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The number of kings in a multipartite tournament

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Abstract

We show that in any n-partite tournament, where $n \geq 3$, with no transmitters and no 3-kings, the number of 4-kings is at least eight. All n-partite tournaments, where $n \geq 3$, having eight 4-kings and no 3-kings are completely characterized. This solves the problem proposed in Koh and Tan (accepted).

1. Introduction

An *orientation* of a graph G is a digraph obtained from G by assigning a direction to each edge in G. Let $K(p_1, p_2, \ldots, p_n)$ denote the complete n-partite graph, where $n \geq 2$ and p_i is the number of vertices in the ith partite set for each $i = 1, 2, \ldots, n$. Any orientation of $K(p_1, p_2, \ldots, p_n)$ is called an *n-partite tournament*. An n-partite tournament is called a *tournament* of order n if $p_1 = p_2 = \cdots = p_n = 1$. A 2-partite tournament is better known as a *bipartite tournament*.

Let D be a digraph with vertex set $V(D)$. Given $u, v \in V(D)$, the *length* of a u-v dipath is the number of arcs contained in the path. The *distance* $d(u,v)$ from u to v is defined as the minimum of the lengths of all u-v dipaths. By convention, $d(u,v) = \infty$ if there exists no u-v dipath. Following [10], a vertex w in D is called an *r-king*, where r is a positive integer, if $d(w,x) \leqslant r$ for each $x \in V(D)$. The set and the number of r-kings in D are, respectively, denoted by $K_r(D)$ and $k_r(D)$. The concept of an r-king is closely related to that of the *eccentricity* $e(v)$ of a vertex v defined by $e(v) = \max\{d(v,x) \mid x \in V(D)\}$, which is a fundamental notion in the applications of graphs and digraphs (see, for instance, [1, 4]).

Given a vertex v in a digraph D, we shall denote, respectively, by $s(v)$ and $s^-(v)$ the outdegree and indegree of v. A vertex v is called a *transmitter* if $s^-(v) = 0$. Let T

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be a tournament. The integer $s(v)$ is also called the *score* of v in T. Note that a vertex w is in $K_1(T)$ if and only if $s^-(w) = 0$. Thus $k_1(T) \leqslant 1$. In 1953, the mathematical sociologist Landau pointed out in [9] that every vertex of maximum score in T is a 2-king, and so $k_2(T) \geqslant 1$. Answering a question asked by Silverman [15], Moon [11] confirmed that $k_2(T) \neq 2$. Thus if $k_2(T) > 1$, then $k_2(T) \geqslant 3$. It is easy to see that $k_2(T) = 1$ if and only if T contains a (unique) transmitter. On the other hand, Maurer [10] showed that given any two integers n, k with $n \geqslant k \geqslant 3$ and $(n,k) \neq (4,4)$, there exists a tournament T of order n such that $k_2(T) = k$; and Reid [14] proved that, given a tournament T of order $n \geqslant 3$, there exists a tournament T' such that the subdigraph induced by $K_2(T')$ is isomorphic to T if and only if T contains no transmitters.

Given a digraph D, a trivial necessary condition for the existence of r-kings in D for some r is that

$$D \text{ contains at most one transmitter.} \qquad (*)$$

Let T be an n-partite tournament satisfying $(*)$. The first set of results pertaining to the existence of r-kings in T was obtained by Gutin who showed in [3] the following: (1) $k_4(T) \geqslant 1$; (2) $k_3(T) \geqslant 1$ if each partite set of T contains at most 3 vertices; and (3) there exist infinitely many multipartite tournaments T such that $k_3(T) = 0$. Gutin's results (1) and (3) were rediscovered by Petrovic and Thomassen [13]. It is obvious that for $n \geqslant 2$, $k_4(T) = k_2(T) = 1$ if and only if T contains a unique transmitter. To extend the above results, Koh and Tan investigated in [6] certain related problems and (i) obtained some new sufficient conditions for T to have $k_3(T) \geqslant 1$, (ii) showed that if T contains no transmitters, then

$$k_4(T) \geqslant \begin{cases} 4 & \text{if } n = 2, \\ 3 & \text{if } n \geqslant 3 \end{cases}$$

(the case when $n = 2$ was proved independently by Petrovic [12]) and (iii) completely characterized all T with no transmitters such that the equalities in (ii) hold. All T with no transmitters and $n \geqslant 3$ such that $k_4(T) = 4$ were characterized in [7].

In searching for the 4-kings of an n-partite tournament T in [6, 7], it was observed that some of the existing 4-kings of T are actually 3-kings. The following problem thus arises naturally:

> If an n-partite tournament T contains no transmitters and $k_3(T) = 0$, what is the least possible value of $k_4(T)$?

In [8], we made the first move to tackle the problem for the case when $n = 2$ by establishing that $k_4(T) \geqslant 8$ and characterizing all bipartite tournaments T with $k_3(T) = 0$ and $k_4(T) = 8$. How about the more general case when $n \geqslant 3$? We shall give in this paper a complete solution to this question.

2. Notation and basic lemmas

Given an integer $n \geqslant 2$, we denote the n partite sets of an n-partite tournament T by $V_1, V_2, \ldots, V_n$. For each $i = 1, 2, \ldots, n$, let

$$M_i = \{w \in V_i \mid s(w) \geqslant s(x) \text{ for each } x \text{ in } V_i\}.$$

Given two distinct vertices u, v in T, we write '$u \to v$' if u is adjacent to v. For any two subsets A, B of $V(T)$, we write '$A \to B$' to signify that $a \to b$ for each $a \in A$ and $b \in B$. If $A = \{a\}$, then '$A \to B$' is replaced by '$a \to B$'. Likewise, if $B = \{b\}$, then '$A \to B$' is replaced by '$A \to b$'. For $v \in V(T)$, let

$$O(v) = \{x \in V(T) \mid v \to x\} \quad \text{and} \quad I(v) = \{x \in V(T) \mid x \to v\}.$$

Thus, $s(v) = |O(v)|$ and $s^-(v) = |I(v)|$, and for $u, v \in V_i$, $i = 1, 2, \ldots, n$, $O(u) \subseteq O(v)$ if and only if $I(u) \supseteq I(v)$. For $A \subseteq V(T)$, the subdigraph of T induced by A is denoted by $\langle A \rangle$.

We shall now give a series of basic lemmas which will be used to derive our main results in the next section.

We first start with tournaments. In Lemmas 1–3 below, H is a tournament of order $n \geqslant 3$ with no transmitters.

Lemma 1 (Reid [14]). *The subdigraph $\langle K_2(H) \rangle$ of H itself contains no transmitters.*

Lemma 2 (Huang and Li [5]). *For each $u \in V(H) \backslash K_2(H)$, $|I(u) \cap K_2(H)| \geqslant 2$.*

The following lemma can be proved easily.

Lemma 3. *Each vertex u in $K_2(H)$ lies on some 3-cycle of H.*

In the remaining lemmas of this section, we assume that T is an n-partite tournament, where $n \geqslant 2$. Let $x_i \in M_i$, $i = 1, 2, \ldots, n$ and $H = \langle \{x_1, x_2, \ldots, x_n\} \rangle$. Note that H is itself a tournament of order n. We shall call such a tournament H a *maximum-score-tournament* (MS-tournament) of T.

Lemma 4 (Petrovic and Thomassen [13]). *Assume that T contains at most one transmitter. Let H be an MS-tournament of T. Then $K_2(H) \subseteq K_4(T)$, and so $k_4(T) \geqslant k_2(H) \geqslant 1$.*

Lemma 5 (Koh and Tan [6]). *Assume $u, v \in V_i$, $i = 1, 2, \ldots, n$. If $s(u) \geqslant s(v)$ and u lies on a 3-cycle of T, then $d(u, v) \leqslant 3$.*

Lemma 6 (Koh and Tan [6]). *Assume $u \in V_i$ and $v \in V_j$, $i \neq j$ and let $w \in V_j \backslash \{v\}$. If $u \to v$ and $s(v) \geqslant s(w)$, then $d(u, w) \leqslant 3$.*

Lemma 7 (Koh and Tan [6]). *Assume $u \in V_i$ and $v \in M_j$. If $d(u,v) \leqslant 2$, then $d(u,x) \leqslant 4$ for each $x \in V_j$.*

Lemma 8. *Assume T has no transmitters. Let $u \in V(T)$. Suppose $d(u,x) \leqslant r$ for all $x \in V(T) \backslash V_i$. Then $u \in K_{r+1}(T)$.*

Proof. Let $y \in V_i$. Since T has no transmitters, there exists $x \in V(T) \backslash V_i$ such that $x \to y$. Thus $d(u,y) \leqslant d(u,x) + d(x,y) \leqslant r+1$. Hence $u \in K_{r+1}(T)$. □

Lemma 9 (Koh and Tan [6]). *Assume $u \in M_i$ for some i. If*

(i) *u lies on a 3-cycle of T and*

(ii) *for each j, $j \neq i$, there exists $v_j \in M_j$ such that $u \to v_j$, then $u \in K_3(T)$.*

Lemma 10. *Let $u, v \in V(T)$ such that $O(u) \subseteq O(v)$. If $u \in K_r(T)$ for some $r \geqslant 3$, then $v \in K_r(T)$.*

Proof. Let $z \in V(T) \backslash \{u\}$. Since $u \in K_r(T)$, $d(u,z) \leqslant r$. As $O(u) \nsubseteq O(v)$, we have $d(v,z) \leqslant r$. It remains to show that $d(v,u) \leqslant r$. If $u \in V_i$ and $v \in V_j$ with $j \neq i$, then $v \to u$; otherwise, $O(u) \nsubseteq O(v)$. Thus $d(v,u) = 1$. Assume now $u, v \in V_i$ for some $i = 1, 2, \ldots, n$. As $u \in K_r(T)$, $d(u,v) \leqslant r$, let $u \to x_1 \to x_2 \to \cdots \to x_{k-1} \to v$, $k \leqslant r$ be a u-v path of length k. Since $O(u) \subseteq O(v)$, $v \to x_1$. Since $I(v) \subseteq I(u)$, $x_{k-1} \to u$. Hence, $v \to x_1 \to x_2 \to \cdots \to x_{k-1} \to u$ is a path of length k from v to u and so $d(v,u) \leqslant r$. □

Lemma 11. *Assume that $n \geqslant 3$, T contains no transmitters and $k_3(T) = 0$. Let $H = \langle \{x_1, x_2, \ldots, x_n\} \rangle$ be an MS-tournament of T. Suppose H contains no transmitters. If $x_i \in K_2(H)$, then there exists $u \in V_j \backslash \{x_j\}$ for some $j = 1, 2, \ldots, n$, $j \neq i$, such that*

(i) *$d(x_i, u) = 4$,*

(ii) *$x_j \to x_i$,*

(iii) *$u \to x_k$ for all $k \neq j$, and*

(iv) *$u \in K_4(T)$.*

Futhermore, for such a u, there exists $v \in K_4(T) \cap (V_j \backslash \{x_j, u\})$ such that $d(u,v) = 4$ and $O(u) \subseteq O(v)$.

Proof. Let $x_i \in K_2(H)$. By Lemma 4, $x_i \in K_4(T)$. Since $k_3(T) = 0$, there exists $u \in V_j$, $j \in \{1, 2, \ldots, n\}$, such that $d(x_i, u) = 4$. By Lemma 3, x_i lies on some 3-cycle of H. Hence x_i lies on some 3-cycle of T. By Lemma 5, $d(x_i, z) \leqslant 3$ for each $z \in V_i$. Thus $j \neq i$. Since $x_i \in K_2(H)$, $d(x_i, x_j) \leqslant 2$. Thus $u \neq x_j$. Observe that $x_j \to x_i$; otherwise, by Lemma 6, $d(x_i, u) \leqslant 3$. Note also that $u \to x_s$ for all $s \neq j$; otherwise, $d(x_i, u) \leqslant d(x_i, x_s) + d(x_s, u) \leqslant 2 + 1 = 3$. By Lemma 6, we have

(a) $d(u,z) \leqslant 3$ for all $z \in V_s$ and for each $s \neq j$.

Since T has no transmitters, for each $y \in V_j$, there exists $z \in V(T) \backslash V_j$ such that $z \to y$. Thus $d(u,y) \leqslant d(u,z) + d(z,y) \leqslant 4$. Hence $u \in K_4(T)$. Since $k_3(T) = 0$, by (a), there exists $v \in V_j \backslash \{u\}$ such that $d(u,v) = 4$. Since $d(u,x_j) \leqslant d(u,x_i) + d(x_i,x_j) = 1 + 2 = 3$, $v \neq x_j$. As $u, v \in V_j$ and $d(u,v) = 4$, $O(u) \subseteq O(v)$. By Lemma 10, $v \in K_4(T)$. □

3. The main results

In this section, we shall solve the problem stated in Section 1. We begin with the following result.

Theorem 1. *Let T be an n-partite tournament, where $n \geqslant 3$, with no transmitters and $k_3(T) = 0$. If T contains an MS-tournament $H = \langle \{x_1, x_2, \ldots, x_n\} \rangle$ such that H itself has no transmitters, then $k_4(T) \geqslant 9$.*

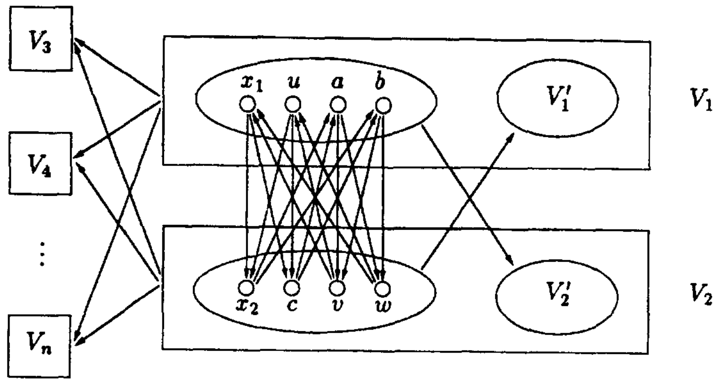
Proof. By assumption, $k_2(H) \geqslant 3$. We consider two cases:

Case 1: $k_2(H) = 3$. We may assume $K_2(H) = \{x_1, x_2, x_3\}$. By Lemma 1, we may also assume $x_1x_2x_3x_1$ is a 3-cycle. By Lemma 4, $K_2(H) \subseteq K_4(T)$. Since $k_3(T) = 0$, by Lemma 11, for each $i = 1,2,3$, there exists $\{u_{p_i}, v_{p_i}\} \subseteq K_4(T) \cap (V_{p_i} \backslash \{x_{p_i}\})$, where $p_i \neq i$, such that $d(x_i, u_{p_i}) = d(u_{p_i}, v_{p_i}) = 4$ and $x_{p_i} \rightarrow x_i$. Now as $x_1x_2x_3x_1$ is a 3-cycle in T, it follows that $p_1 \in \{3,4,\ldots,n\}$, $p_2 \in \{1,4,5,\ldots,n\}$ and $p_3 \in \{2,4,5,\ldots,n\}$. By Lemma 2, for $i = 1,2,3$, if $p_i \geqslant 4$, then $(\{x_1,x_2,x_3\} \backslash \{x_i\}) \rightarrow x_{p_i}$. Thus, p_1, p_2, p_3 are pairwise distinct. Since $\{x_i, u_{p_i}, v_{p_i}\} \subseteq K_4(T)$ for $i = 1,2,3$, we have $k_4(T) \geqslant 9$.

Case 2: $k_2(H) \geqslant 4$. By Lemma 4, $K_2(H) \subseteq K_4(T)$. We may assume $x_1 \in K_2(H)$. Since $k_3(T) = 0$, by Lemma 11, there exists $\{u_p, v_p\} \subseteq K_4(T) \cap (V_p \backslash \{x_p\})$, $p \neq 1$, such that $d(x_1, u_p) = d(u_p, v_p) = 4$ and $O(u_p) \subseteq O(v_p)$. We may assume $p = n$. By Lemma 11, we also have $x_n \rightarrow x_1$. If $x_n \notin K_2(H)$, then by Lemma 2, $|I(x_n) \cap K_2(H)| \geqslant 2$. If $x_n \in K_2(H)$, then by Lemma 1, $\langle K_2(H) \rangle$ has no transmitters and so $I(x_n) \neq \emptyset$ in $\langle K_2(H) \rangle$. In either case, $I(x_n) \cap K_2(H) \neq \emptyset$. We may assume $x_2 \in I(x_n) \cap K_2(H)$. By Lemma 11, there exists $\{u_q, v_q\} \subseteq K_4(T) \cap V_q \backslash \{x_q\}$, $q \neq 2$, such that $d(x_2, u_q) = d(u_q, v_q) = 4$ and $O(u_q) \subseteq O(v_q)$. By Lemma 11, we also have $x_q \rightarrow x_2$. Thus $q \neq n$, and we have $\{u_q, v_q, u_n, v_n\} \subseteq K_4(T) \backslash K_2(H)$. Observe that $u_q \rightarrow x_n$; otherwise, $d(x_2, u_q) = 2$. Also, as $O(u_q) \subseteq O(v_q)$, we have $v_q \rightarrow x_n$. Note that $u_n \rightarrow x_2$; otherwise, $d(x_1, u_n) \leqslant d(x_1, x_2) + d(x_2, u_n) \leqslant 2 + 1 = 3$. Now as $O(u_n) \subseteq O(v_n)$, we have $v_n \rightarrow x_2$. Since $v_n \rightarrow x_2 \rightarrow x_n$ and $x_n \in M_n$, there exists $w \in V(T) \backslash V_n$ such that $x_n \rightarrow w \rightarrow v_n$. Since $O(u_n) \subseteq O(v_n)$, we have $I(v_n) \subseteq I(u_n)$. Thus $w \rightarrow u_n$. Note that $w \notin \{u_q, v_q\}$ since $\{u_q, v_q\} \rightarrow x_n$. By Lemma 11, $u_n \rightarrow V(H) \backslash \{x_n\}$. Thus $w \notin V(H)$. Suppose $w \in K_4(T)$. Then $\{u_n, v_n, u_q, v_q, w\} \cup K_2(H) \subseteq K_4(T)$ and since $k_2(H) \geqslant 4$, we have $k_4(T) \geqslant 9$. Assume now $w \notin K_4(T)$. Since $w \rightarrow u_n \rightarrow x_i$ for each $i = 1,2,\ldots,n-1$, by Lemma 7,

(g) $d(w,z) \leqslant 4$ for all $z \in V_i$ and for each $i = 1,2,\ldots,n-1$.

Since $w \notin K_4(T)$, by (g), there exists $v \in V_n$ such that $d(w,v) \geqslant 5$. Since $w \rightarrow \{u_n, v_n\}$, $v \notin \{u_n, v_n\}$. Also $v \neq x_n$ as $w \rightarrow u_n \rightarrow x_2 \rightarrow x_n$. Note that $O(u_n) \subseteq O(v)$; otherwise, $d(w,v) \leqslant 3$. By Lemma 10, $v \in K_4(T)$. Thus, $\{u_n, v_n, u_q, v_q, v\} \cup K_2(H) \subseteq K_4(T)$. As $k_2(H) \geqslant 4$, we have $k_4(T) \geqslant 9$. The proof is now complete. □



$V_1 = V_1' \cup \{x_1, u, a, b\}$ and $V_2 = V_2' \cup \{x_2, c, v, w\}$

Fig. 1.

Finally, we have:

Theorem 2. *Let T be an n-partite tournament, where $n \geqslant 3$, with no transmitters and $k_3(T) = 0$. Then*

(i) $k_4(T) \geqslant 8$;

(ii) $k_4(T) = 8$ *if and only if T is isomorphic to a multipartite tournament of Fig.* 1, *where $\langle V_1' \cup V_2' \rangle$ and $\langle V_i \cup V_j \rangle$ for $i, j \in \{3, 4, \ldots, n\}$, $i \neq j$, are arbitrary bipartite tournaments.*

Proof. (i) By Theorem 1, we may assume that every MS-tournament of T has a transmitter. Let $H = \langle \{x_1, x_2, \ldots, x_n\} \rangle$ be an MS-tournament of T. We may assume x_1 is a transmitter of H. Then $x_1 \in K_2(H)$. By Lemma 4, $x_1 \in K_4(T)$. Since $x_1 \to x_j$ for all $j \geqslant 2$, by Lemma 6, $d(x_1, x) \leqslant 3$ for all $x \in V(T) \backslash V_1$. As $k_3(T) = 0$, there exists $u \in V_1$ such that $d(x_1, u) = 4$. It follows that $O(x_1) \subseteq O(u)$. Since $x_1 \in M_1$, $O(u) = O(x_1)$. By Lemma 10, $u \in K_4(T)$. Since $K_3(T) = 0$, by Lemma 9, u and x_1 lie on no 3-cycles in T. Since T has no transmitters, $I(x_1) \neq \emptyset$. Let $y \in I(x_1)$. Then $d(y, x_i) \leqslant 2$ for each $i = 1, 2, \ldots, n$. By Lemma 7, $y \in K_4(T)$, and so $I(x_1) \subseteq K_4(T)$.

Claim 1. *If* $|I(x_1) \cap V_i \backslash \{x_i\}| \geqslant 1$, *then* $|I(x_1) \cap V_i \backslash \{x_i\}| \geqslant 2$.

We may assume $I(x_1) \cap V_2 \backslash \{x_2\} \neq \emptyset$. Among the vertices in $I(x_1) \cap V_2 \backslash \{x_2\}$, let v have maximum score. Since x_1 is not on any 3-cycle, $v \to x_i$ for all $i \neq 2$. By Lemma 6, $d(v, x) \leqslant 3$ for all $x \in V(T) \backslash V_2$. Now as $k_3(T) = 0$, there exists $w \in V_2 \backslash \{v\}$ such that $d(v, w) = 4$. Again, we have $O(v) \subseteq O(w)$. Thus $w \neq x_2$. Hence $w \in I(x_1)$, and so $|I(x_1) \cap V_2 \backslash \{x_2\}| \geqslant 2$. In addition, from the choice of v, we have $O(v) = O(w)$.

Claim 2. $I(v) \subseteq K_4(T)$.

Since T has no transmitters, $I(v) \neq \emptyset$. Let $y \in I(v)$. Since $v \to x_i$ for all $i \neq 2$ and $y \to v$, we have $d(y,x_i) \leqslant 2$ for all $i \neq 2$. By Lemma 7, $d(y,x) \leqslant 4$ for all $x \in V(T) \setminus V_2$. Let $z \in V_2 \setminus \{v\}$. If $d(v,z) = 2$, then $d(y,z) \leqslant d(y,v) + d(v,z) = 1+2 = 3$. If $O(v) \subseteq O(z)$, then as $s(v) \geqslant s(z)$, we have $O(v) = O(z)$. Thus $I(z) = I(v)$ and so $y \to z$. In either case, $d(y,z) \leqslant 3$. Hence $y \in K_4(T)$. This shows that $I(v) \subseteq K_4(T)$.

Claim 3. $O(x_2) \cap I(v) \subseteq V_1$.

Let $a \in O(x_2) \cap I(v)$. Then $x_2 \to a \to v$. Since $v \to x_1 \to x_2$ and x_1 lies on no 3-cycles in T, we must have $a \in V_1$. Thus $O(x_2) \cap I(v) \subseteq V_1$, as required.

Since $v \to \{x_1, u\} \to x_2$ and $x_2 \in M_2$, $|O(x_2) \cap I(v)| \geqslant 2$. Thus $s^-(v) \geqslant 2$. By Claims 2 and 3, $|K_4(T) \cap V_1| \geqslant 4$. Observe that we have actually proved the following claim:

Claim 4. *If V_i contains a transmitter of some MS-tournament, then $|V_i \cap K_4(T)| \geqslant 4$.*

Claim 5. *If $s^-(v) \geqslant 3$, then $k_4(T) \geqslant 9$.*

Assume $s^-(v) \geqslant 3$. Suppose $s(v) = s(x_2)$. Then as $v \to x_i$ for all $i \neq 2$, $\langle V(H) \setminus \{x_2\} \cup \{v\} \rangle$ is an MS-tournament with v as a transmitter. By Claim 4, $|V_2 \cap K_4(T)| \geqslant 4$. Now as $|\{x_1, u\} \cup I(v)| \geqslant 5$, we have $k_4(T) \geqslant 9$. Assume now $s(v) < s(x_2)$. Since $v \to \{x_1, u\} \to x_2$, we have $|O(x_2) \cap I(v)| \geqslant 3$. By Claim 3, $O(x_2) \cap I(v) \subseteq V_1$. Suppose $I(x_1) \cap V_i \setminus \{x_i\} \neq \emptyset$ for some $i \geqslant 3$. By Claim 1, $|I(x_1) \cap V_i \setminus \{x_i\}| \geqslant 2$. Now as $I(x_1) \subseteq K_4(T)$, we have $|V_i \cap K_4(T)| \geqslant 2$, and so $k_4(T) \geqslant 9$. Assume now $x_1 \to V_i$ for all $i \geqslant 3$. Then $v \to V_i$ for all $i \geqslant 3$; otherwise, x_1 lies on some 3-cycle in T. Thus $I(v) \subseteq V_1$. If $s^-(v) \geqslant 5$, then $k_4(T) \geqslant 9$. Assume now $3 \leqslant s^-(v) \leqslant 4$. Suppose $I(v) \cap I(x_2) \neq \emptyset$. Then as $|O(x_2) \cap I(v)| \geqslant 3$ and $s^-(v) \leqslant 4$, we have $|O(x_2) \cap I(v)| = 3$ and $|I(x_2) \cap I(v)| = 1$. Let $O(x_2) \cap I(v) = \{a,b,c\}$ and $I(x_2) \cap I(v) = \{e\}$. Note that $e \to v \to V(T) \setminus (V_2 \cup \{a,b,c,e\})$ and $e \to x_2 \to \{a,b,c\}$. By Lemma 8, $e \in K_3(T)$, a contradiction. Thus, $I(x_2) \cap I(v) = \emptyset$. Note that $x_2 \to I(v) \to v \to V(T) \setminus (V_2 \cup I(v))$. By Lemma 8, $x_2 \in K_4(T)$. Now as $k_3(T) = 0$, there exists $z \in V_2 \setminus \{x_2\}$ such that $d(x_2, z) = 4$. Again, $O(x_2) \subseteq O(z)$. Note that $z \notin \{v, w\}$. By Lemma 10, $z \in K_4(T)$. Thus, $\{x_1, u, v, w, x_2, z\} \cup I(v) \subseteq K_4(T)$, and so $k_4(T) \geqslant 9$. This proves Claim 5.

We now consider $s^-(v) = 2$. Since $|O(x_2) \cap I(v)| \geqslant 2$, $I(v) = O(x_2) \cap I(v)$. Note that $x_2 \to I(v) \to v \to V(T) \setminus (V_2 \cup I(v))$. By Lemma 8, $x_2 \in K_4(T)$. As $k_3(T) = 0$, there exists $c \in V_2 \setminus \{x_2\}$ such that $d(x_2, c) = 4$. Thus $O(x_2) \subseteq O(c)$. Since $x_2 \in M_2$, $O(c) = O(x_2)$. By Lemma 10, $c \in K_4(T)$, and so $k_4(T) \geqslant 8$. This proves part (i).

(ii) The sufficiency is obvious. We shall prove the necessity. Assume that $k_3(T) = 0$ and $k_4(T) = 8$. By Theorem 1, we may assume that every MS-tournament of T has a transmitter. Let x_1, x_2, u, v, w be the vertices as described in the proof of part (i). Then $\{x_1, u, v, w\} \subseteq K_4(T)$. Since $k_4(T) = 8$, it follows from the proof of part (i) that $s^-(v) = 2$, $x_2 \in K_4(T)$, and that there exists $c \in K_4(T) \cap V_2 \setminus \{x_2\}$ such that $d(x_2, c) = 4$

and $O(c) = O(x_2)$. Now as $x_2 \in M_2$, we have $s^-(x_2) = 2$ and $x_2 \to V(T)\backslash(V_2 \cup \{x_1, u\})$. Since $|O(x_2) \cap I(v)| \geqslant 2$, we have $I(v) = O(x_2) \cap I(v)$. Let $O(x_2) \cap I(v) = \{a, b\}$. Then $\{v, w\} \to V(T)\backslash(V_2 \cup \{a, b\})$. By Claims 2 and 3 in (i), $\{a, b\} \subseteq K_4(T) \cap V_1$. Thus, $K_4(T) = \{x_1, u, v, w, x_2, a, b, c\}$. Since $I(x_1) \subseteq K_4(T)$, we have $s^-(x_1) = s^-(u) = 2$ and $\{x_1, u\} \to V(T)\backslash(V_1 \cup \{v, w\})$. Note that $\{a, b\} \to \{v, w\} \to V(T)\backslash(V_2 \cup \{a, b\})$ and $\{a, b\} \to v \to x_1 \to V_2\backslash\{v, w\}$. Thus, $d(a, x) \leqslant 3$ for all $x \in V(T)\backslash\{b\}$ and $d(b, x) \leqslant 3$ for all $x \in V(T)\backslash\{a\}$. Now as $k_3(T) = 0$, we must have $d(a, b) = d(b, a) = 4$. Thus $O(a) = O(b)$. Suppose $s^-(a) \geqslant 3$. Let $z \in I(a)\backslash\{x_2, c\}$. Then $z \to \{a, b\} \to \{v, w\} \to V(T)\backslash(V_2 \cup \{a, b\})$. By Lemma 8, $z \in K_4(T)$, a contradiction. Thus, $s^-(a) = s^-(b) = 2$ and $\{a, b\} \to V(T)\backslash(V_1 \cap \{x_2, c\})$. Combining the above results, we conclude that T is isomorphic to an n-partite tournament of Fig. 1. □

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