An orientation of a graph $G$ is a digraph obtained from $G$ by assigning a direction to each edge in $G$. Let $K(p_1, p_2, \ldots, p_n)$ denote the complete $n$-partite graph, where $n \geq 2$ and $p_i$ is the number of vertices in the $i$th partite set for each $i = 1, 2, \ldots, n$. Any orientation of $K(p_1, p_2, \ldots, p_n)$ is called an $n$-partite tournament. An $n$-partite tournament is called a tournament of order $n$ if $p_1 = p_2 = \cdots = p_n = 1$. A 2-partite tournament is better known as a bipartite tournament.

Let $D$ be a digraph with vertex set $V(D)$. Given $u, v \in V(D)$, the length of a $u-v$ dipath is the number of arcs contained in the path. The distance $d(u,v)$ from $u$ to $v$ is defined as the minimum of the lengths of all $u-v$ dipaths. By convention, $d(u,v) = \infty$ if there exists no $u-v$ dipath. Following [10], a vertex $w$ in $D$ is called an $r$-king, where $r$ is a positive integer, if $d(w,x) \leq r$ for each $x \in V(D)$. The set and the number of $r$-kings in $D$ are, respectively, denoted by $K_r(D)$ and $k_r(D)$. The concept of an $r$-king is closely related to that of the eccentricity $e(v)$ of a vertex $v$ defined by $e(v) = \max\{d(v,x) | x \in V(D)\}$, which is a fundamental notion in the applications of graphs and digraphs (see, for instance, [1, 4]).

Given a vertex $v$ in a digraph $D$, we shall denote, respectively, by $s(v)$ and $s^{-}(v)$ the outdegree and indegree of $v$. A vertex $v$ is called a transmitter if $s^{-}(v) = 0$. Let $T$
be a tournament. The integer $s(v)$ is also called the score of $v$ in $T$. Note that a vertex $w$ is in $K_1(T)$ if and only if $s^-(w) = 0$. Thus $k_1(T) \leq 1$. In 1953, the mathematical sociologist Landau pointed out in [9] that every vertex of maximum score in $T$ is a 2-king, and so $k_2(T) \geq 1$. Answering a question asked by Silverman [15], Moon [11] confirmed that $k_2(T) \neq 2$. Thus if $k_2(T) \geq 1$, then $k_2(T) \geq 3$. It is easy to see that $k_2(T) = 1$ if and only if $T$ contains a (unique) transmitter. On the other hand, Mau- rer [10] showed that given any two integers $n,k$ with $n \geq k \geq 3$ and $(n,k) \neq (4,4)$, there exists a tournament $T$ of order $n$ such that $k_2(T) = k$; and Reid [14] proved that, given a tournament $T$ of order $n \geq 3$, there exists a tournament $T'$ such that the subdigraph induced by $K_2(T')$ is isomorphic to $T$ if and only if $T$ contains no transmitters.

Given a digraph $D$, a trivial necessary condition for the existence of $r$-kings in $D$ for some $r$ is that

$$D \text{ contains at most one transmitter.}$$

Let $T$ be an $n$-partite tournament satisfying (*). The first set of results pertaining to the existence of $r$-kings in $T$ was obtained by Gutin who showed in [3] the following:

(1) $k_4(T) \geq 1$; (2) $k_3(T) \geq 1$ if each partite set of $T$ contains at most 3 vertices; and (3) there exist infinitely many multipartite tournaments $T$ such that $k_3(T) = 0$. Gutin’s results (1) and (3) were rediscovered by Petrovic and Thomassen [13]. It is obvious that for $n \geq 2$, $k_4(T) = k_2(T) = 1$ if and only if $T$ contains a unique transmitter. To extend the above results, Koh and Tan investigated in [6] certain related problems and (i) obtained some new sufficient conditions for $T$ to have $k_3(T) \geq 1$, (ii) showed that if $T$ contains no transmitters, then

$$k_4(T) \geq \begin{cases} 4 & \text{if } n = 2, \\ 3 & \text{if } n \geq 3 \end{cases}$$

(the case when $n = 2$ was proved independently by Petrovic [12]) and (iii) completely characterized all $T$ with no transmitters such that the equalities in (ii) hold. All $T$ with no transmitters and $n \geq 3$ such that $k_4(T) = 4$ were characterized in [7].

In searching for the 4-kings of an $n$-partite tournament $T$ in [6, 7], it was observed that some of the existing 4-kings of $T$ are actually 3-kings. The following problem thus arises naturally:

If an $n$-partite tournament $T$ contains no transmitters and $k_3(T) = 0$, what is the least possible value of $k_4(T)$?

In [8], we made the first move to tackle the problem for the case when $n = 2$ by establishing that $k_4(T) \geq 8$ and characterizing all bipartite tournaments $T$ with $k_3(T) = 0$ and $k_4(T) = 8$. How about the more general case when $n \geq 3$? We shall give in this paper a complete solution to this question.
2. Notation and basic lemmas

Given an integer \( n \geq 2 \), we denote the \( n \) partite sets of an \( n \)-partite tournament \( T \) by \( V_1, V_2, \ldots, V_n \). For each \( i = 1, 2, \ldots, n \), let

\[
M_i = \{ w \in V_i \mid s(w) \geq s(x) \text{ for each } x \in V_i \}.
\]

Given two distinct vertices \( u, v \) in \( T \), we write \( 'u \rightarrow v' \) if \( u \) is adjacent to \( v \). For any two subsets \( A, B \) of \( V(T) \), we write \( 'A \rightarrow B' \) to signify that \( a \rightarrow b \) for each \( a \in A \) and \( b \in B \). If \( A = \{ a \} \), then \( 'A \rightarrow B' \) is replaced by \( 'a \rightarrow B' \). Likewise, if \( B = \{ b \} \), then \( 'A \rightarrow B' \) is replaced by \( 'A \rightarrow b' \). For \( v \in V(T) \), let

\[
O(v) = \{ x \in V(T) \mid v \rightarrow x \} \quad \text{and} \quad I(v) = \{ x \in V(T) \mid x \rightarrow v \}.
\]

Thus, \( s(v) = |O(v)| \) and \( s^{-}(v) = |I(v)| \), and for \( u, v \in V_i, i = 1, 2, \ldots, n \), \( O(u) \subseteq O(v) \) if and only if \( I(u) \supseteq I(v) \). For \( A \subseteq V(T) \), the subdigraph of \( T \) induced by \( A \) is denoted by \( \langle A \rangle \).

We shall now give a series of basic lemmas which will be used to derive our main results in the next section.

We first start with tournaments. In Lemmas 1–3 below, \( H \) is a tournament of order \( n \geq 3 \) with no transmitters.

**Lemma 1 (Reid [14]).** The subdigraph \( \langle K_2(H) \rangle \) of \( H \) itself contains no transmitters.

**Lemma 2 (Huang and Li [5]).** For each \( u \in V(H) \backslash K_2(H) \), \( |I(u) \cap K_2(H)| \geq 2 \).

The following lemma can be proved easily.

**Lemma 3.** Each vertex \( u \) in \( K_2(H) \) lies on some 3-cycle of \( H \).

In the remaining lemmas of this section, we assume that \( T \) is an \( n \)-partite tournament, where \( n \geq 2 \). Let \( x_i \in M_i, i = 1, 2, \ldots, n \) and \( H = \langle \{x_1, x_2, \ldots, x_n\} \rangle \). Note that \( H \) is itself a tournament of order \( n \). We shall call such a tournament \( H \) a maximum-score-tournament (MS-tournament) of \( T \).

**Lemma 4 (Petrovic and Thomassen [13]).** Assume that \( T \) contains at most one transmitter. Let \( H \) be an MS-tournament of \( T \). Then \( K_2(H) \subseteq K_4(T) \), and so \( k_4(T) \geq k_2(H) \geq 1 \).

**Lemma 5 (Koh and Tan [6]).** Assume \( u, v \in V_i, i = 1, 2, \ldots, n \). If \( s(u) \geq s(v) \) and \( u \) lies on a 3-cycle of \( T \), then \( d(u, v) \leq 3 \).

**Lemma 6 (Koh and Tan [6]).** Assume \( u \in V_i \) and \( v \in V_j, i \neq j \) and let \( w \in V_j \backslash \{v\} \). If \( u \rightarrow v \) and \( s(v) \geq s(w) \), then \( d(u, w) \leq 3 \).
Lemma 7 (Koh and Tan [6]). Assume \( u \in V_i \) and \( v \in M_j \). If \( d(u, v) \leq 2 \), then \( d(u, x) \leq 4 \) for each \( x \in V_j \).

Lemma 8. Assume \( T \) has no transmitters. Let \( u \in V(T) \). Suppose \( d(u, x) \leq r \) for all \( x \in V(T) \setminus V_i \). Then \( u \in K_{r+1}(T) \).

Proof. Let \( y \in V_i \). Since \( T \) has no transmitters, there exists \( x \in V(T) \setminus V_i \) such that \( x \rightarrow y \). Thus \( d(u, y) \leq d(u, x) + d(x, y) \leq r + 1 \). Hence \( u \in K_{r+1}(T) \). □

Lemma 9 (Koh and Tan [6]). Assume \( u \in M_i \) for some \( i \). If

(i) \( u \) lies on a 3-cycle of \( T \) and

(ii) for each \( j, j \neq i \), there exists \( v_j \in M_j \) such that \( u \rightarrow v_j \), then \( u \in K_3(T) \).

Lemma 10. Let \( u, v \in V(T) \) such that \( O(u) \subseteq O(v) \). If \( u \in K_r(T) \) for some \( r \geq 3 \), then \( v \in K_r(T) \).

Proof. Let \( z \in V(T) \setminus \{ u \} \). Since \( u \in K_r(T) \), \( d(u, z) \leq r \). As \( O(u) \subseteq O(v) \), we have \( d(v, z) \leq r \). It remains to show that \( d(v, u) \leq r \). If \( u \in V_i \) and \( v \in V_j \) with \( j \neq i \), let \( u \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_{k-1} \rightarrow v \), \( k \leq r \) be a \( u \rightarrow v \) path of length \( k \). Since \( O(u) \subseteq O(v) \), \( v \rightarrow x_1 \). Since \( I(v) \subseteq I(u) \), \( x_{k-1} \rightarrow u \). Hence, \( v \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_{k-1} \rightarrow u \) is a path of length \( k \) from \( v \) to \( u \) and so \( d(v, u) \leq r \). □

Lemma 11. Assume that \( n \geq 3 \), \( T \) contains no transmitters and \( k_3(T) = 0 \). Let \( H = \langle \{x_1, x_2, \ldots, x_n\} \rangle \) be an MS-tournament of \( T \). Suppose \( H \) contains no transmitters. If \( x_i \in K_2(H) \), then there exists \( u \in V_j \setminus \{ x_j \} \) for some \( j = 1, 2, \ldots, n \), \( j \neq i \), such that

(i) \( d(x_i, u) = 4 \),

(ii) \( x_j \rightarrow x_i \),

(iii) \( u \rightarrow x_k \) for all \( k \neq j \), and

(iv) \( u \in K_4(T) \).

Furthermore, for such a \( u \), there exists \( v \in K_4(T) \cap (V_j \setminus \{ x_j, u \}) \) such that \( d(u, v) = 4 \) and \( O(u) \subseteq O(v) \).

Proof. Let \( x_i \in K_2(H) \). By Lemma 4, \( x_i \in K_4(T) \). Since \( k_3(T) = 0 \), there exists \( u \in V_j \), \( j \in \{ 1, 2, \ldots, n \} \), such that \( d(x_i, u) = 4 \). By Lemma 3, \( x_i \) lies on some 3-cycle of \( H \). Hence \( x_i \) lies on some 3-cycle of \( T \). By Lemma 5, \( d(x_i, z) \leq 3 \) for each \( z \in V_i \). Thus \( j \neq i \). Since \( x_i \in K_2(H) \), \( d(x_i, x_j) \leq 2 \). Thus \( u \neq x_j \). Observe that \( x_j \rightarrow x_i \); otherwise, by Lemma 6, \( d(x_i, u) \leq 3 \). Note also that \( u \rightarrow x_s \) for all \( s \neq j \); otherwise, \( d(x_i, u) \leq d(x_i, x_s) + d(x_s, u) \leq 2 + 1 = 3 \). By Lemma 6, we have

(a) \( d(u, z) \leq 3 \) for all \( z \in V_s \) and for each \( s \neq j \).

Since \( T \) has no transmitters, for each \( y \in V_j \), there exists \( z \in V(T) \setminus V_j \) such that \( z \rightarrow y \). Thus \( d(u, y) \leq d(u, z) + d(z, y) \leq 4 \). Hence \( u \in K_4(T) \). Since \( k_3(T) = 0 \), by (a), there
exists $v \in V_j \setminus \{u\}$ such that $d(u,v) = 4$. Since $d(u,x_i) \leq d(u,x_i) + d(x_i,x_j) = 1 + 2 = 3$, $v \neq x_j$. As $u,v \in V_j$ and $d(u,v) = 4$, $O(u) \subseteq O(v)$. By Lemma 10, $v \in K_4(T)$. □

3. The main results

In this section, we shall solve the problem stated in Section 1. We begin with the following result.

Theorem 1. Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters and $k_3(T) = 0$. If $T$ contains an MS-tournament $H = \{x_1,x_2,\ldots,x_n\}$ such that $H$ itself has no transmitters, then $k_4(T) \geq 9$.

Proof. By assumption, $k_2(H) \geq 3$. We consider two cases:

Case 1: $k_2(H) = 3$. We may assume $K_2(H) = \{x_1,x_2,x_3\}$. By Lemma 1, we may also assume $x_1x_2x_3x_1$ is a 3-cycle. By Lemma 4, $K_2(H) \subseteq K_4(T)$. Since $k_3(T) = 0$, by Lemma 11, for each $i = 1,2,3$, there exists $\{u_{p_i},v_{p_i}\} \subseteq K_4(T) \cap (V_p \setminus \{x_{p_i}\})$, where $p_i \neq i$, such that $d(x_i,u_{p_i}) = d(u_{p_i},v_{p_i}) = 4$ and $x_{p_i} \rightarrow x_i$. Now as $x_1x_2x_3x_1$ is a 3-cycle in $T$, it follows that $p_1 \in \{3,4,\ldots,n\}$, $p_2 \in \{1,4,5,\ldots,n\}$ and $p_3 \in \{2,4,5,\ldots,n\}$. By Lemma 2, for $i = 1,2,3$, if $p_i > 4$, then $\{\{x_1,x_2,x_3\}\setminus\{x_i\}\} \rightarrow x_{p_i}$. Thus, $p_1,p_2,p_3$ are pairwise distinct. Since $\{x_i,u_{p_i},v_{p_i}\} \subseteq K_4(T)$ for $i = 1,2,3$, we have $k_4(T) \geq 9$.

Case 2: $k_2(H) \geq 4$. By Lemma 4, $K_2(H) \subseteq K_4(T)$. We may assume $x_1 \in K_2(H)$. Since $k_3(T) = 0$, by Lemma 11, there exists $\{u_q,v_q\} \subseteq K_4(T) \cap (V_p \setminus \{x_{q}\})$, $p \neq 1$, such that $d(x_1,u_q) = d(u_q,v_q) = 4$ and $O(u_q) \subseteq O(v_q)$. We may assume $p = n$. By Lemma 11, we also have $x_n \rightarrow x_1$. If $x_n \notin K_2(H)$, then by Lemma 2, $|I(x_n) \cap K_2(H)| \geq 2$. If $x_n \in K_2(H)$, then by Lemma 1, $(K_2(H))$ has no transmitters and so $I(x_n) \neq \emptyset$ in $(K_2(H))$. In either case, $I(x_n) \cap K_2(H) \neq \emptyset$. We may assume $x_2 \in I(x_n) \cap K_2(H)$. By Lemma 11, there exists $\{u_q,v_q\} \subseteq K_4(T) \cap V_q \setminus \{x_q\}$, $q \neq 2$, such that $d(x_2,u_q) = d(u_q,v_q) = 4$ and $O(u_q) \subseteq O(v_q)$. By Lemma 11, we also have $x_q \rightarrow x_2$. Thus $q \neq n$, and we have $\{u_q,v_q,u_n,v_n\} \subseteq K_4(T) \setminus K_2(H)$. Observe that $u_q \rightarrow x_n$, otherwise, $d(x_2,u_q) = 2$. Also, as $O(u_q) \subseteq O(v_q)$, we have $v_q \rightarrow x_n$. Note that $u_n \rightarrow x_2$; otherwise, $d(x_1,u_n) \leq d(x_1,x_2) + d(x_2,u_n) \leq 2 + 1 = 3$. Now as $O(u_n) \subseteq O(v_n)$, we have $v_n \rightarrow x_2$. Since $v_q \rightarrow x_2 \rightarrow x_n$ and $x_n \in M_n$, there exists $w \in V(T) \setminus V_n$ such that $x_n \rightarrow w \rightarrow v_n$. Since $O(u_n) \subseteq O(v_n)$, we have $I(v_n) \subseteq I(u_n)$. Thus $w \rightarrow u_n$. Note that $w \notin \{u_q,v_q\}$ since $\{u_q,v_q\} \rightarrow x_n$. By Lemma 11, $u_n \rightarrow V(H) \setminus \{x_n\}$. Thus $w \notin V(H)$. Suppose $w \in K_4(T)$. Then $\{u_n,v_n,u_q,v_q,w\} \subseteq K_2(H) \subseteq K_4(T)$ and since $k_2(H) \geq 4$, we have $k_4(T) \geq 9$. Assume now $w \notin K_4(T)$. Since $w \rightarrow u_n \rightarrow x_i$ for each $i = 1,2,\ldots,n-1$, by Lemma 7,\[\text{d}(w,z) \leq 4\text{ for all } z \in V_i\text{ and for each } i = 1,2,\ldots,n-1.\]

Since $w \notin K_4(T)$, by (g), there exists $v \in V_n$ such that $d(w,v) \geq 5$. Since $w \notin \{u_n,v_n\}$, $v \notin \{u_n,v_n\}$. Also $v \neq x_n$ as $w \rightarrow u_n \rightarrow x_2 \rightarrow x_n$. Note that $O(u_n) \subseteq O(v)$; otherwise, $d(w,v) \leq 3$. By Lemma 10, $v \in K_4(T)$. Thus, $\{u_n,v_n,u_q,v_q,v\} \subseteq K_2(H) \subseteq K_4(T)$. As $k_2(H) \geq 4$, we have $k_4(T) \geq 9$. The proof is now complete. □
Finally, we have:

**Theorem 2.** Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters and $k_3(T) = 0$. Then

(i) $k_4(T) \geq 8$;

(ii) $k_4(T) = 8$ if and only if $T$ is isomorphic to a multipartite tournament of Fig. 1, where $\langle V'_1 \cup V'_2 \rangle$ and $\langle V_i \cup V_j \rangle$ for $i, j \in \{3, 4, \ldots, n\}$, $i \neq j$, are arbitrary bipartite tournaments.

**Proof.** (i) By Theorem 1, we may assume that every MS-tournament of $T$ has a transmitter. Let $H = \langle \{x_1, x_2, \ldots, x_n\} \rangle$ be an MS-tournament of $T$. We may assume $x_1$ is a transmitter of $H$. Then $x_1 \in K_2(H)$. By Lemma 4, $x_1 \in K_4(T)$. Since $x_1 \rightarrow x_j$ for all $j \geq 2$, by Lemma 6, $d(x_1, x) \leq 3$ for all $x \in V(T) \setminus V_1$. As $k_3(T) = 0$, there exists $u \in V_1$ such that $d(x_1, u) = 4$. It follows that $O(x_1) \subseteq O(u)$. Since $x_1 \in M_1$, $O(u) = O(x_1)$. By Lemma 10, $u \in K_4(T)$. Since $K_3(T) = 0$, by Lemma 9, $u$ and $x_1$ lie on no 3-cycles in $T$. Since $T$ has no transmitters, $I(x_1) \neq \emptyset$. Let $y \in I(x_1)$. Then $d(y, x_1) \leq 2$ for each $i = 1, 2, \ldots, n$. By Lemma 7, $y \in K_4(T)$, and so $I(x_1) \subseteq K_4(T)$.

**Claim 1.** If $|I(x_1) \cap V'_1 \setminus \{x_1\}| \geq 1$, then $|I(x_1) \cap V'_1 \setminus \{x_1\}| \geq 2$.

We may assume $I(x_1) \cap V'_2 \setminus \{x_2\} \neq \emptyset$. Among the vertices in $I(x_1) \cap V'_2 \setminus \{x_2\}$, let $v$ have maximum score. Since $x_1$ is not on any 3-cycle, $v \rightarrow x_i$ for all $i \neq 2$. By Lemma 6, $d(v, x) \leq 3$ for all $x \in V(T) \setminus V_2$. Now as $k_3(T) = 0$, there exists $w \in V'_2 \setminus \{v\}$ such that $d(v, w) = 4$. Again, we have $O(v) \subseteq O(w)$. Thus $w \neq x_2$. Hence $w \in I(x_1)$, and so $|I(x_1) \cap V'_2 \setminus \{x_2\}| \geq 2$. In addition, from the choice of $v$, we have $O(v) = O(w)$. 

\[ V_1 = V'_1 \cup \{x_1, u, a, b\} \quad \text{and} \quad V_2 = V'_2 \cup \{x_2, c, v, w\} \]

Fig. 1.
Claim 2. $I(v) \subseteq K_4(T)$.

Since $T$ has no transmitters, $I(v) \neq \emptyset$. Let $y \in I(v)$. Since $v \rightarrow x_i$ for all $i \neq 2$ and $y \rightarrow v$, we have $d(y, x_i) \leq 2$ for all $i \neq 2$. By Lemma 7, $d(y, x_i) \leq 4$ for all $x \in V(T) \setminus V_2$. Let $z \in V_2 \setminus \{v\}$. If $d(v, z) = 2$, then $d(y, z) \leq d(y, v) + d(v, z) = 1 + 2 = 3$. If $O(v) \subseteq O(z)$, then as $s(v) \geq s(z)$, we have $O(v) = O(z)$. Thus $I(z) = I(v)$ and so $y \rightarrow z$. In either case, $d(y, z) \leq 3$. Hence $y \in K_4(T)$. This shows that $I(v) \subseteq K_4(T)$.

Claim 3. $O(x_2) \cap I(v) \subseteq V_1$.

Let $a \in O(x_2) \cap I(v)$. Then $x_2 \rightarrow a \rightarrow v$. Since $v \rightarrow x_1 \rightarrow x_2$ and $x_1$ lies on no 3-cycles in $T$, we must have $a \in V_1$. Thus $O(x_2) \cap I(v) \subseteq V_1$, as required.

Since $v \rightarrow \{x_1, u\} \rightarrow x_2$ and $x_2 \in M_2$, $|O(x_2) \cap I(v)| \geq 2$. Thus $s^-(v) \geq 2$. By Claims 2 and 3, $|K_4(T) \cap V_1| \geq 4$. Observe that we have actually proved the following claim:

Claim 4. If $V_i$ contains a transmitter of some MS-tournament, then $|V_i \cap K_4(T)| \geq 4$.

Claim 5. If $s^-(v) \geq 3$, then $k_4(T) \geq 9$.

Assume $s^-(v) \geq 3$. Suppose $s(v) = s(x_2)$. Then as $v \rightarrow x_i$ for all $i \neq 2$, $(V(H) \setminus \{x_2\} \cup \{v\})$ is an MS-tournament with $v$ as a transmitter. By Claim 4, $|V_2 \cap K_4(T)| \geq 4$. Now as $|\{x_1, u\} \cup I(v)| \geq 5$, we have $k_4(T) \geq 9$. Assume now $s(v) < s(x_2)$. Since $v \rightarrow \{x_1, u\} \rightarrow x_2$ and $x_1$ lies on no 3-cycle in $T$, we have $|O(x_2) \cap I(v)| \geq 3$. By Claim 3, $O(x_2) \cap I(v) \subseteq V_1$. Suppose $I(x_1) \cap V_1 \setminus \{x_1\} \neq \emptyset$ for some $i \geq 3$. By Claim 1, $|I(x_1) \cap V_1 \setminus \{x_1\}| \geq 2$. Now as $I(x_1) \subseteq K_4(T)$, we have $|V_1 \cap K_4(T)| \geq 2$, and so $k_4(T) \geq 9$. Assume now $x_1 \rightarrow V_i$ for all $i \geq 3$. Then $v \rightarrow V_i$ for all $i \geq 3$; otherwise, $x_1$ lies on some 3-cycle in $T$. Thus $I(v) \subseteq V_1$. If $s^-(v) \geq 5$, then $k_4(T) \geq 9$. Assume now $3 \leq s^-(v) \leq 4$. Suppose $I(v) \cap I(x_2) \neq \emptyset$. Then as $|O(x_2) \cap I(v)| \geq 3$ and $s^-(v) \leq 4$, we have $|O(x_2) \cap I(v)| = 3$ and $|I(x_2) \cap I(v)| = 1$. Let $O(x_2) \cap I(v) = \{a, b, c\}$ and $I(x_2) \cap I(v) = \{e\}$. Note that $e \rightarrow v \rightarrow V(T) \setminus (V_2 \cup \{a, b, c, e\})$ and $e \rightarrow x_2 \rightarrow \{a, b, c\}$. By Lemma 8, $e \in K_3(T)$, a contradiction. Thus, $I(x_2) \cap I(v) = \emptyset$. Note that $x_2 \rightarrow I(v) \rightarrow v \rightarrow V(T) \setminus (V_2 \cup I(v))$.

By Lemma 8, $x_2 \in K_4(T)$. Now as $k_3(T) = 0$, there exists $z \in V_2 \setminus \{x_2\}$ such that $d(x_2, z) = 4$. Again, $O(x_2) \subseteq O(z)$. Note that $z \notin \{v, w\}$. By Lemma 10, $z \in K_4(T)$. Thus, $\{x_1, u, v, w, x_2, z\} \cup I(v) \subseteq K_4(T)$, and so $k_4(T) \geq 9$. This proves Claim 5.

We now consider $s^-(v) = 2$. Since $|O(x_2) \cap I(v)| \geq 2$, $I(v) = O(x_2) \cap I(v)$. Note that $x_2 \rightarrow I(v) \rightarrow v \rightarrow V(T) \setminus (V_2 \cup I(v))$.

By Lemma 8, $x_2 \in K_4(T)$. As $k_3(T) = 0$, there exists $c \in V_2 \setminus \{x_2\}$ such that $d(x_2, c) = 4$. Thus $O(x_2) \subseteq O(c)$. Since $x_2 \in M_2$, $O(c) = O(x_2)$. By Lemma 10, $c \in K_4(T)$, and so $k_4(T) \geq 8$. This proves part (i).

(ii) The sufficiency is obvious. We shall prove the necessity. Assume that $k_3(T) = 0$ and $k_4(T) = 8$. By Theorem 1, we may assume that every MS-tournament of $T$ has a transmitter. Let $x_1, x_2, u, v, w$ be the vertices as described in the proof of part (i). Then $\{x_1, u, v, w\} \subseteq K_4(T)$. Since $k_4(T) = 8$, it follows from the proof of part (i) that $s^-(v) = 2$, $x_2 \in K_4(T)$, and that there exists $c \in K_4(T) \cap V_2 \setminus \{x_2\}$ such that $d(x_2, c) = 4$. Again, $O(x_2) \subseteq O(z)$. Note that $z \notin \{v, w\}$. By Lemma 10, $z \in K_4(T)$. Thus, $\{x_1, u, v, w, x_2, z\} \cup I(v) \subseteq K_4(T)$, and so $k_4(T) \geq 9$. This proves Claim 5.

We now consider $s^-(v) = 2$. Since $|O(x_2) \cap I(v)| \geq 2$, $I(v) = O(x_2) \cap I(v)$. Note that $x_2 \rightarrow I(v) \rightarrow v \rightarrow V(T) \setminus (V_2 \cup I(v))$. By Lemma 8, $x_2 \in K_4(T)$. As $k_3(T) = 0$, there exists $c \in V_2 \setminus \{x_2\}$ such that $d(x_2, c) = 4$. Thus $O(x_2) \subseteq O(c)$. Since $x_2 \in M_2$, $O(c) = O(x_2)$. By Lemma 10, $c \in K_4(T)$, and so $k_4(T) \geq 8$. This proves part (i).
and \( O(c) = O(x_2) \). Now as \( x_2 \in M_2 \), we have \( s^-(x_2) = 2 \) and \( x_2 \rightarrow V(T) \backslash (V_2 \cup \{x_1, u\}) \).

Since \( |O(x_2) \cap I(v)| \geq 2 \), we have \( I(v) = O(x_2) \cap I(v) \). Let \( O(x_2) \cap I(v) = \{a, b\} \). Then \( \{v, w\} \rightarrow V(T) \backslash (V_2 \cup \{a, b\}) \). By Claims 2 and 3 in (i), \( \{a, b\} \subseteq K_4(T) \cap V_1 \). Thus, \( K_4(T) = \{x_1, u, v, w, x_2, a, b, c\} \). Since \( I(x_1) \subseteq K_4(T) \), we have \( s^-(x_1) = s^-(u) = 2 \) and \( \{x_1, u\} \rightarrow V(T) \backslash (V_1 \cup \{v, w\}) \). Note that \( \{a, b\} \rightarrow \{v, w\} \rightarrow V(T) \backslash (V_2 \cup \{a, b\}) \) and \( \{a, b\} \rightarrow v \rightarrow x_1 \rightarrow V_2 \backslash \{v, w\} \). Thus, \( d(a, x) \leq 3 \) for all \( x \in V(T) \backslash \{b\} \) and \( d(b, x) \leq 3 \) for all \( x \in V(T) \backslash \{a\} \). Now as \( k_3(T) = 0 \), we must have \( d(a, b) = d(b, a) = 4 \). Thus \( O(a) = O(b) \). Suppose \( s^-(a) \geq 3 \). Let \( z \in I(a) \backslash \{x_2, c\} \). Then \( z \rightarrow \{a, b\} \rightarrow \{v, w\} \rightarrow V(T) \backslash (V_2 \cup \{a, b\}) \). By Lemma 8, \( z \in K_4(T) \), a contradiction. Thus, \( s^-(a) = s^-(b) = 2 \) and \( \{a, b\} \rightarrow V(T) \backslash (V_1 \cap \{x_2, c\}) \). Combining the above results, we conclude that \( T \) is isomorphic to an \( n \)-partite tournament of Fig. 1. \( \square \)

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References