JOURNAL OF COMBINATORIAL THEORY, Series A 24, 367–376 (1978)

A Combinatorial Proof of the Mehler Formula

DOMINIQUE FOATA*

Department of Mathematics, University of California at San Diego, La Jolla, California 92903

Communicated by the Managing Editors

Received October 26, 1977

DEDICATED TO JOHN RIORDAN WITH RESPECT AND ADMIRATION ON THE OCCASION OF HIS 75TH BIRTHDAY

A combinatorial proof of the Mehler formula on Hermite polynomials is given that is based upon the techniques of the partitional complex.

1. INTRODUCTION

The Hermite polynomials $(H_n(a))_{(n\geqslant 1)}$ are hypergeometric functions defined by

$$H_n(a) = (2a)^n\, {}_2F_0\left({-n/2, (1-n)/2 \atop } ; -1/a^2\right) \qquad (n \geqslant 1).$$

Expanding ${}_2F_0$ yields

$$H_n(a) = \sum_{0\leqslant 2k\leqslant n} (-2)^k (2a)^{n-2k} \frac{n!}{(2!)^k\, k!\, (1!)^{n-2k} (n-2k)!} \qquad (n \geqslant 1). \qquad (1)$$

As is readily verified, the fraction in the above summation is equal to the number of *involutions* of the set $[n] = \{1, 2,..., n\}$ having exactly k transpositions (cycles with length 2) and $(n-2k)$ fixed points. Accordingly, each Hermite polynomial $H_n(a)$ may be viewed as the *generating function for the number of fixed points over the set* $\mathscr{V}_n$ *of involutions of* $[n]$. Actually, let

$$\mu_a(\sigma) = (-2)^k (2a)^{n-2k} \qquad (2)$$

if the involution σ has k transpositions and $(n-2k)$ fixed points. Then

$$H_n(a) = \sum \{\mu_a(\sigma) \colon \sigma \in \mathscr{V}_n\}. \qquad (3)$$

This combinatorial interpretation is not new and has been observed many times (see, e.g., [9, p. 65]). It yields almost immediately (for instance, by

* On leave 1977–78 from the Université de Strasbourg, France.

0097-3165/78/0243–0367$02.00/0

using the partitional complex techniques described below) the well-known closed formula for the exponential generating function for the Hermite polynomials

$$1 + \sum_{n \geqslant 1} H_n(a)\, u^n/n! = \exp(2au - u^2).$$

The following bilinear expansion is known as Mehler's formula

$$1 + \sum_{n \geqslant 1} H_n(a)\, H_n(b)\, u^n/n! = (1 - 4u^2)^{-1/2} \exp\left[\frac{4abu - 4(a^2 + b^2)\, u^2}{1 - 4u^2}\right]. \quad (4)$$

There exist several classical analytic proofs of (4) (see, e.g., Watson [12]). The identity plays an essential role in the study of the positivity of the Poisson kernels for series of orthogonal polynomials (see Askey [1]). It has also been extended to the multilinear case by Carlitz [3] and Slepian [11].

The purpose of this paper is to show how the above interpretation of the Hermite polynomials in terms of statistical distributions over involution sets also provides a combinatorial proof of Mehler's formula.

2. The Exponential Formula

For each ordered pair (σ, τ) of involutions let

$$\mu(\sigma, \tau) = \mu_a(\sigma) \cdot \mu_b(\tau)$$

(using notation (2)). Then the product of two Hermite polynomials may be rewritten as

$$H_n(a)\, H_n(b) = \sum \{\mu(\sigma, \tau) : (\sigma, \tau) \in \mathscr{V}_n \times \mathscr{V}_n\},$$

or

$$H_n(a)\, H_n(b) = \mu\{\mathscr{V}_n \times \mathscr{V}_n\}$$

in an abbreviated form. The left-hand side of (4) is then

$$1 + \sum_{n \geqslant 1} \mu\{\mathscr{V}_n \times \mathscr{V}_n\}\, u^n/n!.$$

As for the right-hand side the factor $(1 - 4u^2)^{-1/2}$ can be expressed as the exponential of a series and $(1 - 4u^2)^{-1}$ can be expanded in power series. Altogether (4) takes the form

$$1 + \sum_{n \geqslant 1} \mu\{\mathscr{V}_n \times \mathscr{V}_n\}\, u^n/n! = \exp\Big[\sum_{n \geqslant 1} (2u)^{2n}/(2n) + \sum_{n \geqslant 0} (2ab(2u)^{2n+1}$$
$$- (a^2 + b^2)(2u)^{2n+2})\Big]. \quad (5)$$

Identity (5) has the usual form of the exponential formula

$$1 + \sum_{n \geqslant 1} a_n u^n/n! = \exp \sum_{n \geqslant 1} b_n u^n/n!, \tag{6}$$

that has been studied in many different contexts. Roughly speaking, (6) says that if the argument of exp is the exponential generating function for certain structures, say $(Y_n)_{(n \geqslant 1)}$, then the left-hand side member of (6) is the exponential generating function for collections of these structures. The sets Y_n $(n \geqslant 1)$ occur as *connected* components of those collections.

Several authors (e.g., Mullin and Rota [10], Bender and Goldman [2], Doubilet, Rota and Stein [4], Foata and Schützenberger [5, 6], Garsia and Joni [7], and Gessel [8]) have proposed computational models for handling the exponential formula. Here the partitional complex ("composé partitionnel") model developed in [5, 6] will be used, but any one of the other proposed models would fit in perfectly. The originality of the proof given here lies less in the computational model chosen than in the actual combinatorial interpretation found for the connected structures Y_n $(n \geqslant 1)$.

3. The Partitional Complex

Let $Y = (Y_n)_{(n \geqslant 1)}$ be a sequence of finite sets. With each $n \geqslant 1$, each y in Y_n and each finite subset I of $\mathbb{N}\backslash\{0\}$ with cardinality n is associated an *indeterminate* denoted by (y, I). The indeterminates (y, I) are assumed to *commute* with each other. The set of all the monomials in the indeterminates (y, I) together with the usual monomial product is the free abelian monoid generated by the (y, I)'s, also called the monoid of the multisets. It will be denoted by Y^+. For each $n \geqslant 1$ the *partitional complex of Y with degree n* is defined to be the subset of Y^+ of all the monomials

$$(y_1, I_1)(y_2, I_2) \cdots (y_r, I_r) \qquad (1 \leqslant r \leqslant n)$$

with the property that $\{I_1, I_2, \ldots, I_r\}$ is a *partition* of the interval $[n] = \{1, 2, \ldots, n\}$. It will be denoted by $Y_n^{(+)}$. The sequence $Y^{(+)} = (Y_n^{(+)})_{(n \geqslant 1)}$ is called the *partitional complex of Y*.

Let μ: $Y^+ \to \Omega$ be a mapping into an algebra of polynomials Ω. Then μ is *multiplicative* if the following two conditions hold

(i) for every pair (w, w') of monomials, then $\mu(ww') = \mu(w)\,\mu(w')$;

(ii) for every indeterminate (y, I), then $\mu(y, I) = \mu(y, [\text{card } I])$.

Condition (ii) is the crucial property. If y belongs to Y_n, define

$$\mu(y) = \mu(y, [n]). \tag{7}$$

Then with a monomial

$$w = (y_1, I_1)(y_2, I_2) \cdots (y_r, I_r)$$

in $Y_n^{(+)}$ we have the *multiplicative property*

$$\mu(w) = \mu(y_1)\mu(y_2) \cdots \mu(y_r).$$

The following theorem was proved in [5, p. 67; 6, p. 59]. The notations $\mu\{Y_n^{(+)}\}$ and $\mu\{Y_n\}$ stand for $\sum \{\mu(w): w \in Y_n^{(+)}\}$ and $\sum \{\mu(y): y \in Y_n\}$, respectively.

Theorem (the exponential formula). *If μ is multiplicative, the following identity holds*

$$1 + \sum_{n \geqslant 1} \mu\{Y_n^{(+)}\}\, u^n/n! = \exp \sum_{n \geqslant 1} \mu\{Y_n\}\, u^n/n!. \tag{8}$$

Comparing (5) and (8) we see that we still need to: interpret the sequence $(Y_n)_{(n \geqslant 1)}$ in Mehler's formula, express $\mathscr{V}_n \times \mathscr{V}_n$ as $Y_n^{(+)}$ and prove that μ is multiplicative. This will be achieved by means of the partitional complex of involutionary graphs.

4. Involutionary Graphs

As can be expected, the language of Graph Theory will appear to be convenient in the description of our model. *A labeled graph G* is a triple (X, E, L) with X a finite totally ordered set, E a finite family of doubletons $\{x, x'\}$ with x and x' two distinct elements of X, and L a finite family of singletons $\{x\}$ in X. The elements of X, E and L are called *vertices*, *edges* and *loops*, respectively. A *path* is a sequence $(e_1, e_2, \ldots, e_m)$ of edges with the property that either $m = 1$, or $m \geqslant 2$ and for each $i = 1, 2, \ldots, m-1$ the two edges e_i and e_{i+1} have exactly one vertex in common. If $m = 1$ and $e_1 = \{x, x'\}$, the path is said to go from x to x'. If $m \geqslant 2$, $e_m = \{z, z'\}$ and x (resp. z') is not in e_2 (resp. in e_{m-1}), then the path is said to go from x to z'. If $x = z'$ the path is a *cycle*. By convention, if two edges e, e' are equal to the same doubleton $\{x, x'\}$, the sequence (e, e') is also a cycle. A path or a cycle $(e_1, e_2, \ldots, e_m)$ is *eulerian* if it contains once and only once every edge of the graph. Two vertices x and x' are equivalent if either $x = x'$ or $x \neq x'$ and there exists a path going from x to x'. The equivalence classes are the connected components of the graph. A graph is *connected* if it has only one connected component. Clearly, a graph is an (unordered) collection of its connected components (say) $\{G_1, G_2, \ldots, G_r\}$. For each $i = 1, 2, \ldots, r$ let I_i be the set of all vertices of the component G_i. If card $I_i = m_i$, denote by

$\omega_i : I_i \to [m_i]$ the unique increasing map of I_i onto $[m_i]$. Denote by y_i the graph with m_i vertices labeled $1, 2,..., m_i$ whose edges and loops are defined as follows: if $\{x, x\}$ (resp. $\{x\}$) is an edge (resp. loop) of G_i, then $\{\omega_i(x), \omega_i(x')\}$ (resp. $\{\omega_i(x)\}$) is an edge (resp. loop) of y_i. The graph G_i is then completely determined by the ordered pair $(y_i, I_i)(1 \leqslant i \leqslant r)$, and the graph G itself by the monomial

$$(y_1, I_1)(y_2, I_2) \cdots (y_r, I_r). \tag{9}$$

An *involutionary* graph is a labeled graph with two kinds of edges or loops, say, blue and red, having the further property that each vertex is *incident to exactly one blue edge or loop, and one red edge or loop*. Clearly, a graph is involutionary if and only if its connected components are involutionary. For each $n \geqslant 1$ denote by Y_n the set of all connected involutionary graphs with n vertices labeled $1, 2,..., n$ and let $Y = (Y_n)_{(n \geqslant 1)}$. As noted above, each involutionary graph with n vertices labeled $1, 2,..., n$ may be identified with the monomial written in (9). Each graph y_i belongs to Y_{m_i} ($m_i = \text{card } I_i$) $(1 \leqslant i \leqslant r)$ and $\{I_1, I_2,..., I_r\}$ is a partition of $[n]$. Thus

LEMMA 1. *For each $n \geqslant 1$ the set of all involutionary graphs with n vertices labeled $1, 2,..., n$ is identical with the partitional complex of Y with degree n.*

5. CONNECTED INVOLUTIONARY GRAPHS

In order to have an explicit formula for the argument of exp in identity (8) we must determine and classify the connected involutionary graphs. This is the purpose of the next lemma.

LEMMA 2. *A graph y with n vertices labeled $1, 2,..., n$ having edges and loops of two different colors, blue and red, is a connected involutionary graph if and only if it has one of the following forms*

(A-type) n even and y is an Eulerian cycle $(\{x_1, x_2\}, \{x_2, x_3\},..., \{x_{n-1}, x_n\}, \{x_n, x_1\})$ with the property that $x_1x_2 \cdots x_n$ is a permutation of $12 \cdots n$ and two successive edges have different colors;

(B-type) (resp. C-type) n even and y is an Eulerian path $(\{x_1, x_2\}, \{x_2, x_3\},..., \{x_{n-1}, x_n\})$ with the property that $x_1x_2 \cdots x_n$ is a permutation of $12 \cdots n$, there are one blue (resp. red) loop about x_1 and one blue (resp. red) loop about x_n, and for each $i = 1, 2,..., n-1$ the ith edge $\{x_i, x_{i+1}\}$ is red or blue (resp. blue or red) according as i is odd or even;

(D-type) n odd and y is an Eulerian path $(\{x_1, x_2\}, \{x_2, x_3\},..., \{x_{n-1}, x_n\})$ *with the property that* $x_1x_2 \cdots x_n$ *is a permutation of* $12 \cdots n$, *there are one blue loop about* x_1 *and one red loop about* x_n, *and for each* $i = 1, 2,..., n - 1$ *the ith edge* $\{x_i, x_{i+1}\}$ *is red or blue according as i is odd or even.*

As an illustration graphs of the four types are shown in Table I.

Proof. By induction on n. let Y_n be the set of all connected involutionary graphs with n vertices labeled $1, 2,..., n$. Clearly Y_1 contains only one graph shown in Fig. 1. The set Y_2 has three elements drawn in Fig. 2. Assume $n \geqslant 2$

FIGURE 1

FIGURE 2

and that the lemma holds for every integer $m \leqslant n$. Consider a graph y of the set Y_{n+1} and examine the following four cases: (i) two loops about vertex $(n + 1)$; (ii) (resp. (iii)) one blue (resp. red) loop about $(n + 1)$ and one red (resp. blue) edge incident to $(n + 1)$; (iv) two edges incident to $(n + 1)$. Remember that in an involutionary graph every vertex is incident to exactly one blue edge or loop, and one red edge or loop. If case (i) holds, there is no edge incident to $(n + 1)$. Hence y belongs to Y_1, that has been excluded.

By symmetry it remains to study cases (ii) and (iv). In case (ii) let $\{(n + 1), x\}$ be the unique red edge incident to $(n + 1)$. As $n + 1 \geqslant 3$ there is no blue loop about x. Hence there exists a blue edge $\{x, x'\}$ with $x' \neq n + 1$. Remove the edge $\{x, n + 1\}$ and the loop $\{n + 1\}$ from graph y and put a red loop about x as shown in Fig. 3. We clearly obtain an element y' of Y_n.

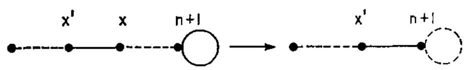


FIGURE 3

As y' has a red loop, it is either of C-type or D-type, according as n is even or odd. Hence, y is of D-type or B-type according as $(n + 1)$ is odd or even.

In case (iv) let $\{x, n + 1\}$ and $\{x', n + 1\}$ be the blue and red edges incident to $(n + 1)$, respectively. Note that $x \neq x'$ for $n + 1 \geqslant 3$. Remove the two edges $\{x, n + 1\}$ and $\{x', n + 1\}$ from y and retract vertices x and x' to a single vertex labeled $z = \min\{x, x'\}$, as shown in Fig. 4. We then obtain a connected involutionary graph y' whose vertex set is $[n] \backslash \max\{x, x'\}$. By induction, graph

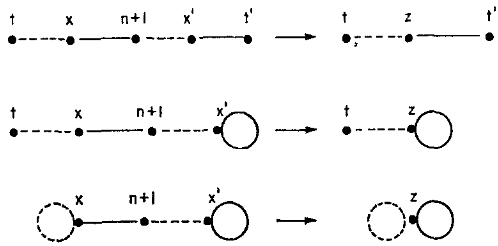


FIGURE 4

y' has one of the four types A, B, C or D, with the restriction that the labels of its vertices are taken from the set $[n]\backslash\max\{x, y\}$. Hence, y is of the same type as y'. Q.E.D.

For each $n \geqslant 1$ denote by A_n, B_n, C_n, D_n the subset of Y_n of all the graphs of A-, B-, C-, D-type, respectively. Clearly

$$\begin{aligned} \text{card } A_n &= \begin{cases} 0 & \text{if } n \text{ odd} \\ (n-1)! & \text{if } n \text{ even,} \end{cases} \\ \text{card } B_n = \text{card } C_n &= \begin{cases} 0 & \text{if } n \text{ odd} \\ n!/2 & \text{if } n \text{ even,} \end{cases} \\ \text{card } D_n &= \begin{cases} n! & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases} \end{aligned} \tag{10}$$

6. THE MULTIPLICATIVE FUNCTION

Let (y, I) be a connected involutionary graph with vertex set I. If y has k blue (resp. l red) edges, k' blue (resp. l' red) loops, let

$$\mu(y, I) = (-2)^k (2a)^{k'} (-2)^l (2b)^{l'} \tag{11}$$

where a and b are two commuting variables. The map μ is extended to all of Y^+ in a natural manner, so that condition (i) for a multiplicative function holds. Moreover the definition of μ in (11) depends only on the number of edges and loops and not on the labels of the vertices. Accordingly condition (ii)

$$\mu(y, I) = \mu(y, [\text{card } I])$$

holds, and μ is *multiplicative*. In particular, if

$$w = (y_1, I_1)(y_2, I_2) \cdots (y_r, I_r)$$

is an involutionary graph with n vertices labeled 1, 2,..., n, that is, an element of $Y_n^{(+)}$, we have

$$\mu(w) = \mu(y_1)\,\mu(y_2)\cdots\mu(y_r)$$
$$= (-2)^k(2a)^{k'}(-2)^l(2b)^{l'}$$

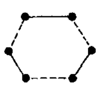
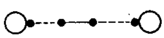
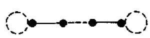
with k, k', l and l' being the number of blue edges, blue loops, red edges, red loops *of w itself*, respectively.

Let us determine the value of μ at y when y runs over A_n, B_n, C_n, D_n. We first note that μ is constant on each of these subsets so that, for instance,

$$\mu\{A_n\} = \mu(y)\cdot\text{card}\,A_n$$

for any y in A_n. Taking (10) into account we obtain Table I.

TABLE I

Type	n	Set	$\mu(y)$	card set	$\mu\{\text{set}\}$
	Even	A_n	$(-2)^n$	$(n-1)!$	$\mu\{A_n\} = 2^n(n-1)!$
	Odd	ϕ	—	0	$= 0$
	Even	B_n	$(-2)^{n-1}(2a)^2$	$n!/2$	$\mu\{B_n\} = -a^2\,2^n\,n!$
	Odd	ϕ	—	0	$= 0$
	Even	C_n	$(-2)^{n-1}(2b)^2$	$n!/2$	$\mu\{C_n\} = -b^2\,2^n\,n!$
	Odd	ϕ	—	0	$= 0$
	Even	ϕ	—	0	$\mu\{D_n\} = 0$
	Odd	D_n	$(-2)^{n-1}2a\,2b$	$n!$	$= 2ab\,a^n n!$

It follows from the computations made in Table I that if n is even ($n \geqslant 2$)

$$\mu\{Y_n\} = \mu\{A_n\} + \mu\{B_n\} + \mu\{C_n\}$$
$$= 2^n n!(1/n - (a^2 + b^2)),$$

and if n is odd ($n \geqslant 1$)

$$\mu\{Y_n\} = \mu\{D_n\} = 2^n n!\,2ab.$$

Hence

$$1 + \sum_{n\geqslant 1} \mu\{Y_n\}\, u^n/n! = \sum_{n\geqslant 1} (2u)^{2n}/(2n) + \sum_{n\geqslant 0} (2ab(2u)^{2n+1} - (a^2 + b^2)(2u)^{2n+2}),$$

which is precisely the argument of exp in Mehler's formula (5).

It remains to show that the expression $\mu\{\mathscr{V}_n \times \mathscr{V}_n\}$ of (5) is equal to $\mu\{Y_n^{(+)}\}$ ($n \geqslant 1$). But this follows immediately from the graphical representation of a pair of involutions. For suppose that the pair (σ, τ) of involutions is such that σ (resp. τ) has k (resp. l) transpositions and $(n - 2k)$ (resp. $(n - 2l)$) fixed points. With (σ, τ) we can associate a graph with n vertices labeled $1, 2,..., n$, the edges and loops being defined as follows: there is a blue (resp. red) edge between vertices i and j if and only if $i \neq j$ and $\sigma(i) = j$ (resp. $\tau(i) = j$), while there is a blue (resp. red) loop about vertex i if $\sigma(i) = i$ (resp. $\tau(i) = i$). Clearly, the graph associated with a pair (σ, τ) of involutions is involutionary, and conversely, to an involutionary graph there corresponds one and only one pair of involutions. Furthermore, as transpositions and fixed points of the pair correspond to edges and loops of the graph, in a one-to-one manner, we have

$$\mu\{\mathscr{V}_n \times \mathscr{V}_n\} = \mu\{Y_n^{(+)}\}.$$

The combinatorial proof of Mehler's formula is then completed.

Acknowledgments

The author wishes to thank professor Schützenberger for conveying him his belief that Mehler's formula was to be proved combinatorially, and professor Askey for encouraging him to write up the proof.

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