difference approximation to (1.1). Our specific goals are (1) to establish the existence and uniqueness of the finite difference solution, (2) to show the convergence of the finite difference solution to the analytic solution with the fourth-order accuracy, and (3) to develop an efficient monotone iterative algorithm for solving the resulting nonlinear finite difference systems. To achieve the above goals we use the method of upper and lower solutions and its associated monotone iterations. The proposed approach has several advantages. Firstly, the proposed finite difference method possesses the fourth-order accuracy and thus provides precise numerical results. Next, the suggested monotone iterative algorithm offers two monotone sequences which converge to the extremal solutions or a unique solution of the resulting nonlinear finite difference system, and the iterative sequences improve the upper and lower bounds of the solution, step by step. Finally, these processes do not require any monotonicity of the function $f$ and so essentially enlarge their applications.

The outline of the paper is as follows. In Section 2, we transform (1.1) into a coupled system of nonlinear second-order equations and then construct the compact finite difference scheme. In Section 3, we deal with the existence and uniqueness of the finite difference solution, without any monotone requirement on the function $f$, by using the method of upper and lower solutions. The convergence of the finite difference solution is discussed in Section 4, where the fourth-order accuracy of the method is proved. Section 5 is devoted to an efficient monotone iterative algorithm for solving the resulting nonlinear finite difference system. In Section 6, we discuss some techniques for the construction of upper and lower solutions. In Section 7, we give some applications to two model problems and present some numerical results demonstrating the monotone convergence of the iterative sequences and the fourth-order accuracy of the method. We also compare our method with the standard finite difference method and show its advantages. The final section contains some concluding remarks.

2. Compact finite difference scheme

To derive a compact finite difference approximation, we let

$$u_0 = u, \quad u_i = -u_{i-1}^{(2)} \quad (1 \leq i \leq n - 1),$$

and transform problem (1.1) into the coupled system

$$
\begin{align}
-u_i^{(2)} &= u_{i+1}(x) \quad (0 \leq i \leq n - 2), \\
n_{i-1}(x) &= f(x, u_0(x), u_1(x), \ldots, u_{n-1}(x)), \quad 0 < x < 1, \\
u_i(0) &= \sum_{j=1}^{p} \alpha_{ij}u_i(\xi_{ij}), \quad u_i(1) = \sum_{j=1}^{p} \beta_{ij}u_i(\eta_{ij}) \quad (0 \leq i \leq n - 1).
\end{align}
$$

(2.2)

Obviously, $u$ is a solution of (1.1) if and only if $(u_0, u_1, u_2, \ldots, u_{n-1})$ is a solution of (2.2). Our compact finite difference approximation to (1.1) is based on the coupled system (2.2).

Let $h = 1/L$ be the mesh size and let $x_k = kh (0 \leq k \leq L)$ be the mesh points in $[0, 1]$. Assume that for all $0 \leq i \leq n - 1$ and $1 \leq j \leq p$, the points $\xi_{ij}$ and $\eta_{ij}$ in the boundary condition of (1.1) (or (2.2)) serve as mesh points. This assumption is always satisfied by a proper choice of mesh size $h$. For convenience, we use the following notation:

$$
\begin{align}
S^{[i]}[u(\xi_{ij})] &= \sum_{j=1}^{p} \alpha_{ij}u(\xi_{ij}), \\
S^{[i]}[u(\eta_{ij})] &= \sum_{j=1}^{p} \beta_{ij}u(\eta_{ij}),
\end{align}
$$

(2.3)

and introduce the finite difference operators $\delta_{h}^{3}$ and $P_{h}$ as follows:

$$
\begin{align}
\delta_{h}^{3} v(x_{k}) &= v(x_{k-1}) - 2v(x_{k}) + v(x_{k+1}), \quad 1 \leq k \leq L - 1, \\
P_{h} v(x_{k}) &= \frac{h^2}{12} (v(x_{k-1}) + 10v(x_{k}) + v(x_{k+1})), \quad 1 \leq k \leq L - 1.
\end{align}
$$

(2.4)
Using the following Numerov’s formula (cf. [48–50])
\[
\delta_h^2 v(x_k) = \mathcal{P}_h v^{(2)}(x_k) + \mathcal{O}(h^6), \quad 1 \leq k \leq L - 1,
\]  
(2.5)
we have from (2.2) and (2.3) that
\[
\begin{align*}
-\delta_h^2 u_i(x_k) &= \mathcal{P}_h u_{i+1}(x_k) + \mathcal{O}(h^6) \quad (0 \leq i \leq n - 2), \\
-\delta_h^2 u_n-1(x_k) &= \mathcal{P}_h f(x_k, u_k) + \mathcal{O}(h^6), \quad 1 \leq k \leq L - 1, \\
u_i(0) &= S_i^\alpha [u_i(\xi_i)], \quad u_i(1) = S_i^\beta [u_i(\eta_i)] \quad (0 \leq i \leq n - 1).
\end{align*}
\]  
(2.6)
After dropping the \( \mathcal{O}(h^6) \) term, we derive a compact finite difference approximation to (2.2) as follows:
\[
\begin{align*}
-\delta_h^2 u_i, h(x_k) &= \mathcal{P}_h u_{i+1, h}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_h^2 u_n-1, h(x_k) &= \mathcal{P}_h f(x_k, u_{h,k}), \quad 1 \leq k \leq L - 1, \\
u_{i, h}(0) &= S_i^\alpha [u_i, h(\xi_i)], \quad u_{i, h}(1) = S_i^\beta [u_i, h(\eta_i)] \quad (0 \leq i \leq n - 1),
\end{align*}
\]  
(2.7)
where \( u_{i, h} = (u_{0, h}(x_k), u_{1, h}(x_k), \ldots, u_{n-1, h}(x_k)) \) represents the approximation of \( u(x_k) \).

For two constants \( \underline{M} \) and \( \overline{M} \) satisfying \( \underline{M} \geq \overline{M} > -\pi^2 \) we define
\[
h(M, \overline{M}) = \begin{cases} 
\frac{12}{\sqrt{\overline{M}}}, & M > -8, \overline{M} > 0, \\
1, & M > -8, \overline{M} \leq 0, \\
\min \left\{ \frac{12}{\sqrt{\overline{M}}} \sqrt{\frac{12}{\pi^2}} \left(1 + \frac{M}{\pi^2}\right) \right\}, & M \leq -8, \overline{M} > 0, \\
\frac{12}{\pi^2} \left(1 + \frac{M}{\pi^2}\right), & M \leq -8, \overline{M} \leq 0.
\end{cases}
\]  
(2.8)
A fundamental and useful property of the operators \( \delta_h^2 \) and \( \mathcal{P}_h \) is stated below.

**Lemma 2.1** (See Lemma 3.1 of [51]). Let \( \underline{M}, \overline{M} \) and \( M_k \) be some constants satisfying
\[
-\pi^2 < \underline{M} \leq M_k \leq \overline{M}, \quad 0 \leq k \leq L.
\]  
(2.9)
If
\[
\begin{align*}
-\delta_h^2 u_i(x_k) + \mathcal{P}_h (M_k u_k(x_k)) &\geq 0, \quad 1 \leq k \leq L - 1, \\
u_k(0) &\geq 0, \quad u_k(1) \geq 0,
\end{align*}
\]  
(2.10)
and \( h < h(M, \overline{M}) \), then \( u_k(x_k) \geq 0 \) for all \( 0 \leq k \leq L \).

The following results are also useful for our forthcoming discussions. Their proofs will be given in Appendix.

**Lemma 2.2.** Assume
\[
\sigma_i = \max \left\{ \sum_{j=1}^{p} \alpha_{i,j}, \sum_{j=1}^{p} \beta_{i,j} \right\} < 1.
\]  
(2.11)
Let \( \underline{M}, \overline{M} \) and \( M_k \) be the given constants such that
\[
-8(1 - \sigma_i) < \underline{M} \leq M_k \leq \overline{M}, \quad 0 \leq k \leq L.
\]  
(2.12)
If
\[
\begin{align*}
-\delta_h^2 u_i(x_k) + \mathcal{P}_h (M_k u_k(x_k)) &\geq 0, \quad 1 \leq k \leq L - 1, \\
u_k(0) &\geq S_i^\alpha [u_k(\xi_i)], \quad u_k(1) \geq S_i^\beta [u_k(\eta_i)],
\end{align*}
\]  
(2.13)
and \( h < h(M, \overline{M}) \), then \( u_k(x_k) \geq 0 \) for all \( 0 \leq k \leq L \).

**Remark 2.1.** It is clear from Lemma 2.1 that if \( \sigma_i = 0 \) then the condition (2.12) in Lemma 2.2 can be replaced by the weaker condition (2.9). Lemmas 2.1 and 2.2 guarantee that the linear problems based on (2.10) and (2.13) with the inequality relation “\( \geq \)” replaced by the equality relation “\( = \)” are well-posed.
Lemma 2.3. Let the condition (2.11) be satisfied, and let \( M, \overline{M} \) and \( M_k \) be the given constants satisfying (2.12). Assume that the functions \( u_h(x_k) \) and \( g(x) \) satisfy

\[
\begin{aligned}
-\delta_h^2 u_h(x_k) + \partial_{x_k} (M u_h(x_k)) &= g(x_k), \quad 1 \leq k \leq L - 1, \\
u_h(0) &= S^0[u_h(x_0)], \quad u_h(1) = S^0[u_h(\eta_1)].
\end{aligned}
\]  

(2.14)

Then when \( h < h(M, \overline{M}) \),

\[
\|u_h\|_{\infty} \leq \|g\|_{\infty} / ((8(1 - \sigma) + \min(M, 0))h^2),
\]  

(2.15)

where \( \|u_h\|_{\infty} = \max_{1 \leq k \leq L - 1} |u_h(x_k)| \) denotes discrete infinity norm for any mesh function \( u_h(x_k) \).

3. The existence and uniqueness of the solution

By writing the vector \( u = (u_0, u_1, \ldots, u_{n-1}) \) in the split form \( u = ([u]_{\mu_1}, [u]_{\mu_2}, u_{n-1}) \), where \( \mu_1 \) and \( \mu_2 \) are nonnegative integers satisfying \( \mu_1 + \mu_2 = n - 1 \) and \([u]_{\mu}\) denotes a vector with \( \mu \) components of \( u \), we sometimes write

\[
f(\cdot, u) = f(\cdot, [u]_{\mu_1}, [u]_{\mu_2}, u_{n-1}).
\]

In particular, we have \( f(\cdot, u) = f(\cdot, [u]_{\mu_1}, u_{n-1}) \) when \( \mu_1 = 0 \) or \( \mu_2 = 0 \). To investigate the existence and uniqueness of the solution of (2.7), we use the method of upper and lower solutions. The definition of the upper and lower solutions is given as follows.

Definition 3.1. Two vector functions \( \tilde{u}_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k), \ldots, \tilde{u}_{n-1,h}(x_k)) \) and \( \tilde{u}_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k), \ldots, \tilde{u}_{n-1,h}(x_k)) \) are called a pair of coupled upper and lower solutions of (2.7) if \( \tilde{u}_{h,k} \geq \tilde{u}_{h,k} \), and \( \tilde{u}_{h,k} \) satisfies

\[
\begin{aligned}
-\delta_h^2 \tilde{u}_{i,h}(x_k) &\geq \partial_{x_k} \tilde{u}_{i+1,h}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_h^2 \tilde{u}_{n-1,h}(x_k) &\geq \partial_{x_k} \tilde{u}_{n-1,h}(x_k) \quad \text{for all } \tilde{u}_{h,k} \leq \tilde{u}_{h,k}, \\
\tilde{u}_{i,h}(0) &\geq S^0[\tilde{u}_{i,h}(x_0)], \quad \tilde{u}_{i,h}(1) \leq S^0[\tilde{u}_{i,h}(\eta_1)] \quad (0 \leq i \leq n - 1), \quad 1 \leq k \leq L - 1,
\end{aligned}
\]

(3.1)

while \( \tilde{u}_{h,k} \) satisfies (3.1) with the reversed inequalities.

Here and below, inequalities between vectors are in the componentwise sense. Notice that the above definition does not depend on any monotone property of the function \( f \).

For a given pair of coupled upper and lower solutions \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} \), we set

\(
\tilde{u}_{h,k}, \tilde{u}_{h,k} = \{ u \in \mathbb{R}^n; \tilde{u}_{h,k} \leq u \leq \tilde{u}_{h,k} \},
\)

and make the following basic hypotheses:

\( (H_1) \) For each \( 0 \leq k \leq L \), there exists a constant \( M_k \) such that \( M_k > -\pi^2 \) and

\[
f(x_k, [v]_{n-1}, v_{n-1}) - f(x_k, [v]_{n-1}, v'_{n-1}) \geq -M_k (v_{n-1} - v'_{n-1})
\]

whenever \( \tilde{u}_{h,k} \leq ([v]_{n-1}, v'_{n-1}) \leq ([v]_{n-1}, v_{n-1}) \leq \tilde{u}_{h,k} \};

\( (H_2) \) \( h < h(M, \overline{M}) \), where \( \overline{M} = \max_k M_k \) and \( M = \min_k M_k \).

The existence of the constant \( M_k \) in \( (H_1) \) is trivial if \( f(x_k, u) \) is a \( C^1 \)-function of \( u_{n-1} \) for \( u \in (\tilde{u}_{h,k}, \tilde{u}_{h,k}) \). In fact, \( M_k \) may be taken as any nonnegative constant satisfying

\[
M_k \geq \max \{ -f_{u_{n-1}}(x_k, u); u \in (\tilde{u}_{h,k}, \tilde{u}_{h,k}) \}.
\]

Based on Lemma 2.1 we have the following existence result for (2.7).

Theorem 3.1. Let \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} \) be a pair of coupled upper and lower solutions of (2.7), and let hypotheses \( (H_1) \) and \( (H_2) \) be satisfied. Then system (2.7) has at least one solution \( u_{h,k} \) in \( (\tilde{u}_{h,k}, \tilde{u}_{h,k}) \).

Proof. For any \( u_{h,k} \in (\tilde{u}_{h,k}, \tilde{u}_{h,k}) (0 \leq k \leq L) \), Lemma 2.1 ensures that the uncoupled linear problem

\[
\begin{aligned}
-\delta_h^2 u_{i,h}(x_k) + \partial_{x_k} (M u_{i,h}(x_k)) &= g(x_k), \quad 0 \leq i \leq n - 2, \\
-\delta_h^2 u_{n-1,h}(x_k) &= \partial_{x_k} f(x_k, u_{n-1,h}), \quad 1 \leq k \leq L - 1, \\
u_{i,h}(0) &= S^0[u_{i,h}(x_0)], \quad u_{i,h}(1) = S^0[u_{i,h}(\eta_1)], \quad 0 \leq i \leq n - 1,
\end{aligned}
\]

(3.3)

has a unique solution \( u_{h,k} = (u_{0,h}(x_k), u_{1,h}(x_k), \ldots, u_{n-1,h}(x_k)) \) in \( \mathbb{R}^n \). Now, we define a map \( \mathcal{T} : (\tilde{u}_{h,k}, \tilde{u}_{h,k}) \rightarrow \mathbb{R}^n \) by

\[
\mathcal{T} u_{h,k} = u_{h,k}, \quad \forall u_{h,k} \in (\tilde{u}_{h,k}, \tilde{u}_{h,k}).
\]

(3.4)
Let \( \tilde{u}_{i,h}(x_k) = \hat{u}_{i,h}(x_k) - u_{i,h}(x_k) \) (0 ≤ i ≤ n − 1). It follows from (3.1)–(3.3) that
\[
\begin{align*}
-\delta_{n-1}^2 \tilde{u}_{i,h}(x_k) &\geq 0 \quad (0 \leq i \leq n - 2), \\
-\delta_{n-1}^2 \tilde{u}_{n-1,h}(x_k) + \mathcal{P}_h(M_h \tilde{u}_{n-1,h}(x_k)) &\geq 0, \quad 1 \leq k \leq L - 1, \\
\tilde{u}_{i,h}(0) &\geq 0, \quad \tilde{u}_{i,h}(1) \geq 0 \quad (0 \leq i \leq n - 1).
\end{align*}
\] (3.5)

By Lemma 2.1, \( \tilde{u}_{i,h}(x_k) \geq 0 \) (0 ≤ i ≤ n − 1), i.e., \( \mathbf{u}_{h,k} \leq \mathbf{u}_{h,k} \). Similarly, using the property of a lower solution, we have \( \mathbf{u}_{h,k} \geq \mathbf{u}_{h,k} \). Hence, \( \mathcal{T} \) maps \( \mathbf{u}_{h,k} \) into itself. This with the continuity of \( f \) implies that \( \mathcal{T} \) is a bounded continuous map on \( \mathbf{u}_{h,k} \). By virtue of Brower's fixed point theorem, there exists \( \mathbf{u}_{h,k} \) such that
\[ \mathcal{T} \mathbf{u}_{h,k} = \mathbf{u}_{h,k}, \quad 0 \leq k \leq L. \]

This proves that \( \mathbf{u}_{h,k} \) is a solution of system (2.7) in \( \mathbf{u}_{h,k} \).

**Theorem 3.1** shows that (2.7) has at least one solution, provided that it possesses a pair of coupled upper and lower solutions, which also serve as upper and lower bounds of this solution. Next, we consider the uniqueness of the solution by developing a monotone iterative scheme, which also gives a computational algorithm and improves, step by step, the upper and lower bounds of the solution.

Using the coupled upper and lower solutions \( \tilde{u}_{h,k} \) and \( \hat{u}_{h,k} \) as the initial iterations, we construct two vector sequences \( \{\mathbf{u}^{(m)}_{h,k}\} \) and \( \{\mathbf{u}^{(m)}_{h,k}\} \) from the following iterative schemes:
\[
\begin{align*}
-\delta_{n-1}^2 \mathbf{u}^{(m)}_{i,h}(x_k) &= \mathcal{P}_h \mathbf{u}^{(m-1)}_{i+1,h}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_{n-1}^2 \mathbf{u}^{(m)}_{n-1,h}(x_k) + \mathcal{P}_h(M_h \mathbf{u}^{(m)}_{n-1,h}(x_k)) &= \mathcal{P}_h \left( M_h \mathbf{u}^{(m-1)}_{n-1,h}(x_k) + \max_{\mathbf{u}_{h,k} \in [\mathbf{u}_{h,k}^{-1}, \mathbf{u}_{h,k}^{+}]} f(x_k, [\mathbf{w}_{h,k}]_{h-1}, \tilde{u}^{(m-1)}_{n-1,h}(x_k)) \right), \quad 1 \leq k \leq L - 1, \\
\mathbf{u}^{(m)}_{i,h}(0) &= S^i [\mathbf{u}^{(m-1)}_{i,h}(x_0)], \quad \mathbf{u}^{(m)}_{i,h}(1) = S^i [\mathbf{u}^{(m-1)}_{i,h}(x_1)] \quad (0 \leq i \leq n - 1)
\end{align*}
\] (3.6)
and
\[
\begin{align*}
-\delta_{n-1}^2 \mathbf{u}^{(m)}_{i,h}(x_k) &= \mathcal{P}_h \mathbf{u}^{(m-1)}_{i+1,h}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_{n-1}^2 \mathbf{u}^{(m)}_{n-1,h}(x_k) + \mathcal{P}_h(M_h \mathbf{u}^{(m)}_{n-1,h}(x_k)) &= \mathcal{P}_h \left( M_h \mathbf{u}^{(m-1)}_{n-1,h}(x_k) + \min_{\mathbf{v}_{h,k} \in [\mathbf{u}_{h,k}^{-1}, \mathbf{u}_{h,k}^{+}]} f(x_k, [\mathbf{w}_{h,k}]_{h-1}, \tilde{u}^{(m-1)}_{n-1,h}(x_k)) \right), \quad 1 \leq k \leq L - 1, \\
\mathbf{u}^{(m)}_{i,h}(0) &= S^i [\mathbf{u}^{(m-1)}_{i,h}(x_0)], \quad \mathbf{u}^{(m)}_{i,h}(1) = S^i [\mathbf{u}^{(m-1)}_{i,h}(x_1)] \quad (0 \leq i \leq n - 1),
\end{align*}
\] (3.7)
where \( M_h \) is the constant in hypothesis (H1). The following lemma shows that the above two sequences are well-defined and monotone.

**Lemma 3.1.** Let \( \mathbf{u}_{h,k} \) and \( \mathbf{u}_{h,k} \) be a pair of coupled upper and lower solutions of (2.7), and let hypotheses (H1) and (H2) hold. Then the sequences \( \{\mathbf{u}^{(m)}_{h,k}\} \) and \( \{\mathbf{u}^{(m)}_{h,k}\} \) from (3.6) and (3.7) with \( \mathbf{u}^{(0)}_{h,k} = \tilde{u}_{h,k} \) and \( \mathbf{u}^{(0)}_{h,k} = \hat{u}_{h,k} \) are well-defined and possess the monotone property
\[
\mathbf{u}^{(m)}_{h,k} \leq \mathbf{u}^{(m+1)}_{h,k} \leq \mathbf{u}^{(m+1)}_{h,k} \leq \mathbf{u}^{(m)}_{h,k}, \quad 0 \leq k \leq L, \quad m = 0, 1, \ldots
\] (3.8)

**Proof.** Let \( m = 1 \) in (3.6) and (3.7). Since \( \mathbf{u}^{(0)}_{h,k} = \tilde{u}_{h,k}, \mathbf{u}^{(0)}_{h,k} = \hat{u}_{h,k} \), and \( \mathbf{u}_{h,k} \geq \tilde{u}_{h,k} \), the right-hand sides of (3.6) and (3.7) are known. Hence by Lemma 2.1, the first iterations \( \mathbf{u}^{(1)}_{h,k} \) and \( \mathbf{u}^{(1)}_{h,k} \) exist uniquely. Let \( w^{(1)}_{i,h}(x_k) = \mathbf{u}^{(1)}_{i,h}(x_k) - \mathbf{u}^{(1)}_{i,h}(x_k) \) (0 ≤ i ≤ n − 1). Since by hypothesis (H1), the right-hand side of (3.6) is greater than or equal to that of (3.7), we have from (3.6) and (3.7) that
\[
\begin{align*}
-\delta_{n-1}^2 w^{(1)}_{i,h}(x_k) &\geq 0 \quad (0 \leq i \leq n - 2), \\
-\delta_{n-1}^2 w^{(1)}_{n-1,h}(x_k) + \mathcal{P}_h(M_h w^{(1)}_{n-1,h}(x_k)) &\geq 0, \quad 1 \leq k \leq L - 1, \\
w^{(1)}_{i,h}(0) &\geq 0, \quad w^{(1)}_{i,h}(1) \geq 0 \quad (0 \leq i \leq n - 1).
\end{align*}
\] (3.9)

It follows from Lemma 2.1 that \( w^{(1)}_{i,h}(x_k) \geq 0 \) (0 ≤ i ≤ n − 1), i.e., \( \tilde{u}^{(1)}_{h,k} \geq \mathbf{u}^{(1)}_{h,k} \).

Let \( w^{(0)}_{i,h}(x_k) = \tilde{u}_{i,h}(x_k) - \tilde{u}^{(1)}_{i,h}(x_k) \) (0 ≤ i ≤ n − 1). Since the inequality for \( \tilde{u}^{(n-1)}_{n-1,h}(x_k) \) in (3.1) is equivalent to
\[
-\delta_{n-1}^2 \tilde{u}^{(n-1)}_{n-1,h}(x_k) + \mathcal{P}_h(M_h \tilde{u}^{(n-1)}_{n-1,h}(x_k)) \geq \mathcal{P}_h \left( M_h \tilde{u}^{(n-1)}_{n-1,h}(x_k) + \max_{\mathbf{u}_{h,k} \in [\mathbf{u}_{h,k}^{-1}, \mathbf{u}_{h,k}^{+}]} f(x_k, [\mathbf{w}_{h,k}]_{h-1}, \tilde{u}^{(n-1)}_{n-1,h}(x_k)) \right),
\] (3.10)
we have from (3.1) and (3.6) that \(\overline{u}^{(0)}_{i,h}(x_k)\) \((0 \leq i \leq n - 1)\) satisfy (3.9). This ensures that \(\overline{u}^{(0)}_{i,h}(x_k) \geq 0\) \((0 \leq i \leq n - 1)\), i.e., \(\overline{u}_{h,k} \geq \overline{u}^{(1)}_{h,k}\). A similar argument using (3.7) and the property of a lower solution shows \(\overline{u}_{h,k} \leq \underline{u}^{(1)}_{h,k}\). This proves \(\overline{u}^{(0)}_{h,k} \leq \underline{u}^{(1)}_{h,k} \leq \overline{u}^{(0)}_{h,k}\) which implies the monotone property (3.8) for \(m = 0\). The conclusion of the lemma follows by an induction argument. □

It follows from the monotone property (3.8) that the limits
\[
\lim_{m \to \infty} \overline{u}^{(m)}_{h,k} = \overline{u}_{h,k}, \quad \lim_{m \to \infty} \underline{u}^{(m)}_{h,k} = \underline{u}_{h,k}
\]
exist and
\[
\underline{u}^{(m)}_{h,k} \leq \overline{u}^{(m+1)}_{h,k} \leq \underline{u}^{(m)}_{h,k} \leq \overline{u}^{(m+1)}_{h,k}, \quad 0 \leq k \leq L, \quad m = 0, 1, \ldots .
\]

We next show that the limits \(\overline{u}_{h,k}\) and \(\underline{u}_{h,k}\) satisfy
\[
\begin{align*}
-\beta^2_h \overline{u}_{h,k}(x_k) & = \mathcal{P}_h \overline{u}_{h,k+1}(x_k) \quad (0 \leq i \leq n - 2), \\
-\beta^2_h \overline{u}_{h,n-1,k}(x_k) & = \mathcal{P}_h \left( \max_{v_{h,k} \in [\overline{u}^{(m)}_{h,k}, \underline{u}^{(m)}_{h,k}]} f(x_k, [\overline{v}_{h,k}]_{n-1}, \overline{u}_{h,n-1,k}(x_k)) \right), \quad 1 \leq k \leq L - 1,
\end{align*}
\]
and
\[
\begin{align*}
-\beta^2_h \underline{u}_{h,k}(x_k) & = \mathcal{P}_h \underline{u}_{h,k+1}(x_k) \quad (0 \leq i \leq n - 2), \\
-\beta^2_h \underline{u}_{h,n-1,k}(x_k) & = \mathcal{P}_h \left( \min_{v_{h,k} \in [\overline{u}^{(m)}_{h,k}, \underline{u}^{(m)}_{h,k}]} f(x_k, [\overline{v}_{h,k}]_{n-1}, \underline{u}_{h,n-1,k}(x_k)) \right), \quad 1 \leq k \leq L - 1,
\end{align*}
\]

Theorem 3.2. Let the conditions in Lemma 3.1 hold. Then the sequences \([\overline{u}^{(m)}_{h,k}]\) and \([\underline{u}^{(m)}_{h,k}]\) from (3.6) and (3.7) with \(\overline{u}^{(0)}_{h,k} = \overline{u}_{h,k}\) and \(\underline{u}^{(0)}_{h,k} = \underline{u}_{h,k}\) converge monotonically to the respective limits \(\overline{u}_{h,k}\) and \(\underline{u}_{h,k}\) that satisfy (3.12)–(3.14). Moreover, for any solution \(u'_{h,k}\) of system (2.7) in \(\overline{u}_{h,k}, \underline{u}_{h,k}\) we have \(u'_{h,k} \in [\overline{u}_{h,k}, \underline{u}_{h,k}]\).

Proof. To prove (3.13), it suffices to show
\[
\lim_{m \to \infty} \left( M_k \overline{u}^{(m)}_{n-1,k}(x_k) + \max_{v_{h,k} \in [\overline{u}^{(m)}_{h,k}, \underline{u}^{(m)}_{h,k}]} f(x_k, [\overline{v}_{h,k}]_{n-1}, \overline{u}^{(m)}_{n-1,k}(x_k)) \right) = M_k \overline{u}_{n-1,k}(x_k) + \max_{v_{h,k} \in [\overline{u}^{(m)}_{h,k}, \underline{u}^{(m)}_{h,k}]} f(x_k, [\overline{v}_{h,k}]_{n-1}, \underline{u}_{n-1,k}(x_k)).
\]
Indeed, the above fact can be verified by exactly the same argument as that in proving Lemma A of the Appendix in [52] (also see Appendix of [47]). The relation (3.14) can be similarly proved.

Let \(u_{h,k}\) be any solution of system (2.7) in \(\overline{u}_{h,k}, \underline{u}_{h,k}\). Then \(\overline{u}^{(m)}_{h,k} \leq u'_{h,k} \leq \underline{u}^{(m)}_{h,k}\). Assume, by induction, that for certain \(m = m_0 \geq 0\),
\[
\underline{u}^{(m)}_{h,k} \leq u'_{h,k} \leq \overline{u}^{(m)}_{h,k}.
\]

Let \(w^{(m+1)}_{i,h}(x_k) = \overline{u}^{(m+1)}_{i,h}(x_k) - u'_{i,h}(x_k)\) \((0 \leq i \leq n - 1)\). Since by hypothesis (H1),
\[
M_k \overline{u}^{(m+1)}_{n-1,h}(x_k) + \max_{v_{h,k} \in [\overline{u}^{(m)}_{h,k}, \underline{u}^{(m)}_{h,k}]} f(x_k, [\overline{v}_{h,k}]_{n-1}, \overline{u}^{(m)}_{n-1,h}(x_k)) \geq M_k u'_{n-1,h}(x_k) + f(x_k, [\overline{u}_{h,k}]_{n-1}, u'_{n-1,h}(x_k)) \geq M_k u'_{n-1,h}(x_k) + f(x_k, [\overline{u}_{h,k}]_{n-1}, u'_{n-1,h}(x_k)),
\]
we have from (2.7) and (3.6) that
\[
\begin{align*}
-\beta^2_h w^{(m+1)}_{i,h}(x_k) & \geq 0 \quad (0 \leq i \leq n - 2), \\
-\beta^2_h w^{(m+1)}_{n-1,h}(x_k) & + \mathcal{P}_h (M_k w^{(m+1)}_{n-1,h}(x_k)) \geq 0, \quad 1 \leq k \leq L - 1,
\end{align*}
\]
This gives rise to \(w^{(m+1)}_{h,k}(x_k) \geq 0\) \((0 \leq i \leq n - 1)\), i.e., \(\overline{u}^{(m+1)}_{h,k} \geq \underline{u}^{(m+1)}_{h,k}\). In the same manner, we verify that \(u'_{h,k} \geq \underline{u}^{(m+1)}_{h,k}\). This completes the induction and so the inequality (3.16) is valid for all \(m \geq 0\). Letting \(m \to \infty\) in (3.16), we conclude that
\(u'_{h,k} \in [\overline{u}_{h,k}, \underline{u}_{h,k}]\). □

It is clear from Theorem 3.2 that if \(f(\cdot, [\overline{u}]_{n-1}, u_{n-1})\) is monotone nondecreasing in \([\overline{u}]_{n-1}\) then the limits \(\overline{u}_{h,k}\) and \(\underline{u}_{h,k}\) in (3.11) are the maximal and the minimal solutions of (2.7) in \(\overline{u}_{h,k}, \underline{u}_{h,k}\), respectively. This leads to the following conclusion.
Corollary 3.1. Let the conditions in Lemma 3.1 hold. If, in addition, \( f(x_k, [u]_{n-1}, u_{n-1}) \) is monotone nondecreasing in \([u]_{n-1}\) for all \( u \in (\tilde{u}_{h,k}, \bar{u}_{h,k}) \), then system (2.7) has a maximal solution \( \bar{u}_{h,k} \) and a minimal solution \( \tilde{u}_{h,k} \) in \((\tilde{u}_{h,k}, \bar{u}_{h,k})\). Moreover, the sequences \((\tilde{u}_{h,k}^{(m)})\) and \((\bar{u}_{h,k}^{(m)})\) from (3.6) and (3.7) with \( \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k} \) and \( \bar{u}_{h,k}^{(0)} = \bar{u}_{h,k} \) converge monotonically from above and below, respectively, to \( \tilde{u}_{h,k} \) and \( \bar{u}_{h,k} \).

In general, the limit \( \tilde{u}_{h,k} \) or \( \bar{u}_{h,k} \) is the unique solution of (2.7) in \((\tilde{u}_{h,k}, \bar{u}_{h,k})\) if \( u_{h,k} = u_{h,k}^{*} \). To ensure this it is necessary to impose some additional conditions. Assume that \( f(x, u) \) is a \( C^1 \)-function of \( u \) and denote

\[
\begin{align*}
\bar{M}_i &= \max_{0 \leq k \leq L} \max \{ |f_{i_0}(x_k, u)| : u \in (\tilde{u}_{h,k}, \bar{u}_{h,k}) \} \quad (0 \leq i \leq n - 1), \\
\bar{M}^*_i &= \max_{0 \leq k \leq L} \max \{ -f_{i_{n-1}}(x_k, u) : u \in (\tilde{u}_{h,k}, \bar{u}_{h,k}) \}, \\
\bar{M}^*_i &= \min_{0 \leq k \leq L} \min \{ -f_{i_{n-1}}(x_k, u) : u \in (\tilde{u}_{h,k}, \bar{u}_{h,k}) \}. 
\end{align*}
\]

The following theorems give several sufficient conditions for the uniqueness of the solution.

Theorem 3.3. Let the conditions in Lemma 3.1 hold. If, in addition,

\[
\alpha_{i,j} \equiv \alpha_j, \quad \beta_{i,j} \equiv \beta_j, \quad \xi_{i,j} \equiv \xi_j, \quad \eta_{i,j} \equiv \eta_j \quad (0 \leq i \leq n - 1; \ j = 1, 2, \ldots, p), \\
\sigma = \max \left\{ \sum_{j=1}^{p} \alpha_{i,j}, \sum_{j=1}^{p} \beta_{i,j} \right\} < 1, \quad \max_{1 \leq i \leq n-2} \{ \bar{M}_0, 1 + \bar{M}_i, 1 - \bar{M}^*_i \} < 8(1 - \sigma),
\]

then system (2.7) has a unique solution \( u_{h,k}^* \) in \((\tilde{u}_{h,k}, \bar{u}_{h,k})\). Moreover, the sequences \((\tilde{u}_{h,k}^{(m)})\) and \((\bar{u}_{h,k}^{(m)})\) given by (3.6) and (3.7) with \( \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k} \) and \( \bar{u}_{h,k}^{(0)} = \bar{u}_{h,k} \) converge monotonically from above and below, respectively, to \( u_{h,k}^* \).

Proof. It suffices to show \( \tilde{u}_{h,k} = \bar{u}_{h,k} \), where \( \tilde{u}_{h,k} \) and \( \bar{u}_{h,k} \) are the limits in (3.11). Let \( w_{i,h}(x_k) = \bar{u}_{h,k}(x_k) - \tilde{u}_{h,k}(x_k) \) \((0 \leq i \leq n - 1)\). Then \( w_{i,h}(x_k) \geq 0 \) and by (3.13) and (3.14),

\[
\begin{align*}
-\delta_0 w_{i,h}(x_k) &= \mathcal{P}_{h} w_{i+1,h}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_k^2 w_{1,h}(x_k) &= \mathcal{P}_{h} g(x_k), \quad 1 \leq k \leq L - 1, \\
w_{1,h}(0) &= S_{h}^0 w_{1,h}(\xi_0), \quad w_{1,h}(1) = S_{h}^1 w_{1,h}(\eta_1) \quad (0 \leq i \leq n - 1),
\end{align*}
\]

where

\[
g(x_k) = \max_{v_h,k \in (\tilde{u}_{h,k}, \bar{u}_{h,k})} f(x_k, [v_h,k]_{n-1}, \bar{u}_{n-1,h}(x_k)) - \min_{v_h,k \in (\tilde{u}_{h,k}, \bar{u}_{h,k})} f(x_k, [v_h,k]_{n-1}, \bar{u}_{n-1,h}(x_k)).
\]

Using the intermediate value and the mean-value theorems and then using the notation in (3.18), we have the estimate

\[
g(x_k) \leq \sum_{i=0}^{n-2} \bar{M}_i w_{i,h}(x_k) - \bar{M}^*_i w_{n-1,h}(x_k).
\]

Then by (3.20),

\[
\begin{align*}
-\delta_0^2 w_{i,h}(x_k) - \max_{1 \leq i \leq n-2} \{ \bar{M}_0, 1 + \bar{M}_i, 1 - \bar{M}^*_i \} \mathcal{P}_h g(x_k) &\leq 0, \quad 1 \leq k \leq L - 1, \\
w_{1,h}(0) &= S_{h}^0 w_{1,h}(\xi_0), \quad w_{1,h}(1) = S_{h}^1 w_{1,h}(\eta_1) \quad (0 \leq i \leq n - 1).
\end{align*}
\]

Applying Lemma 2.2 gives \( w_{i,h}(x_k) \leq 0 \) \((0 \leq k \leq L)\). This implies \( \tilde{u}_{h,k} = \bar{u}_{h,k} \). □

The following uniqueness results are for the more general boundary conditions.

Theorem 3.4. Let the conditions in Lemma 3.1 hold and let \( \alpha_i \) be defined by (2.11). If, in addition, (2.11) holds for all \( 0 \leq i \leq n - 1 \) and

\[
\sum_{i=0}^{n-1} \prod_{k=0}^{n-1} 8(1 - \sigma_k) \bar{M}_i < \prod_{k=0}^{n-1} 8(1 - \sigma_k),
\]

then the conclusions of Theorem 3.3 are also valid.
Proof. Let \( w_{i,h} = \overline{u}_{i,h}(x_k) - u_{i,h}(x_k) \) (0 \( \leq i \leq n - 1 \)). An application of Lemma 2.3 to (3.20) yields
\[
\|w_{i,h}\|_{\infty} \leq \|w_{i+1,h}\|_{\infty}/(8(1 - \sigma_i)) \quad (0 \leq i \leq n - 2), \quad \|w_{n-1,h}\|_{\infty} \leq \|g\|_{\infty}/(8(1 - \sigma_{n-1})).
\] (3.24)
Since
\[
\|g\|_{\infty} \leq \sum_{i=0}^{n-1} M_i \|w_{i,h}\|_{\infty},
\] (3.25)
the estimate (3.24) implies that
\[
\|w_{n-1,h}\|_{\infty} \leq \sum_{i=0}^{n-1} M_i \left( \prod_{k<i}^{n-1} (1 - \sigma_k) \right)^{-1} \|w_{n-1,h}\|_{\infty}.
\] (3.26)
In view of (3.23), this is possible only when \( \|w_{n-1,h}\|_{\infty} = 0 \) which in turn, by (3.24), implies \( \|w_{i,h}\|_{\infty} = 0 \) for all \( 0 \leq i \leq n - 2 \). This proves \( w_{i,h}(x_k) = 0 \) for all \( i \) and \( k \). The proof is completed. \( \square \)

Theorem 3.5. Let the conditions in Lemma 3.1 hold and let \( \sigma_i \) be defined by (2.11). If, in addition, (2.11) holds for all \( 0 \leq i \leq n - 1 \) and
\[
M^*_n = -8(1 - \sigma_{n-1}), \quad h < h(M^*_n, M^*_n),
\]
\[
\sum_{i=0}^{n-1} M_i \left( \prod_{k<i}^{n-1} (1 - \sigma_k) \right) < \left( 8(1 - \sigma_{n-1}) + \min(M^*_n, 0) \right) \prod_{k=0}^{n-1} (1 - \sigma_k),
\] (3.27)
then the conclusions of Theorem 3.3 are also valid.

Proof. Let \( w_{i,h} = \overline{u}_{i,h}(x_k) - u_{i,h}(x_k) \) (0 \( \leq i \leq n - 1 \)). By the intermediate value and the mean-value theorems, the function \( g(x_k) \) in (3.20) may be written as
\[
g(x_k) = g^*(x_k) - M_{n-1,k} w_{n-1,h}(x_k),
\] (3.28)
where the functions \( g^*(x_k) \) and \( M_{n-1,k} \) satisfy
\[
\|g^*\|_{\infty} \leq \sum_{i=0}^{n-2} M_i \|w_{i,h}\|_{\infty}, \quad M^*_n \leq M_{n-1,k} \leq M^*_n.
\] (3.29)
Thus system (3.20) is equivalent to
\[
\begin{cases}
-\delta^2 w_{i,h}(x_k) = \partial_h w_{i+1,h}(x_k) & (0 \leq i \leq n - 2), \\
-\delta^2 w_{n-1,h}(x_k) + \partial_h w_{n-1,h}(x_n) = \partial_h g^*(x_n) & (0 \leq i < n - 1), \\
w_{i,h}(0) = S^i \{ u_{i,h}(\xi_i) \} & (0 \leq i \leq n - 1).
\end{cases}
\] (3.30)
Since \( -8(1 - \sigma_{n-1}) < M^*_n \leq M_{n-1,k} \leq M^*_n \) and \( h < h(M^*_n, M^*_n) \), we have from Lemma 2.3 that
\[
\begin{cases}
\|w_{i,h}\|_{\infty} \leq \|w_{i+1,h}\|_{\infty}/(8(1 - \sigma_i)) & (0 \leq i \leq n - 2), \\
\|w_{n-1,h}\|_{\infty} \leq \|g^*\|_{\infty}/(8(1 - \sigma_{n-1}) + \min(M^*_n, 0)).
\end{cases}
\] (3.31)
The remaining proof is similar as that for Theorem 3.4 by using the estimate of \( \|g^*\|_{\infty} \) given in (3.29). \( \square \)

4. Convergence of the compact scheme

In this section, we deal with the convergence of the finite difference solution and show the fourth-order accuracy of the scheme (2.7). Throughout this section we assume that the function \( f(x, u) \) and the solution \( u(x) \) of (1.1) are sufficiently smooth.

Let \( u(x_k) = (u_0(x_k), u_1(x_k), \ldots, u_{n-1}(x_k)) \) be the value of the solution of (2.2) at the mesh point \( x_k \), and let \( u_{h,k} \) be the solution of (2.7). We consider the error \( e_{i,h}(x_k) = u_i(x_k) - u_{i,h}(x_k) \). In fact, we have from (2.6) and (2.7) that
\[
\begin{cases}
-\delta^2 e_{i,h}(x_k) = \partial_h e_{i+1,h}(x_k) + \partial^2 e_{i,h}(x_k) & (0 \leq i \leq n - 2), \\
-\delta^2 e_{n-1,h}(x_k) = \partial_h \{ f(x_k, u_{h,k}) - f(x_k, u_{h,k}) \} + \partial^2 e_{n-1,h}(x_k) & (0 \leq i < n - 1), \\
e_{i,h}(0) = S^i \{ e_{i,h}(\xi_i) \} & (0 \leq i \leq n - 1).
\end{cases}
\] (4.1)

Theorem 4.1. Let \( \{ u_{h,k}, u^h \} \) be the set in \( R^n \) such that \( u(x_k), u_{h,k} \in \{ u_{h,k}, u^h \} \). Also let \( M_i \) (0 \( \leq i \leq n - 1 \)), \( M^*_n \) and \( M^*_n \) be the constants defined by (3.18) with respect to \( \{ u_{h,k}, u^h \} \). Assume that (2.11) holds for all \( 0 \leq i \leq n - 1 \), and let either the
condition (3.23) or the condition (3.27) hold. Then for sufficiently small h,
\[
\max_{0 \leq k \leq 1} \|u(x_k) - u_{h,k}\|_\infty \leq C^* h^4,
\]
(4.2)
where \(C^*\) is a positive constant independent of h.

**Proof.** Assume that the condition (3.23) holds. Let \(g(x_k) = f(x_k, u(x_k)) - f(x_k, u_{h,k})\). Applying Lemma 2.3 to (4.1) we obtain
\[
\left\{ \begin{array}{l}
\|e_{i,h}\|_\infty \leq \|e_{i-1,h}\|_\infty / (8(1 - \sigma_k)) + \Theta(h^4) \quad (0 \leq i \leq n - 2), \\
\|e_{n-1,h}\|_\infty \leq \|g\|_\infty / (8(1 - \sigma_{n-1})) + \Theta(h^4).
\end{array} \right.
\]
(4.3)
Since by the mean-value theorem,
\[
\|g\|_\infty \leq \sum_{i=0}^{n-1} M_{i} \|e_{i,h}\|_\infty,
\]
we have from (4.3) that
\[
\left\{ \begin{array}{l}
\|e_{i,h}\|_\infty \leq \left( \prod_{k=1}^{n-2} 8(1 - \sigma_k) \right)^{-1} \sum_{i=0}^{n-1} M_{i} \left( \prod_{k=1}^{n-1} 8(1 - \sigma_k) \right)^{-1} \|e_{n-1,h}\|_\infty + \Theta(h^4).
\end{array} \right.
\]
(4.4)
In view of the condition (3.23), there exists a positive constant \(C^*\) independent of h such that \(\|e_{i,h}\|_\infty \leq C^* h^4\) for all \(0 \leq i \leq n - 2\). This proves the estimate (4.2).

We next assume that the condition (3.27) is satisfied. In this case, the estimate (4.2) can be similarly proved by using the argument for the proof of Theorem 3.5. We omit the details. \(\square\)

Theorem 4.1 shows that the proposed scheme (2.7) possesses the fourth-order accuracy under the conditions of the theorem.

5. An efficient monotone iterative algorithm

To develop a more efficient monotone iterative algorithm for solving (2.7), we replace the set \((u_{h,k}^{(m-1)}, \overline{u}_{h,k}^{(m-1)})\) and \(M_k\) in (3.6) and (3.7) by \(\delta_k^{(m)}\) and
\[
M_k^{(m)} = \max_{u \in \delta_k^{(m)}} \left\{ -f_{u_{n-1}}(x_k, u) \right\},
\]
(5.1)
respectively, where
\[
\delta_k^{(m)} = \left\{ u \in \mathbb{R}^n; \overline{u}_{h,k}^{(m)}(x_k) \leq u_i \leq \underline{u}_{h,k}^{(m)}(x_k) \ (0 \leq i \leq n - 2), \overline{u}_{h,k}^{(m-1)}(x_k) \leq u_{n-1} \leq \underline{u}_{h,k}^{(m-1)}(x_k) \right\}.
\]
(5.2)
This leads to the following iterative schemes:
\[
\left\{ \begin{array}{l}
-\delta_{k}^{2} \overline{u}_{h,k}^{(m)}(x_k) = \mathcal{P}_h \overline{u}_{h+1,k}^{(m-1)}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_{h}^{2} \overline{u}_{h,k}^{(m)}(x_k) = \mathcal{P}_h (M_k^{(m)} u_{h+1,k}^{(m-1)}(x_k)) \\
\end{array} \right.
\]
\[
\left\{ \begin{array}{l}
\overline{u}_{h,k}^{(m)}(0) = S_{i}^{\delta}[\overline{u}_{h,k}^{(m-1)}(\xi)], \quad \overline{u}_{h,k}^{(m)}(1) = S_{i}^{\delta}[\overline{u}_{h,k}^{(m-1)}(\eta_i)] \quad (0 \leq i \leq n - 1)
\end{array} \right.
\]
(5.3)
and
\[
\left\{ \begin{array}{l}
-\delta_{k}^{2} \underline{u}_{h,k}^{(m)}(x_k) = \mathcal{P}_h \underline{u}_{h+1,k}^{(m-1)}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_{h}^{2} \underline{u}_{h,k}^{(m)}(x_k) = \mathcal{P}_h (M_k^{(m)} u_{h+1,k}^{(m-1)}(x_k)) \\
\end{array} \right.
\]
\[
\left\{ \begin{array}{l}
\underline{u}_{h,k}^{(m)}(0) = S_{i}^{\delta}[\underline{u}_{h,k}^{(m-1)}(\xi)], \quad \underline{u}_{h,k}^{(m)}(1) = S_{i}^{\delta}[\underline{u}_{h,k}^{(m-1)}(\eta_i)] \quad (0 \leq i \leq n - 1)
\end{array} \right.
\]
(5.4)
As we shall see later, the above new iterative schemes maintain the monotone convergence of sequences while exhibiting a faster rate of convergence than the original iterative schemes (3.6) and (3.7). Thus, they indeed provide a more efficient monotone iterative algorithm.
In order to show that the sequences from (5.3) and (5.4) are well-defined and monotone, we let $M^*_{n-1}$ and $M^*_{n-1}$ be the constants defined by (3.18) and assume

$$M^*_{n-1} > -\pi^2, \quad h < h(M^*_{n-1}, M^*_{n-1}).$$

(5.5)

**Lemma 5.1.** Let $\tilde{u}_{h,k}$ and $\tilde{u}_{h,k}$ be a pair of coupled upper and lower solutions of (2.7), and let condition (5.5) be satisfied. Then the sequences $\{\tilde{u}_{h,k}\}$ and $\{\tilde{u}_{h,k}\}$ from (5.3) and (5.4) with $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$ and $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$ are well-defined and possess the monotone property (3.8).

**Proof.** Using the same argument as that for the proof of Lemma 3.1, we show that $\tilde{u}_{h,k}(x_k)$ and $\tilde{u}_{h,k}(x_k)$ ($0 \leq i \leq n - 2$) exist uniquely, and

$$\tilde{u}_{h,k}(x_k) \leq \tilde{u}_{h,k}(x_k) \leq \tilde{u}_{h,k}(x_k), \quad (0 \leq i \leq n - 2).$$

(5.6)

This implies that $\tilde{u}_{h,k}$ and $\tilde{u}_{h,k}$ are well-defined, and thus the right-hand sides of the second equations in (5.3) and (5.4) are known. Since $-\pi^2 < M^*_{n-1} \leq M^*_{n-1}$, we again use the argument for the proof of Lemma 3.1 to conclude that $\tilde{u}_{h,k}(x_k)$ and $\tilde{u}_{h,k}(x_k)$ exist uniquely, and the monotone property (5.6) holds also for $i = n - 1$. This proves that the first iterations $\tilde{u}_{h,k}$ and $\tilde{u}_{h,k}$ are well-defined and the monotone property (3.8) is true for $m = 0$. Then the conclusion of the lemma follows by an induction argument. □

As a consequence of Lemma 5.1 we have the following analogous result as that in Theorem 3.2.

**Theorem 5.1.** Let the conditions in Lemma 5.1 hold. Then the sequences $\{\tilde{u}_{h,k}\}$ and $\{\tilde{u}_{h,k}\}$ from (5.3) and (5.4) with $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$ and $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$ converge monotonically to the respective limits $\tilde{u}_{h,k}$ and $\tilde{u}_{h,k}$ that satisfy (3.12)–(3.14). Moreover, for any solution $\tilde{u}_{h,k}$ of system (2.7) in $(\tilde{u}_{h,k}, \tilde{u}_{h,k})$ we have $\tilde{u}_{h,k} \in (\tilde{u}_{h,k}, \tilde{u}_{h,k})$.

**Proof.** The monotone property (3.8) ensures that the sequence $\{M^*_k\}$ is monotone nonincreasing and is bounded from below by $M^*_{n-1}$ given in (3.18). Therefore, the sequence $\{M^*_k\}$ converges as $m \to \infty$. Finally, the proof follows from the similar argument as that for the proof of Theorem 3.2. □

In view of Theorem 5.1, all the conclusions in Corollary 3.1 and Theorems 3.3–3.5 hold true for the iterative schemes (5.3) and (5.4). We summarize these conclusions for (5.3) and (5.4) in the following theorem.

**Theorem 5.2.** Let the conditions in Lemma 5.1 hold, and let $\{\tilde{u}_{h,k}\}$ and $\{\tilde{u}_{h,k}\}$ be the sequences from (5.3) and (5.4) with $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$ and $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$. Then (i) the sequences $\{\tilde{u}_{h,k}\}$ and $\{\tilde{u}_{h,k}\}$ converge monotonically from above and below, respectively, to the maximal solution $\tilde{u}_{h,k}$ and the maximal solution $\tilde{u}_{h,k}$ of (2.7) in $(\tilde{u}_{h,k}, \tilde{u}_{h,k})$ if $f(x_s, [u]_{m-1}, u_{m-1})$ is monotone nondecreasing in $[u]_{m-1}$ for all $u \in (\tilde{u}_{h,k}, \tilde{u}_{h,k})$, and (ii) the sequences $\{\tilde{u}_{h,k}\}$ and $\{\tilde{u}_{h,k}\}$ converge monotonically from above and below, respectively, to a unique solution $\tilde{u}_{h,k}$ of (2.7) in $(\tilde{u}_{h,k}, \tilde{u}_{h,k})$ if the condition (2.11) holds for all $0 \leq i \leq n - 1$ and one of the conditions (3.19), (3.23) and (3.27) is satisfied.

We now compare the iterative schemes (5.3) and (5.4) with the iterative schemes (3.6) and (3.7).

**Theorem 5.3.** Let the conditions in Lemmas 3.1 and 5.1 hold. Denote by $(\tilde{u}_{h,k}^{(m)}, \tilde{u}_{h,k}^{(m)})$ the sequences from (3.6) and (3.7), and by $(\tilde{u}_{h,k}^{(m)}, \tilde{u}_{h,k}^{(m)})$ the sequences from (5.3) and (5.4), where $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$ and $\tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}$. Then

$$\tilde{u}_{h,k}^{(m)} \leq \tilde{u}_{h,k}^{(m)}, \quad \tilde{u}_{h,k}^{(m)} \geq \tilde{u}_{h,k}^{(m)}, \quad 0 \leq k \leq L, \quad m = 0, 1, 2, \ldots .$$

(5.7)

**Proof.** Let $\tilde{u}_{h,k}^{(m)}(x_k) = \tilde{u}_{h,k}^{(m)}(x_k) - \tilde{u}_{h,k}^{(m)}(x_k)$ ($0 \leq i \leq n - 1$). It is clear from the monotone property of the sequences that $M_k \geq M_k^{(m)}$ and $M_k^{(m)}(x_k) \geq M_k^{(m)}(x_k)$ for every $m$ and $k$. Making use of this result and

$$M_k \tilde{u}_{h,k}^{(m)}(x_k) = M_k^{(m)}(x_k) - M_k^{(m)}(x_k) + (M_k^{(m)} - M_k^{(m)})(x_k),$$

we obtain from (3.6) and (5.3) that

$$-\delta^2 \tilde{u}_{h,k}^{(m)}(x_k) + \beta h(M_k^{(m)}(x_k)) \geq \beta h(M_k^{(m)}(x_k) - \tilde{u}_{h,k}^{(m)}(x_k)) + g^{(m)}(x_k),$$

(5.8)

where

$$g^{(m)}(x_k) = \max_{\tilde{v}_{h,k} \in (\tilde{u}_{h,k}^{(m)}, \tilde{u}_{h,k}^{(m)})} f(x_k, [v_{h,k}]_{m-1}, \tilde{u}_{h,k}^{(m)}(x_k)) - \max_{\tilde{v}_{h,k} \in (\tilde{u}_{h,k}^{(m)}, \tilde{u}_{h,k}^{(m)})} f(x_k, [v_{h,k}]_{m-1}, \tilde{u}_{h,k}^{(m)}(x_k)).$$
We use the induction to prove (5.7). Since \( \bar{u}_{h,k}^{(0)} = \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}^{0} \) and \( \bar{u}_{h,k}^{(0)} = \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k}^{0} \), it follows from the monotone property of the sequences that \( s_{k}^{(1)} \leq (\bar{u}_{h,k}^{0} , \bar{u}_{h,k}^{0}) \). This implies that the right-hand side of (5.8) is nonnegative when \( m = 1 \). Therefore by (3.6), (5.3) and (5.8),

\[
\begin{align*}
-\delta_{h}^{2} \bar{w}_{l,h}^{(1)}(x_{k}) &= 0 \quad (0 \leq i \leq n - 2), \\
-\delta_{h}^{2} \bar{w}_{n-1,h}^{(1)}(x_{k}) + P_{h}(M_{k} \bar{w}_{n-1,h}^{(1)}(x_{k})) &\geq 0, \quad 1 \leq k \leq L - 1, \\
\bar{w}_{l,h}^{(1)}(0) &= 0, \quad \bar{w}_{l,h}^{(1)}(1) = 0 \quad (0 \leq i \leq n - 1).
\end{align*}
\]

(5.9)

This yields \( \bar{w}_{l,h}^{(1)}(x_{k}) = 0 \quad (0 \leq i \leq n - 2) \) and \( \bar{w}_{n-1,h}^{(1)}(x_{k}) \geq 0 \), i.e., \( \bar{u}_{l,h}^{(1)}(x_{k}) = \bar{u}_{h,k}^{(1)}(x_{k}) \) \( (0 \leq i \leq n - 2) \) and \( \bar{u}_{n-1,h}^{(1)}(x_{k}) \geq \bar{u}_{n-1,h}^{(1)}(x_{k}) \). By a similar argument, \( \bar{u}_{l,h}^{(1)}(x_{k}) = \bar{u}_{h,k}^{(1)}(x_{k}) \) \( (0 \leq i \leq n - 2) \) and \( \bar{u}_{n-1,h}^{(1)}(x_{k}) \geq \bar{u}_{n-1,h}^{(1)}(x_{k}) \). This proves (5.7) for \( m = 1 \). Assume, by induction, that (5.7) holds true for some \( m_{0} \geq 1 \). Then we have \( s_{k}^{(m_{0}+1)} \leq (\bar{u}_{h,k}^{(m_{0})} , \bar{u}_{h,k}^{(m_{0})}) \). The similar reasoning as that for \( m = 1 \) gives that (5.7) holds also for \( m = m_{0} + 1 \). The induction for (5.7) is completed. \( \square \)

The comparison result (5.7) states that with the same initial iterations, which are coupled upper and lower solutions of (2.7), the sequences from (5.3) and (5.4) converge faster than the corresponding sequences from (3.6) and (3.7).

**Remark 5.1.** Since we adopt the locally extreme values of \( f \), at the right-hand sides of the iterative schemes (5.3) and (5.4) (also (3.6) and (3.7)), the monotone convergence of the produced sequences follows without any monotone requirement on \( f \). This extends the usual monotone iterative method and enlarges its applications.

**Remark 5.2.** If the function \( f(\cdot, [u]_{n-1}, u_{n-1}) \) is monotone in \( [u]_{n-1} \), the computation of the maximum and minimum values of the nonlinear function \( f \) in (5.3) and (5.4) (also (3.6) and (3.7)) is trivial. Otherwise, the maximum and minimum values can be determined by considering the system \( f_{u} = 0 \) \( (0 \leq i \leq n - 2) \). A similar remark is also valid for the computation of \( M_{k}^{(0)} \) in (5.1). On the other hand, we see that in order to obtain the sequences \( [\bar{u}_{h,k}^{(m)}] \) and \( [\tilde{u}_{h,k}^{(m)}] \) from (5.3) and (5.4) (also (3.6) and (3.7)), one needs only to solve certain linear tridiagonal systems of equations for each \( m \), and so the Thomas algorithm is applicable.

6. Construction of upper and lower solutions

It is seen from the previous section that in order to implement the monotone iterative schemes (5.3) and (5.4), it is necessary to find a pair of coupled upper and lower solutions of (2.7). The construction of this pair depends mainly on the function \( f(\cdot, u) \), and much discussion on the subject can be found in [53] for the continuous problems. In this section, we discuss some techniques for the construction of coupled upper and lower solutions of (2.7). It is assumed, throughout this section, that (2.11) holds for all \( 0 \leq i \leq n - 1 \).

6.1. Bounded function \( f \)

Assume that

\[
f(x, [u]_{n-1}, 0) \geq 0, \quad f(x, u) \leq \rho \quad \text{for } x \in [0, 1], \ u \geq 0,
\]

(6.1)

where \( \rho \) is a positive constant. Consider the following linear system:

\[
\begin{align*}
-\delta_{h}^{2} \tilde{u}_{n-1,h}(x_{k}) &= h^{2} \rho, \quad 1 \leq k \leq L - 1, \\
\tilde{u}_{n-1,h}(0) &= S_{n-1}^{0} \tilde{u}_{n-1,h}(\xi_{n-1}), \quad \tilde{u}_{n-1,h}(1) = S_{n-1}^{0} \tilde{u}_{n-1,h}(\eta_{n-1}).
\end{align*}
\]

(6.2)

**Lemma 2.2.** implies that the solution \( \tilde{u}_{n-1,h}(x_{k}) \) of the above system exists uniquely and is nonnegative. Using \( \tilde{u}_{n-1,h}(x_{k}) \) in the following linear system:

\[
\begin{align*}
-\delta_{h}^{2} \tilde{u}_{h,k}(x_{k}) &= P_{h}\tilde{u}_{h,k}(x_{k}), \quad 1 \leq k \leq L - 1, \\
\tilde{u}_{h,k}(0) &= S_{h}^{0} \tilde{u}_{h,k}(\xi_{k}), \quad \tilde{u}_{h,k}(1) = S_{h}^{0} \tilde{u}_{h,k}(\eta_{k}),
\end{align*}
\]

(6.3)

we have again from **Lemma 2.2** that for each \( i = n - 2, n - 1, \ldots, 0 \), the solution \( \tilde{u}_{h,k}(x_{k}) \) exists uniquely and is nonnegative. Let \( \tilde{u}_{h,k} = (\tilde{u}_{0,h}(x_{k}), \tilde{u}_{1,h}(x_{k}), \ldots, \tilde{u}_{n-1,h}(x_{k})) \). Since by (6.1),

\[
-\delta_{h}^{2} \tilde{u}_{0,h} = 0 \leq P_{h}f(\tilde{u}_{0,h}, [v_{h,k}]_{n-1}, 0),
\]

(6.4)

it is clear from (6.2) and (6.3) that the pair \( \tilde{u}_{h,k} = \tilde{0} \) and \( \tilde{u}_{h,k} = \tilde{0} \) are coupled upper and lower solutions of (2.7).

On the other hand, if there exists a positive constant \( c \) such that

\[
f(x, [u]_{n-1}, 0) \geq 0, \quad f(x, [u]_{n-1}, c) \leq 0 \quad \text{for } x \in [0, 1], \ u \geq 0,
\]

(6.5)

then

\[
-\delta_{h}^{2} \tilde{u}_{0,h} = 0 \leq P_{h}f(\tilde{u}_{0,h}, [v_{h,k}]_{n-1}, 0), \quad -\delta_{h}^{2} c \geq 0 \geq P_{h}f(\tilde{u}_{0,h}, [v_{h,k}]_{n-1}, c) \quad \text{for all } v_{h,k} \geq 0.
\]

(6.6)
Let \( \tilde{u}_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k), \ldots, \tilde{u}_{n-2,h}(x_k), c) \), where for each \( i = n - 2, n - 3, \ldots, 0, \tilde{u}_{i,h}(x_k) \) is the nonnegative solution of (6.3) with \( \tilde{u}_{i-1,h}(x_k) \) replaced by the constant \( c \). Then the inequality (6.6) ensures that the pair \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} = 0 \) are coupled upper and lower solutions of (2.7).

Assume that the condition (6.1) is replaced by

\[
\text{if } f(x, [u]_{n-1}, 0) \geq 0, \quad f(x, u) \leq \rho \quad \text{for } x \in [0, 1], \quad 0 \leq u \leq (\delta_0, \delta_1, \ldots, \delta_{n-1}),
\]

(6.7)

where \( \rho \) and \( \delta_i \) are positive constants. Let \( \tilde{u}_{i,h}(x_k) (0 \leq i \leq n - 1) \) be the nonnegative solutions of (6.2) and (6.3). Then by Lemma 2.3,

\[
\|\tilde{u}_{i,h}\|_{\infty} \leq \rho \left( \prod_{k=1}^{n-1} (1 - \sigma_k) \right)^{-1}, \quad 0 \leq i \leq n - 1.
\]

(6.8)

We see from (6.2) and (6.3) that the pair \( \tilde{u}_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k), \ldots, \tilde{u}_{n-1,h}(x_k)) \) and \( \tilde{u}_{h,k} = 0 \) are coupled upper and lower solutions of (2.7) if (6.4) holds. By the locally bounded property (6.7) and the estimate (6.8), the latter is true if

\[
\rho \left( \prod_{k=1}^{n-1} (1 - \sigma_k) \right)^{-1} \leq \delta_i, \quad 0 \leq i \leq n - 1.
\]

(6.9)

Similarly, we replace the condition (6.5) by

\[
f(x, [u]_{n-1}, 0) \geq 0, \quad f(x, [u]_{n-1}, c) \leq 0 \quad \text{for } x \in [0, 1], \quad 0 \leq u \leq (\delta_0, \delta_1, \ldots, \delta_{n-1}),
\]

(6.10)

where \( \delta_i \) and \( \delta_i \) are positive constants. Let \( \tilde{u}_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k), \ldots, \tilde{u}_{n-2,h}(x_k), c) \), where for each \( i = n - 2, n - 3, \ldots, 0, \tilde{u}_{i,h}(x_k) \) is the nonnegative solution of (6.3) with \( \tilde{u}_{i-1,h}(x_k) \) replaced by the constant \( c \). Then the pair \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} = 0 \) are coupled upper and lower solutions of (2.7) provided that

\[
c \left( \prod_{k=1}^{n-2} (1 - \sigma_k) \right)^{-1} \leq \delta_i, \quad 0 \leq i \leq n - 2.
\]

(6.11)

To see this we observe that the inequalities in (6.6) are satisfied for all \( 0 \leq v_{h,k} \leq \tilde{v}_{h,k} \) under the conditions (6.10) and (6.11). This ensures that \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} = 0 \) satisfy all the requirements of a pair of coupled upper and lower solutions.

6.2. Function \( f \) satisfying a linear growth

Assume that

\[
f(x, [u]_{n-1}, 0) \geq 0, \quad f(x, u) \leq \rho^*(x)u_{n-1} + \rho(x) \quad \text{for } x \in [0, 1], \quad u \geq 0,
\]

(6.12)

where \( \rho^*(x) \) and \( \rho(x) \) are nonnegative continuous functions in \( [0, 1] \). Let \( \tilde{u}_{n-1,h}(x_k) \) be the solution of the linear system:

\[
\begin{align*}
-\delta_0^2 \tilde{u}_{n-1,h}(x_k) & = \mathcal{P}_h(\rho^*(x_k)\tilde{u}_{n-1,h}(x_k)) = \mathcal{P}_h(\rho(x_k)), \quad 1 \leq k \leq L - 1, \\
\tilde{u}_{n-1,h}(0) & = S^\beta_{n-1}([\tilde{u}_{n-1,h}(\xi_{n-1})]), \quad \tilde{u}_{n-1,h}(1) = S^\beta_{n-1}([\tilde{u}_{n-1,h}(\eta_{n-1})]),
\end{align*}
\]

(6.13)

and let \( \tilde{u}_{i,h}(x_k) (i = n - 2, n - 3, \ldots, 0) \) be the solution of (6.3). By Lemma 2.2, these solutions exist and are nonnegative if \( \rho^*(x) < 8(1 - \sigma_{n-1}) \) for all \( x \in [0, 1] \). Moreover by (6.12),

\[
-\delta_0^2 \tilde{u}_{i,h}(x_k) \geq \mathcal{P}_h(\rho^*(x_k)\tilde{u}_{i-1,h}(x_k) + \rho(x_k)) \geq \mathcal{P}_h(\rho(x_k), [v_{h,k}]_{n-1}, \tilde{u}_{i-1,h}(x_k))
\]

(6.14)

for all \( 0 \leq v_{h,k} \leq \tilde{v}_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k), \ldots, \tilde{u}_{n-1,h}(x_k)) \). This shows that the pair \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} = 0 \) are coupled upper and lower solutions of (2.7) if \( \rho^*(x) < 8(1 - \sigma_{n-1}) \) for all \( x \in [0, 1] \).

6.3. Mixed monotone function \( f \)

Assume that there exist nonnegative integers \( \mu_1 \) and \( \mu_2 \) satisfying \( \mu_1 + \mu_2 = n - 1 \) such that the function \( f(\cdot, [u]_{\mu_1}, [u]_{\mu_2}, u_{n-1}) \) is monotone nondecreasing in \([u]_{\mu_1}\) and is monotone nonincreasing in \([u]_{\mu_2}\). Then the requirements of a pair of coupled upper and lower solutions \( \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k} \) of (2.7) are reduced to

\[
\begin{align*}
-\delta_0^2 \tilde{u}_{i,h}(x_k) & \geq \mathcal{P}_h(\tilde{u}_{i+1,h}(x_k)), \quad 0 \leq i \leq n - 2, \\
-\delta_0^2 \tilde{u}_{i-1,h}(x_k) & \geq \mathcal{P}_h(\tilde{u}_{i,h}(x_k), [\tilde{u}_{i,h}]_{\mu_1}, [\tilde{u}_{i,h}]_{\mu_2}, \tilde{u}_{i-1,h}(x_k)), \quad 1 \leq k \leq L - 1, \\
\tilde{u}_{i,h}(0) & \geq S^\beta_{i}([\tilde{u}_{i,h}(\xi_i)]), \quad \tilde{u}_{i,h}(1) \geq S^\beta_{i}([\tilde{u}_{i,h}(\eta_i)]), \quad 0 \leq i \leq n - 1
\end{align*}
\]

(6.15)
Consider the fourth-order five-point boundary value problem:
\[
\begin{align*}
-\delta_k^2 \bar{u}_i(x_k) &\leq \mathcal{P}_h \bar{u}_{i+1}(x_k) \quad (0 \leq i \leq n - 2), \\
-\delta_k^2 \bar{u}_{i-1}(x_k) &\leq \mathcal{P}_h \bar{u}_i(x_k), \quad 1 \leq k \leq L - 1, \\
\bar{u}_i(0) &\leq S_{\bar{u}_i}^{\alpha}(\xi_i), \quad \bar{u}_i(1) \leq S_{\bar{u}_i}^{\beta}(\eta_i) \quad (0 \leq i \leq n - 1). 
\end{align*}
\]

(6.16)

Assume, in addition, that
\[
\begin{align*}
&f(x, [u]_{j=1}, [u]_{j=2}, 0) \geq 0, \\
&f(x, [u]_{j=1}, [0]_{j=2}, u_{n-1}) \leq \rho^*(x) u_{n-1} + \rho(x) \quad \text{for } x \in [0, 1], \quad u \geq 0,
\end{align*}
\]

(6.17)

where $\rho^*(x)$ and $\rho(x)$ are nonnegative continuous functions satisfying $\rho^*(x) < 8(1-\sigma_{n-1})$ for all $x \in [0, 1]$. Let $\tilde{u}_{i,k}(x_k)$ ($0 \leq i \leq n - 1$) be the nonnegative solutions of (6.13) and (6.3). Then a similar reasoning as that in the previous subsection shows that the pair $(\tilde{u}_{i,k}, \tilde{u}_{i,k}(x_k))$ and $(\bar{u}_{i,k}, 0)$ satisfy (6.15) and (6.16), and they therefore are a pair of coupled upper and lower solutions of (2.7).

### 7. Applications and numerical results

In this section, we give some applications of the results in the previous sections to two model problems. We present some numerical results to demonstrate the monotone convergence of the sequences from (5.3) and (5.4) and to show the fourth-order accuracy of the scheme (2.7), as predicted in the analysis.

In each of the two problems, the analytic solution $u(x)$ of (1.1) is explicitly known, against which we can compare the finite difference solution $u_{i,h}^{(m)}(x_k)$ to show the fourth-order accuracy of the scheme (2.7). The order of accuracy is calculated by

\[
\text{error}_{\infty}(h) = \|u - u_{0,h}^{(m)}\|_{\infty}, \quad \text{order}_{\infty}(h) = \log_2 \left( \text{error}_{\infty}(h)/\text{error}_{\infty}(h/2) \right). 
\]

(7.1)

All computations are carried out by using a MATLAB subroutine on a Pentium-4 computer with 2G memory, and the termination criterion of iterations for (5.3) and (5.4) is given by

\[
\sum_{i=0}^{n-1} \|\bar{u}_{i,h}^{(m)} - u_{i,h}^{(m)}\|_{\infty} < 10^{-14}. 
\]

(7.2)

**Example 1.** Consider the fourth-order five-point boundary value problem:

\[
\begin{align*}
u^{(4)}(x) &= \sigma(x) \frac{u^{(2)}(x)}{1 + u(x)} + q(x), \quad 0 < x < 1, \\
u(0) &= \frac{1}{2} u \left( \frac{1}{2} \right), \quad u(1) = \frac{1}{4} u \left( \frac{1}{2} \right) + \frac{\sqrt{3}}{6} u \left( \frac{3}{4} \right), \\
u^{(2)}(0) &= \frac{\sqrt{3}}{3} u^{(2)} \left( \frac{1}{4} \right), \quad u^{(2)}(1) = \frac{\sqrt{3}}{12} u^{(2)} \left( \frac{1}{4} \right) + \frac{\sqrt{3}}{4} u^{(2)} \left( \frac{3}{4} \right),
\end{align*}
\]

(7.3)

where $\sigma(x)$ is a sign-changing continuous function and $q(x)$ is a nonnegative continuous function. Clearly, problem (7.3) is a special case of (1.1) with $n = 2$ and

\[
f(x, u) = \sigma(x) \frac{-u}{1 + u} + q(x).
\]

(7.4)

To obtain an explicit analytic solution of (7.3), we choose

\[
q(x) = \theta^2 \left( \theta^2 + \frac{\sigma(x)}{1 + \sin(\theta x + \pi/6)} \right) \sin(\theta x + \pi/6), \quad \theta = \frac{2\pi}{3}.
\]

(7.5)

Then $u(x) = \sin(\theta x + \pi/6)$ is a solution of (7.3). Moreover, $q(x) \geq 0$ if $\sigma(x) \geq -3\theta^2/2$ in $[0, 1]$.

For problem (7.3), the corresponding scheme (2.7) is now reduced to

\[
\begin{align*}
\delta_h^2 u_0(x_k) &\leq \mathcal{P}_h u_{1,h}(x_k), \quad -\delta_k^2 u_1(x_k) = \mathcal{P}_h f(x_k, u_{1,k}), \quad 1 \leq k \leq L - 1, \\
u_0(0) &= \frac{1}{2} u_0 \left( \frac{1}{2} \right), \quad u_0(1) = \frac{1}{4} u_0 \left( \frac{1}{2} \right) + \frac{\sqrt{3}}{6} u_0 \left( \frac{3}{4} \right), \\
u_1(0) &= \frac{\sqrt{3}}{3} u_1 \left( \frac{1}{4} \right), \quad u_1(1) = \frac{\sqrt{3}}{12} u_1 \left( \frac{1}{4} \right) + \frac{\sqrt{3}}{4} u_1 \left( \frac{3}{4} \right).
\end{align*}
\]

(7.6)
To find a pair of coupled upper and lower solutions of (7.6) we observe from (7.4) that

\[ f(x, u_0, 0) = q(x) \geq 0, \quad f(x, u) \leq |\sigma(x)|u_1 + q(x) \quad \text{for} \ x \in [0, 1], \ u \geq 0. \]  

(7.7)

This implies that the condition (6.12) is satisfied for the present function \( f \) with \( \rho^*(x) = |\sigma(x)| \) and \( \rho(x) = q(x) \). Let \( \tilde{u}_{1,h}(x_k) \) and \( \tilde{u}_{0,h}(x_k) \) be the respective solutions of (6.13) and (6.3) (corresponding to (7.6)) with \( \rho^*(x) = |\sigma(x)| \) and \( \rho(x) = q(x) \).

Then we have from the construction of upper and lower solutions in Section 6 that the pair \( u_{h,k} = (\tilde{u}_{0,h}(x_k), \tilde{u}_{1,h}(x_k)) \) and \( \tilde{u}_{h,k} = 0 \) are coupled upper and lower solutions of (7.6) if \( |\sigma(x)| < 8(1 - \sqrt{3}/3) \) for all \( x \in [0, 1] \).

Let \( \sigma(x) = \sin(\theta \pi x) \). Using \( \tilde{u}_{h,k}^{(0)} = \tilde{u}_{h,k} \) and \( \tilde{u}_{h,k}^{(0)} = 0 \), we compute the sequences \( \{\tilde{u}_{h,k}^{(m)}\} \) and \( \{u_{h,k}^{(m)}\} \) from the iterative schemes (5.3) and (5.4) for (7.6) and various values of \( h \). In all the numerical computations, the basic feature of monotone convergence of the sequences was observed. Let \( h = 1/32 \). In Fig. 7.1, we present some numerical results of the sequences \( \{\tilde{u}_{0,h}^{(m)}(x_k)\} \) and \( \{u_{0,h}^{(m)}(x_k)\} \) at \( x_k = 0.5 \), where the solid line denotes the sequence \( \{u_{0,h}^{(m)}(x_k)\} \) and the dashed–dotted line stands for the sequence \( \{u_{0,h}^{(m)}(x_k)\} \). Since the sequences \( \{\tilde{u}_{h,k}^{(m)}\} \) and \( \{u_{h,k}^{(m)}\} \) converge to the same limit as \( m \to \infty \), their common limit \( u_{h,k}^* \) is the unique solution of (7.6) in \( 0, \tilde{u}_{h,k}(n = 2) \). Some numerical results of \( u_{h,k}^* \) at various \( x_k \) are explicitly given in Table 7.1. We also list the values of the analytic solution \( u(x_k) \) in this table. Clearly, the finite difference solution \( u_{h,k}^* \) meets the analytic solution \( u(x_k) \) closely.

To further demonstrate the accuracy of the solution \( u_{h,k}^* \), we list the maximum error \( \text{err}_{\infty}(h) \) and the order \( \text{order}_{\infty}(h) \) in the first three columns of Table 7.2 for various values of \( h \). The data demonstrate that the solution \( u_{h,k}^* \) has the fourth-order accuracy. This coincides with the analysis very well.

For comparison, we also solve (7.3) by the standard finite difference (SFD) method. This method leads to a finite difference scheme in the form (7.6) with \( \mathcal{P}_h = I \) (an identical operator). Thus, similar iterative schemes as (5.3) and (5.4) can be used in actual computations. The corresponding maximum error \( \text{err}_{\infty}(h) \) and the order \( \text{order}_{\infty}(h) \) are listed in the last two columns of Table 7.2. We see that the standard finite difference method possesses only the second-order accuracy.

Table 7.1
Solutions \( u_{0,h}^*(x_k) \) and \( u(x_k) \) of Example 1.

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( u_{0,h}^*(x_k) )</th>
<th>( u(x_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>0.60876152207967</td>
<td>0.60876142900872</td>
</tr>
<tr>
<td>1/8</td>
<td>0.70710688929036</td>
<td>0.70710678118655</td>
</tr>
<tr>
<td>3/16</td>
<td>0.79335346157970</td>
<td>0.7933534029124</td>
</tr>
<tr>
<td>1/4</td>
<td>0.86602536183736</td>
<td>0.86602540378444</td>
</tr>
<tr>
<td>5/16</td>
<td>0.92387976717540</td>
<td>0.92387953251129</td>
</tr>
<tr>
<td>3/8</td>
<td>0.96592597396624</td>
<td>0.96592582628907</td>
</tr>
<tr>
<td>7/16</td>
<td>0.99144501295591</td>
<td>0.99144486137381</td>
</tr>
<tr>
<td>1/2</td>
<td>1.00000015289372</td>
<td>1</td>
</tr>
</tbody>
</table>
Theorem 5.1

Table 7.3

(7.9)

σ

(5.3)

37x126
h
the above

where

Table 7.2

(1, 0, 0)

2.28295160553671e−128
5.9717836084835e−16
2.44754945177839e8
3.92413025447347e16
5.5124945177839e−6
0.0073656405996
1.52889736798275e−6
0.0018460296779
9.55463436393072e−6
3.9996704288123
5.917836084835e−10
3.72817316575696e−11
2.8295160553671e−12

Example 2. Our second example is for the following sixth-order five-point boundary value problem:

\[
\begin{align*}
-u^{(6)}(x) &= \sigma_1(x) e^{-u(x)} + \sigma_2(x) u^{(4)}(x) (1-u^{(4)}(x)) + q(x), \quad 0 < x < 1, \\
u(0) &= \sqrt{\frac{3}{2}} u\left(\frac{1}{4}\right), \quad u(1) = \sqrt{\frac{3}{2}} u\left(\frac{3}{4}\right), \\
u^{(2)}(0) &= \frac{13}{17} u^{(2)}\left(\frac{1}{2}\right), \quad u^{(2)}(1) = \frac{13}{17} u^{(2)}\left(\frac{3}{4}\right), \\
u^{(4)}(0) &= \frac{3}{5} u^{(4)}\left(\frac{1}{2}\right) + \frac{1}{5} u^{(4)}\left(\frac{3}{4}\right), \quad u^{(4)}(1) = \frac{3}{5} u^{(4)}\left(\frac{1}{2}\right) + \frac{1}{5} u^{(4)}\left(\frac{3}{4}\right), \\
\end{align*}
\]

where \( \sigma_1(x), \sigma_2(x) \) and \( q(x) \) are nonnegative continuous functions. The corresponding scheme (2.7) for this example is given by

\[
\begin{align*}
-\delta^2_h u_{i,h}(x_k) &= \sigma_1(x_k) e^{-u_{i,h}(x_k)} + \sigma_2(x_k) u_{i-1,h}(x_k) (1-u_{i-1,h}(x_k)) + q(x_k), \\
u_{0,h}(0) &= \sqrt{\frac{3}{2}} u_{0,h}\left(\frac{1}{4}\right), \quad u_{0,h}(1) = \sqrt{\frac{3}{2}} u_{0,h}\left(\frac{3}{4}\right), \\
u_{1,h}(0) &= \frac{13}{17} u_{1,h}\left(\frac{1}{2}\right), \quad u_{1,h}(1) = \frac{13}{17} u_{1,h}\left(\frac{3}{4}\right), \\
u_{2,h}(0) &= \frac{3}{5} u_{2,h}\left(\frac{1}{2}\right) + \frac{1}{5} u_{2,h}\left(\frac{3}{4}\right), \quad u_{2,h}(1) = \frac{3}{5} u_{2,h}\left(\frac{1}{2}\right) + \frac{1}{5} u_{2,h}\left(\frac{3}{4}\right), \\
\end{align*}
\]

where for \( 0 \leq k \leq L \),

\[
f(x_k, u_{i,h}) = \sigma_1(x_k) e^{-u_{i-1,h}(x_k)} + \sigma_2(x_k) u_{i-1,h}(x_k) (1-u_{i-1,h}(x_k)) + q(x_k).
\]

Let

\[
q(x) = \theta^4 \sin(\theta x + \frac{\pi}{6}) \left(\frac{\theta^2}{6} \left(1 + \theta^2 \sigma_2(x) \sin(\theta x + \frac{\pi}{6})\right) - \sigma_2(x) - \sigma_1(x) e^{-z(x)}\right),
\]

\[
z(x) = \frac{\pi^2}{4} \left(x^2 \sin(\frac{\pi}{6}) + \frac{3 \sqrt{3} + \frac{\pi^2}{128}}{2}\right), \quad \theta = \frac{2\pi}{3}.
\]

Then \( q(x) \geq 0 \) in \([0, 1]\) if \( 4 \sigma_1(x) + \sigma_2(x) \leq 4 \theta^6 \sin(\theta x + \pi/6) \), and \( u(x) = z(x) \) is a solution of (7.8). Let \( \bar{\sigma}_i \geq \max_{0 \leq i \leq 2} \sigma_i(x) \) (\( i = 1, 2 \)) and \( \bar{q} \geq \max_{0 \leq i \leq 1} q(x) \). It is clear that the condition (6.1) is satisfied for the present function \( f \) with \( \rho = \bar{\sigma}_1 + \bar{\sigma}_2 / 4 + \bar{q} \). Let \( u_{i,h}(x_k) \) \((0 \leq i \leq 2)\) be the solutions of (6.2) and (6.3) (corresponding to (7.9)) with respect to the above \( \rho \). Then the construction of upper and lower solutions in Section 6 implies that \( u_{i,h} = u_{i-1,h} + u_{i-2,h} + u_{i-3,h} + q \) and \( \bar{u}_{i,h} = 0 \) form a pair of coupled upper and lower solutions of (7.9).

Let

\[
\sigma_1(x) = \sin(\pi x) \sin(\theta x + \pi/6), \quad \sigma_2(x) = x^2 \sin(\theta x + \pi/6) / 2, \quad \bar{\sigma}_1 = 1, \quad \bar{\sigma}_2 = 1 / 2, \quad \bar{q} = \theta^6 (1 + \bar{\sigma}_2^2 \theta^2).
\]

Using \( \bar{u}_{i,h}^m = u_{i,h}^m + u_{i-1,h}^m \) and \( u_{i,h}^m = 0 \), we compute the sequences \( \{u_{i,h}^m(x_k)\} \) and \( \{\bar{u}_{i,h}^m(x_k)\} \) from the iterative schemes (5.3) and (5.4) for (7.9) and various values of \( h \). Let \( h = 1 / 32 \). Some numerical results of the sequences \( \{u_{0,h}^m(x_k)\} \) and \( \{\bar{u}_{1,h}^m(x_k)\} \) at \( x_k = 0.5 \) are plotted in Fig. 7.2, where the solid line denotes the sequence \( \{u_{0,h}^m(x_k)\} \) and the dashed–dotted line stands for the sequence \( \{\bar{u}_{0,h}^m(x_k)\} \). We see that the sequences possess the monotone convergence described in Theorem 5.1. Since the sequences \( \{u_{0,h}^m(x_k)\} \) and \( \{\bar{u}_{0,h}^m(x_k)\} \) converge to the same limit as \( m \to \infty \), their common limit \( u_{0,h}^m \) is the unique solution of (7.9) in \( \{0, \bar{u}_{i,h}(x_k)\} \) with \( n = 3 \). The maximum error \( \text{error}_{\infty, h} \) and the order \( \text{order}_{\infty, h} \) of the finite difference solution \( u_{0,h}^m(x_k) \) by scheme (7.9) and the SFD scheme are presented in Table 7.3. The numerical results clearly indicate that the proposed scheme (7.9) is more efficient than the SFD scheme.
Fig. 7.2. The monotone convergence of \((u^{(m)}_{0,0}(x_k), u^{(m)}_{0,k}(x_k))\) at \(x_k = 0.5\) for Example 2.

Table 7.3
The accuracy of the finite difference solution \(u^*_0(x_k)\) of Example 2.

<table>
<thead>
<tr>
<th>(h)</th>
<th>Scheme (7.9)</th>
<th>SFD scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error(_\infty) ((h))</td>
<td>order(_\infty) ((h))</td>
</tr>
<tr>
<td>1/4</td>
<td>1.38685904151270e−03</td>
<td>4.01177497749615</td>
</tr>
<tr>
<td>1/8</td>
<td>8.59741158216743e−05</td>
<td>4.00294156988937</td>
</tr>
<tr>
<td>1/16</td>
<td>5.36243739190233e−06</td>
<td>4.00073577674078</td>
</tr>
<tr>
<td>1/32</td>
<td>3.34981452354555e−07</td>
<td>4.00019137208009</td>
</tr>
<tr>
<td>1/64</td>
<td>2.09315637713082e−08</td>
<td>3.99062439131334e−02</td>
</tr>
<tr>
<td>1/128</td>
<td>1.30813537779773e−09</td>
<td>3.977348234081354e−02</td>
</tr>
<tr>
<td>1/256</td>
<td>8.22084622598140e−11</td>
<td>3.91229374943258</td>
</tr>
<tr>
<td>1/512</td>
<td>5.46007683510652e−12</td>
<td>3.90262053798140e−12</td>
</tr>
</tbody>
</table>

8. Concluding remarks

In this paper, we proposed a fourth-order compact finite difference method for a class of nonlinear 2nth-order multi-point boundary value problems. The existence and uniqueness of the finite difference solution and the convergence of the method were discussed. An efficient monotone iterative algorithm was developed for solving the resulting nonlinear finite difference systems. The proposed method is more attractive due to its fourth-order accuracy, compared to the standard finite difference method. In this work, we generalized the method of upper and lower solutions to nonlinear higher-order multi-point boundary value problems. We also developed a technique for designing and analyzing compact finite difference schemes of nonlinear higher-order multi-point boundary value problems.

We conclude by taking note that the proposed fourth-order compact discretization methodology may be straightforwardly extended to the following nonhomogeneous multi-point boundary condition:

\[
u^{(2i)}(0) = \sum_{j=1}^{p} \alpha_{j,i} u^{(2i)}(\xi_{j,i}) + \lambda_{0,i}, \quad \nu^{(2i)}(1) = \sum_{j=1}^{p} \beta_{j,i} u^{(2i)}(\eta_{j,i}) + \lambda_{1,i}, \quad 0 \leq i \leq n - 1,
\]

where for each \(i, \lambda_{0,i}\) and \(\lambda_{1,i}\) are two prescribed constants. In particular, all the results in this paper hold true for the above nonhomogeneous multi-point boundary condition.

Acknowledgments

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Appendix

In this Appendix, we prove Lemmas 2.2 and 2.3.
For each $0 \leq i \leq n - 1$ and $1 \leq j \leq L - 1$, we define
\[
\alpha_{ij}^* = \begin{cases} 
\alpha_{ij}, & x_j = \xi_{ij} \text{ for some } i, \\
0, & \text{otherwise,}
\end{cases}
\quad \beta_{ij}^* = \begin{cases} 
\beta_{ij}, & x_j = \eta_{ij} \text{ for some } i, \\
0, & \text{otherwise.}
\end{cases}
\tag{A.1}
\]

Let $A = (a_{ik})$, $B = (b_{ik})$ and $D_i = (d^{(i)}_{kl})$ be the $(L - 1)$th-order matrices with
\[
a_{ik} = 2\delta_{k,i} - \delta_{k,i-1} - \delta_{k,i+1}, \quad b_{ik} = \frac{5}{6}\delta_{k,i} + \frac{1}{12}\delta_{k,i-1} + \frac{1}{12}\delta_{k,i+1}, \quad d^{(i)}_{kl} = \delta_{k,i}\alpha_{ij}^* + \delta_{k,i-1}\beta_{ij}^*,
\tag{A.2}
\]
where $\delta_{k,i}$ is 1 if $k = i$ and $\delta_{k,i} = 0$ if $k \neq i$.

**Lemma A.1.** Let the condition (2.11) be satisfied. Then the inverse $(A - D_i)^{-1} > 0$ and
\[
\|A - D_i\|^{-1}_\infty \leq \frac{1}{8(1 - \sigma_l)h^2}.
\tag{A.3}
\]

**Proof.** It can be checked by Corollary 3.20 on Page 91 of [54] that the inverse $(A - D_i)^{-1} > 0$. Let $E = (1, 1, \ldots, 1)^T \in \mathbb{R}^{L - 1}$ and $S_i = (A - D_i)^{-1}E$. Then $\|A - D_i\|^{-1}_\infty = \|S_i\|_\infty$. It is known that the inverse $A^{-1} = (J_{k,i})$ exists and its elements $J_{k,i}$ are given by
\[
J_{k,i} = \begin{cases} 
\frac{(L - l)k}{L}, & k \leq l, \\
\frac{(L - k)l}{L}, & k > l.
\end{cases}
\]
A simple calculation shows that $\|A^{-1}\|_\infty \leq L^2/8 = 1/(8h^2)$ and $J_{k,l} + J_{k,l-1} = 1$ for each $1 \leq k \leq L - 1$. This implies
\[
S_i = A^{-1}E + A^{-1}D_i S_i \leq \|A^{-1}\|_\infty E + \sigma_l \|S_i\|_\infty E \leq \left( \frac{1}{8h^2} \right) E + \sigma_l \|S_i\|_\infty E.
\]
Thus the estimate (A.3) follows immediately. \hfill \Box

**Proof of Lemma 2.2.** Define the following $(L - 1)$th-order matrices or vectors:
\[
U_h = (u_h(x_1), u_h(x_2), \ldots, u_h(x_L))^T, \\
M = \text{diag}(M_1, M_2, \ldots, M_L), \\
M_b = \text{diag}(M_0, 0, \ldots, 0, M_L), \\
G_b = \left(1 - \frac{h^2}{12}M_0\right)u_h(0), 0, \ldots, 0, \left(1 - \frac{h^2}{12}M_L\right)u_h(1)^T.
\tag{A.4}
\]
Using the matrices $A$ and $B$ defined by (A.2), we have from (2.13) that
\[
(A + h^2BM) U_h \geq G_b.
\tag{A.5}
\]
Since $M > -8(1 - \sigma_l)$ and $h < h(M, \overline{M})$, it is easy to check that $1 - \frac{h^2}{12}M_k \geq 0$ $(k = 0, L)$. Thus by the boundary condition in (2.13),
\[
G_b \geq D_i U_h - \frac{h^2}{12}M_b D_i U_h,
\tag{A.6}
\]
where $D_i$ is the $(L - 1)$th-order matrix defined by (A.2). This leads to
\[
\left(A - D_i + h^2BM + \frac{h^2}{12}M_b D_i\right) U_h \geq 0.
\tag{A.7}
\]
Let $Q_i \equiv A - D_i + h^2BM + \frac{h^2}{12}M_b D_i$. To prove $u_h(x_k) \geq 0$ for all $0 \leq k \leq L$, it suffices to show that the inverse of $Q_i$ exists and is nonnegative.

**Case 1:** $M \geq 0$. In this case, the matrix $Q_i$ satisfies the condition of Corollary 3.20 on Page 91 of [54] and therefore, its inverse $Q_i^{-1}$ exists and is positive.

**Case 2:** $0 > M \geq -8(1 - \sigma_l)$. For this case, we define
\[
M^+ = \text{diag}(M_1^+, M_2^+, \ldots, M_L^+), \\
M_k^+ = \max(M_k, 0), \\
M^- = M - M^+.
\tag{A.8}
\]
and $M_0^+$. $M_0^-$ can be similarly defined. Let $\overline{Q}_i = A - D_i + h^2 BM^+ + \frac{h^2}{12} M_0^+ D_i$. We know from Case 1 that $\overline{Q}_i^{-1}$ exists and is positive. Since

$$Q_i = \overline{Q}_i + h^2 BM^- + \frac{h^2}{12} M_0^- D_i = \overline{Q}_i \left( I + h^2 \overline{Q}_i^{-1} \left( BM^- + \frac{1}{12} M_0^- D_i \right) \right),$$

(A.9)

we need only to prove that the inverse $\left( I + h^2 \overline{Q}_i^{-1} \left( BM^- + \frac{1}{12} M_0^- D_i \right) \right)^{-1}$ exists and is nonnegative. By Theorem 3 on Page 298 of [55], this is true if

$$\left\| h^2 \overline{Q}_i^{-1} \left( BM^- + \frac{1}{12} M_0^- D_i \right) \right\|_\infty < 1.$$

(A.10)

Since $\overline{Q}_i \geq A - D_i$ which implies $0 < \overline{Q}_i^{-1} \leq (A - D_i)^{-1}$, we have from Lemma A.1 that

$$\left\| \overline{Q}_i^{-1} \right\|_\infty \leq \left\| (A - D_i)^{-1} \right\|_\infty \leq \frac{1}{8(1 - \sigma_i)h^2}.$$  

(A.11)

It is clear that $\|B + \frac{1}{12} D_i\|_\infty = 1$, $\|M^-\|_\infty \leq -M$ and $\|M_0^-\|_\infty \leq -M$. Thus we have

$$\left\| h^2 \overline{Q}_i^{-1} \left( BM^- + \frac{1}{12} M_0^- D_i \right) \right\|_\infty \leq \frac{M}{8(1 - \sigma_i)}.$$  

(A.12)

The estimate (A.10) follows from $M > -8(1 - \sigma_i)$.

**Proof of Lemma 2.3.** Using the same notation as before, the system (2.14) can be written as

$$Q_i U_i = G,$$

(A.13)

where $G = (g(x_1), g(x_2), \ldots, g(x_{n-1}))^T$.

**Case 1** $M \geq 0$. Since the matrix $Q_i^{-1}$ exists and is positive, we have $0 < Q_i^{-1} \leq (A - D_i)^{-1}$. This shows

$$\left\| Q_i^{-1} \right\|_\infty \leq \left\| (A - D_i)^{-1} \right\|_\infty \leq \frac{1}{8(1 - \sigma_i)h^2}.$$  

Thus by (A.13), $\|U_i\|_\infty \leq \|G\|_\infty / (8(1 - \sigma_i)h^2)$ which implies the desired estimate (2.15).

**Case 2** $0 > M > -8(1 - \sigma_i)$. It follows from (A.9) that

$$\left\| Q_i^{-1} \right\|_\infty \leq \left\| \overline{Q}_i^{-1} \right\|_\infty \left\| \left( I + h^2 \overline{Q}_i^{-1} \left( BM^- + \frac{1}{12} M_0^- D_i \right) \right)^{-1} \right\|_\infty.$$  

By (A.11) and (A.12),

$$\left\| Q_i^{-1} \right\|_\infty \leq \frac{1}{8(1 - \sigma_i)h^2} \left( \frac{8(1 - \sigma_i)}{8(1 - \sigma_i) + M} + \frac{1}{(8(1 - \sigma_i) + M)h^2} \right).$$

This together with (A.13) leads to the estimate (2.15). \qed

**References**
