On Semilinear Elliptic Boundary Value Problems in Unbounded Domains

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1. Introduction

In this paper we consider nonlinear elliptic boundary value problems (BVPs) of the form

\[ Lu = f(x, u, \nabla u) \quad \text{in } \Omega, \]
\[ Bu = g \quad \text{on } \partial \Omega, \tag{1.1} \]

where \( L \) is a linear, second order, uniformly elliptic differential operator, \( B \) is a first order boundary operator, and \( \Omega \) is an unbounded domain of \( \mathbb{R}^n \) with boundary \( \partial \Omega \). Our main purpose is to prove the existence of classical solutions. We also show that a solution which tends to zero as \( |x| \) tends to \( \infty \) is unique. More explicitly, if \( f \) satisfies a Nagumo condition (Assumption A(a) in Section 2) and if (1.1) has a supersolution \( u_0 \) and a subsolution \( v_0 \), such that \( v_0 \leq u_0 \), Theorem 3.4 establishes the existence of a maximal solution and a minimal solution in the order interval \([v_0, u_0]\). If \( f \) is also strictly monotonically decreasing in \( u \) or is monotonically non-increasing in \( u \) and satisfies a local Lipschitz condition in \( \nabla u \), Theorem 4.1 establishes the existence of a unique solution of (1.1) tending to zero as \( |x| \) tends to \( \infty \). These results are then applied in Section 4 to establish specific existence and uniqueness criteria.

The existence results are essentially known in the case of bounded domains. We refer the reader to the work of Nagumo [10], Ako [2], Amann [3], Amann and Crandall [4], and the survey paper of Schmitt [15], where an extensive bibliography is given.

In the unbounded domain case, the existence results are known for some special cases. When \( f \) is independent of \( \nabla u \), the existence of a maximal and a minimal solution was established by the author in [11] under more restrictive assumptions on \( f \) involving monotonicity with respect to \( u \). In [2]
Ogata established the existence of bounded solutions in exterior domains when $f$ is assumed to be bounded in $\Omega \times R$.

For the case when $f$ has a linear growth in $\nabla u$, we refer the reader to Bose [5] and Meyers and Serrin [9].

2. Preliminaries

Let $\alpha \in (0, 1)$ be fixed. Denote by $\Omega$ an unbounded domain of real $n$-space $R^n$ with boundary $\partial \Omega$ and closure $\bar{\Omega}$.

Let $C^{m+\alpha}(\bar{\Omega})$, $m = 1, 2, \ldots$, denote the usual H"{o}lder space, namely $C^{m+\alpha}(\bar{\Omega}) = \{u: \Omega \rightarrow R/u \in C^{m+\alpha}(M) \text{ for every bounded subset } M \subset \Omega\}$. The norm in this space is denoted by $\| u \|_{m+\alpha}$. We denote by $L$ the real differential operator

$$Lu = - \sum_{i,j=1}^{n} a_{ij}(x) D_i D_j u,$$

with symmetric coefficient matrix. We suppose that $a_{ij} \in C^\alpha(\bar{\Omega})$. The operator $L$ is assumed to be uniformly elliptic on every bounded subdomain of $\Omega$.

We denote by $\beta \in C^{1+\alpha}(\partial \Omega)$ an outward pointing, nowhere tangent vector field on $\partial \Omega$. Then we consider boundary operators of the form

$$Bu = b_0 u + \delta \frac{\partial u}{\partial \beta},$$

where either $\delta = 0$ and $b_0 = 1$ (Dirichlet boundary operator), or $\delta = 1$ and $b_0 \geq 0$ (Neumann or regular oblique derivative boundary operator).

Let $f: \Omega \times (0, \infty) \rightarrow R$ and $g: \partial \Omega \rightarrow R$ be given functions. Then we consider BVPs of the form

$$Lu = f(x, u, \nabla u) \quad \text{in } \Omega;$$

$$Bu = g \quad \text{on } \partial \Omega,$$

where $\nabla u = (D_1 u, D_2 u, \ldots, D_n u)$ denotes the gradient of $u$. By a solution of (2.1) we mean a function $u$ in $\bar{\Omega}$ such that $u \in C^{2+\alpha}(\bar{\Omega})$ and satisfies (2.1) identically.

The functions $f$, $g$, and $b_0$ are required to satisfy the following conditions.

Assumptions A. (a) $f(x, t, s) \in C^\alpha(\Omega \times R \times R^n)$, and is continuously differentiable in $s$ and $t$. 505/41/3-4
(b) For each bounded domain $M \subset \Omega$, there exists a continuous function $p_M \colon [0, \infty) \to [0, \infty)$ such that
\[ |f(x, u, p)| \leq p_M(u) (1 + |p|^2), \quad x \in M, \quad u \in \mathbb{R}, \quad p \in \mathbb{R}^n, \]
\[ (c) \quad b_0, g \text{ belong to } C^{2-\kappa+\alpha}(\partial \Omega). \]

**DEFINITION 2.1.** A function $u \colon \Omega \to \mathbb{R}$ is called a subsolution of (2.1) if
\[ u \in C^{2+a}(\Omega) \text{ and } \]
\[ Lu \leq f(x, u, \nabla u) \quad \text{in } \Omega; \]
\[ Bu \leq g \quad \text{on } \partial \Omega. \]
Supersolutions are defined by reversing the above inequality signs.

### 3. Existence of Solutions in Exterior Domains

In this section we assume that $\Omega$ is an exterior domain with boundary $\partial \Omega$ of class $C^{2+a}$.

Let $a > 0$ be such that $\{x \in \mathbb{R}^n : |x| > a\} \subset \Omega$. The following notation will be used:
\[ \Omega_b = \{x \in \Omega : |x| < b\}; \]
\[ S_b = \{x \in \mathbb{R}^n : |x| = b\}; \]
\[ D_{a,b} = C^{2+a}(\overline{\Omega}_b), \quad b > 0. \]

**LEMMA 3.1.** Let the operators $L, B$, and the functions $f, g$, satisfy the conditions specified in Section 2. If there exist a subsolution $v_0$ and a supersolution $u_0$ of (2.1) such that $v_0 \leq u_0$ on $\overline{\Omega}$, then there exist sequences of functions $u_j$ and $v_j$ on $\Omega$ with the following properties for all $j \geq 1$:

(i) $u_j, v_j \in D_{a,a+j}$;
(ii) $Lu_j - f(x, u_j, \nabla u_j) = 0 = Lv_j - f(x, v_j, \nabla v_j)$ in $\Omega_{a+j}$;
\[ Bu_j = Bv_j = g \quad \text{on } \partial \Omega; \]
\[ u_j = u_0, v_j = v_0 \quad \text{on } S_{a+j}, \]
(iii) $v_0 \leq v_1 \leq \cdots \leq v_{n-1} \leq v_n \leq \cdots \leq u_n \leq u_{n-1} \leq \cdots \leq u_1 \leq u_0$ on $\overline{\Omega}$;
(iv) If $u$ is any solution of
\[ Lu = f(x, u, \nabla u) \quad \text{in } \Omega_{a+j}; \]
\[ Bu = g \quad \text{on } \partial \Omega. \]
satisfying
\[ v_0 \leq u \leq u_0 \quad \text{on } \bar{\Omega}_{a+j}, \]
then
\[ v_j \leq u \leq u_j \quad \text{on } \bar{\Omega}_{a+j}, \]

**Proof.** Since \( v_0 \) is a subsolution and \( u_0 \) is a supersolution of the BVP

\[
Lu = f(x, u, \nabla u) \quad \text{in } \Omega_{a+j};
\]
\[
Bu = g \quad \text{on } \partial \Omega;
\]
\[
u = u_0 \quad \text{on } S_{a+j},
\]

satisfying \( v_0 \leq u_0 \) on \( \Omega_{a+j} \), known results [3, 4, 15] for bounded domains imply the existence of a maximal solution \( U_j \in C^{2+a}(\Omega_{a+j}) \) of (3.1) such that if \( u \) is any solution of (3.1) satisfying \( v_0 \leq u \leq u_0 \) on \( \bar{\Omega}_{a+j} \), then \( v_0 \leq u \leq U_j \leq u_0 \) on \( \bar{\Omega}_{a+j} \). Let \( u_j \) be the extension of \( U_j \) to \( \bar{\Omega} \) defined by \( u_j(x) = u_0(x) \) for \( |x| > a + j \).

To construct \( v_j \), we note first that \( u_0 \) is a supersolution and \( v_0 \) is a solution of the BVP

\[
Lu = f(x, u, \nabla u) \quad \text{in } \Omega_{a+j};
\]
\[
Bu = g \quad \text{on } \partial \Omega;
\]
\[
u = v_0 \quad \text{on } S_{a+j}. \]

Hence the results in [3, 4, 15] imply the existence of a minimal solution \( V_j \) such that if \( v \) is any solution of (3.2) satisfying \( v_0 \leq v \leq u_0 \) on \( \Omega_{a+j} \), then \( v_0 \leq V_j \leq v \leq u_0 \) on \( \bar{\Omega}_{a+j} \). Let \( v_j \) be the extension of \( V_j \) to \( \bar{\Omega} \) defined by \( v_j(x) = v_0(x) \) for \( |x| > a + j \). It is then clear that the sequences \( \{u_j\}, \{v_j\} \) satisfy properties (i) and (ii).

We note that the function \( u_{j+1} \) is a subsolution of (3.1) satisfying \( v_0 \leq u_{j+1} \leq u_0 \) on \( \bar{\Omega}_{a+j} \). Then the BVP (3.1) has a solution \( w_j \) satisfying \( u_{j+1} \leq w_j \leq u_0 \) on \( \bar{\Omega}_{a+j} \) by the results on bounded domains. But \( u_j \) is a maximal solution of (3.1) on \( \bar{\Omega}_{a+j} \). Hence \( u_{j+1} \leq w_j \leq u_j \) on \( \bar{\Omega}_{a+j} \). Similar arguments show that \( v_j \leq v_{j+1} \leq u_0 \) on \( \bar{\Omega}_{a+j} \). We also note that \( v_j \) is a solution of (3.1) satisfying \( v_0 \leq v_j \leq u_0 \) on \( \bar{\Omega}_{a+j} \). Then the BVP (3.1) has a solution \( z_j \) satisfying \( v_j \leq z_j \leq u_0 \). The maximality of the solution \( u_j \) then implies that \( v_j \leq z_j \leq u_j \leq u_0 \) on \( \bar{\Omega}_{a+j} \). Using the definition \( u_j, v_j \) we can then conclude that condition (iii) is satisfied.

Finally, any solution \( u \) of \( Lu = f(x, u, \nabla u) \) in \( \Omega_{a+j} \), \( Bu = g \) on \( \partial \Omega \) satisfying \( v_0 \leq u \leq u_0 \) on \( \bar{\Omega}_{a+j} \), is a supersolution of (3.2) and is a subsolution of (3.1). This existence theorems for bounded domains then imply the existence of a solution \( \alpha_j \) of (3.2), and a solution \( \gamma_j \) of (3.1) such that \( v_0 \leq \alpha_j \leq u \), and \( u \leq \gamma_j \leq u_0 \) on \( \bar{\Omega}_{a+j} \). The maximality of \( u_j \) and the
minimality of $v_j$ then imply that $v_j \leq a_j \leq u \leq \gamma_j \leq u_j$ on $\overline{\Omega}_{a+j}$. This shows that property (iv) is satisfied and completes the proof of Lemma 3.1.

**Lemma 3.2.** Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of functions $z_j: \Omega \to R$ such that $z_j \in D_{a,a+j}$, $v_0 \leq z_j \leq u_0$ on $\overline{\Omega}$, and for all $j \in \mathbb{N}$

$$Lz_j = f(x,z_j,\nabla z_j) \quad \text{in } \Omega_{a+j},$$
$$Bz_j = g \quad \text{on } \partial \Omega.$$

Let $K$ be an arbitrary positive number and $i \in \mathbb{N}$ fixed, and let $w_j$ be the unique solution of the BVP.

$$(L + K)w = 0 \quad \text{in } \Omega_{a+i};$$
$$(1 - \delta)w + \delta \frac{\partial w}{\partial v} = (1 - \delta)z_j + \delta \frac{\partial z_j}{\partial v} \quad \text{on } S_{a+i};$$
$$Bw = g \quad \text{on } \partial \Omega, j \geq i,$$

where $v$ is the outward normal vector on $S_{a+i}$. Then there exists a positive constant $K_0$, independent of $j$, such that

$$\|w_j\|_{2+a,\Omega_{a+i}} \leq K_0 \quad \text{for all } j \geq i.$$

**Proof.** We consider the two cases $\delta = 0$ and $\delta = 1$.

(i) $\delta = 0$. Assumption A(c) and the Schauder estimate (e.g., [6]) imply that

$$\|w_j\|_{2+a,\overline{\Omega}_{a+i}} \leq C_1 \left[ \|z_j\|_{2+a,\Omega_{a+i}} + \|g\|_{2+a,\partial \Omega} \right]$$

for some positive constant $C_1$ independent of $j, j \geq i$.

(ii) $\delta = 1$. Assumptions A(c) and the Schauder-type inequality (e.g., [1]) imply that

$$\|w_j\|_{2+a,\overline{\Omega}_{a+i}} \leq C_2 \left[ \left\| \frac{\partial z_j}{\partial v} \right\|_{1+a,\partial S_{a+i}} + \|g\|_{1+a,\partial \Omega} \right]$$

for some positive constant $C_2$ independent of $j, j \geq i$.

It is clear from (i) and (ii) above that the conclusion of Lemma 3.2 will then follow if we show that $\|z_j\|_{2+a,\Omega_{a+i}}$ is uniformly bounded with respect to $j$. We show this next.

Let $M, Q$ and $R$ be bounded domains such that $S_{a+i} \subset M, \overline{M} \subset Q, \overline{Q} \subset R, \overline{R} \subset \Omega_{a+i+1}, \partial M, \partial Q$ and $\partial R$ of class $C^{1+a}$. By hypotheses $z_j$ satisfies $Lz_j=$
$f(x, z_j, \nabla z_j)$ in $\Omega_{a+i+1}$ for all $j \geq i + 1$. It follows from an a priori interior estimate of Ladyzhenskay and Ural'tseva [8, Theorem 3.1, p. 266] that

$$\max_{x \in R} |\nabla z_j(x)| \leq C_3 \max_{x \in R_{a+i+1}} |z_j(x)|$$

for some positive constant $C_3$ independent of $z_j$. Since $|z_j(x)|$ is uniformly bounded by hypotheses, the above estimate then implies that $\{\nabla z_j\}$ is uniformly bounded on $R$.

Let $v_j, j \geq i + 2$, be the unique solution of the BVP

$$L v = f(x, z_j(x), \nabla z_j(x)) \quad \text{in } R;$$
$$v(x) = 0 \quad \text{on } \partial R.$$  \hspace{1cm} (3.4)

Define $f_j(x) = f(x, z_j(x), \nabla z_j(x))$, which is a uniformly bounded sequence on $R$ on account of the uniform boundedness of $\{z_j\}$ and $\{\nabla z_j\}$. It then follows that for any $p > 1$, $\|f_j\|_{L^p(R)}$ is uniformly bounded. The norm of the solution $v_j$ of (3.4) in the Sobolev space $W^p(R)$ therefore satisfies

$$\|v_j\|_{W^p(R)} \leq C_4$$

for some positive constant $C_4$ independent of $j$ by the $L^p$-estimate of Agmon et al. [1]. With the choice $p = n/(1 - \alpha)$, the Sobolev embedding Lemma shows that

$$\|v_j\|_{1+\alpha,R} \leq C_5 \|v_j\|_{W^p(R)} \leq C_5 C_4$$  \hspace{1cm} (3.5)

for some positive constant $C_5$ independent of $j$.

Let $\theta_j$ be the unique solution of the BVP

$$L \theta = 0 \quad \text{in } R,$$
$$\theta = z_j \quad \text{on } \partial R, j \geq i + 2.$$  \hspace{1cm} (3.6)

Since $\{z_j\}$ is uniformly bounded in $R$, the classical Maximal principle for elliptic equations implies that $\|\theta_j\|_{0,R} \leq C_6$ for some positive constant $C_6$ independent of $j$. Classical interior Schauder estimates [6] for (3.6) then implies that

$$\|\theta_j\|_{2+\alpha,\partial R} \leq C_7 \|\theta_j\|_{0,R} \leq C_6 C_7$$  \hspace{1cm} (3.7)

for some positive constant $C_7$ independent of $j$.

It follows from (3.4) and (3.6) that $\theta_j + v_j$ is a solution of the BVP

$$L u = f(x, e_j, \nabla z_j) \quad \text{in } R;$$
$$u = z_j \quad \text{on } \partial R, j \geq i + 2.$$
However, \( u = z_j \) also is a solution by hypotheses, and hence \( z_j = v_j + \theta_j \) on \( \bar{R} \) by the standard uniqueness theorem. It is then a consequence of (3.5) and (3.7) that \( \| z_j \|_{1 + \alpha, \bar{R}} \leq C_\alpha \) for some positive constant \( C_\alpha \) independent of \( j \). From this and Assumptions A(a) it follows that the function \( f_j(x) = f(x, z_j(x), \nabla z_j(x)) \) satisfies \( \| f_j \|_{\alpha, \theta} \leq C_\theta \) for all \( j \geq i + 2 \), where \( C_\theta \) is again independent of \( j \). Since \( z_j \) is a solution of \( L z_j(x) = \bar{f}(x) \) for \( x \in \bar{Q} \), we then have from the interior Schauder estimate [6] that

\[
\| z_j \|_{2 + \alpha, \bar{R}} \leq C_{10} \left( \| z_j \|_{0, \bar{R}} + \| f_j \|_{\alpha, \theta} \right)
\]

\[
\leq C
\]

for some constant \( C \) independent of \( j \). Since \( S_{a+i} \subset M \), we conclude that \( \{ \| z_j \|_{2 + \alpha, S_{a+i}} \} \) is uniformly bounded. The conclusion of Lemma 3.2 then follows from the Schauder-type estimates in (i) and (ii).

**Lemma 3.3.** Let the sequence \( \{ z_j \} \) be as in Lemma 3.2. Then for each positive integer \( i \) there exists a positive constant \( C \), independent of \( j \), such that

\[
\| z_j \|_{2 + \alpha, \bar{R}_{a+i}} \leq C
\]

(3.8)

for all \( j \geq i \).

**Proof.** Let \( w_j, j \geq i \), be the unique solution of the BVP (3.3). Define \( f_j: \bar{Q} \times R \times R^n \rightarrow R \) by

\[
f_j(x, s, t) = f(x, s + w_j, t + \nabla w_j) + K w_j.
\]

Then \( z_j - w_j \) is a solution of the BVP

\[
L u = f_j(x, v, \nabla v) \quad \text{in} \quad \Omega_{a+i};
\]

\[
(1 - \delta)v + \delta \frac{\partial v}{\partial \sigma} = 0 \quad \text{on} \quad S_{a+i},
\]

\[
B v = 0 \quad \text{on} \quad \partial \Omega
\]

(3.9)

for all \( j \geq i \). Since \( f_j \) satisfies the same regularity and growth conditions as \( f \), a recent result of Amann and Crandall, Lemma 4 in [4] implies that there exists an increasing function \( \gamma: R_+ \rightarrow R_+ \) such that

\[
\| v \|_{w^2_\theta(\Omega_{a+i})} \leq \gamma(C_j(v)), \quad j \geq i,
\]

(3.10)

for all solutions \( v \) of (3.9) in the Sobolev space \( w^2_\theta(\Omega_{a+i}) \), where \( \gamma \) depends only on the coefficients of \( L \) and \( B \), \( \Omega_{a+i} \), \( n \), and \( p \), and

\[
C_j(v) = \max_{\Omega_{a+i}} |(f_j(x, v(x), \nabla v(x)) + v(x))/(1 + |\nabla v(x)|^2)|.
\]
It then follows from the definition of $f_j$ and the growth condition on $f$ that

\begin{align*}
C_j(v) \leq & \max_{\Omega_{a+i}} \left[ \rho \Omega_{a+i} (|v(x) + w_j(x)|) (1 + |\nabla v(x) + \nabla w_j(x)|^2) \right. \\
& \left. + k |w_j(x)| + |v(x)| \right] \\
& \leq \max_{\Omega_{a+i}} \left[ 2\rho \Omega_{a+i} (|v(x) + w_j(x)|) (1 + |\nabla w_j(x)|^2) \right. \\
& \left. + k |w_j(x)| + |v(x)| \right].
\end{align*}

Substituting $v = z_j - w_j$ in the above inequality, we obtain

\begin{align*}
C_j(z_j - w_j) \leq & \max_{\Omega_{a+i}} \left[ 2\rho (|z_j|) (1 + |\nabla w_j(x)|^2) \right. \\
& \left. + |z_j(x)| + (1 + K) |w_j(x)| \right].
\end{align*}

It then follows from Lemma 3.2, the uniform boundedness of $\{z_j\}$ on $\Omega_{a+i}$ for $j \geq a + i$, and (3.10) that

\begin{align*}
\|z_j - w_j\|_{\Omega_{a+i}} \leq K_1
\end{align*}

for all $f \geq i$, where $K_1$ is a positive constant independent of $j$. With the choice $p = n/(1 - \alpha)$, the Sobolev embedding Lemma shows that

\begin{align*}
\|z_j - w_j\|_{L^2(\Omega_{a+i})} \leq K_2, \quad j \geq i,
\end{align*}

(3.11)

for some positive constant $K_2$ independent of $j$. In view of the regularity assumptions of $f$, Lemma 3.2, and the estimate (3.11), the functions $\tilde{f}_j(x) = f(x, z_j(x), z_j(x)) + Kw_j$ satisfy

\begin{align*}
\|\tilde{f}_j\|_{L^2(\Omega_{a+i})} \leq K_3, \quad j \geq i,
\end{align*}

(3.12)

for some positive constant $K_3$ independent of $j$. Since $z_j - w_j$ is a solution of the linear BVP

\begin{align*}
Lu = f_j(x) & \quad \text{in } \Omega_{a+i}; \\
\frac{\partial u}{\partial v_l} + (1 - \delta)u = 0 & \quad \text{on } S_{a+i}; \\
Bu = 0 & \quad \text{on } \partial \Omega, j \geq i,
\end{align*}

it follows from the classical Schauder estimates [6] and (3.12) that

\begin{align*}
\|z_j - w_j\|_{L^2(\Omega_{a+i})} \leq K_4, \quad j \geq i,
\end{align*}

(3.13)

for some positive constant $K_4$ independent of $j$. The conclusion of Lemma 3.3 then follows from Lemma 3.2 and the estimate (3.13).
Theorem 3.4. Under the hypotheses of Lemma 3.1, the BVP (2.1) has a maximal solution \( \bar{u} \) and a minimal solution \( \underline{u} \), \( \underline{u}_0(x) \leq \bar{u}(x) \leq \underline{u}_0(x) \) on \( \overline{\Omega} \); i.e., if \( u \) is any other solution of the BVP (3.1), with \( \underline{u}_0(x) \leq u(x) \leq \underline{u}_0(x) \) on \( \overline{\Omega} \), then \( \bar{u}(x) \leq u(x) \leq \bar{u}(x) \) on \( \overline{\Omega} \).

Proof. Let \( \{ u_j \} \) and \( \{ v_j \} \) be the sequences in Lemma 3.1. Let \( \bar{u}, \underline{u} \) denote the pointwise limits; i.e.,

\[
\bar{u}(x) = \lim_{j \to \infty} u_j(x), \quad \underline{u}(x) = \lim_{j \to \infty} v_j(x), \quad x \in \overline{\Omega},
\]

The above limits exist by the monotonicity of the sequences considered. We show next that \( \bar{u} \) is a solution of the BVP (2.1). The proof for \( \underline{u} \) is similar. First note that the sequence \( \{ z_j \} \) in Lemma 3.3 and Lemma 3.4 can be replaced by the sequence \( \{ u_j \} \). This follows from properties (i) and (ii) of Lemma 3.1. By Lemma 3.3 there exists a positive integer \( K \), independent of \( j \), such that \( \| u_j \|_{C^2(\overline{\Omega}_{a+1})} \leq K \) for all \( j \geq i \), \( i = 1, 2, \ldots \). The compactness of the injection \( C^{2+\alpha}(\overline{\Omega}_{a+1}) \to C^2(\overline{\Omega}_{a+1}) \) then implies that \( \{ u_j : j \geq 1 \} \) has a subsequence \( \{ u_j^i \} \) which converges uniformly in the \( C^2(\overline{\Omega}_{a+1}) \) norm to a function \( u^i \) on \( \overline{\Omega}_{a+1} \). Define \( u_0^i = u_j \) for convenience and define \( \{ u_j^i \} \) inductively to be a subsequence of \( \{ u_j \} \) which converges uniformly in the \( C^2(\overline{\Omega}_{a+1}) \) norm to a function \( u^i \) on \( \overline{\Omega}_{a+1} \), \( i = 1, 2, \ldots \). Obviously, \( \bar{u}(x) = u^i(x) \) if \( x \in \overline{\Omega}_{a+1} \). For any bounded domain \( M \subset \Omega \), \( M \subset \Omega_{a+i} \) for some integer \( i \), and hence the diagonal sequence \( \{ u_j^i \} \) converges uniformly in the \( C^2(M) \) norm to \( u^i \) on \( \overline{M} \). In particular \( u^i \) and \( Lu^i \) converge uniformly on \( \overline{M} \) to \( \bar{u} \) and \( Lu \), respectively. It is also clear that \( B u^i \) converges to \( B \bar{u} \) for each \( x \in \partial \Omega \). Since \( L u^i(x) = f(x, u^i(x), \nabla u^i(x)) \) by Lemma 3.1, it follows that \( \bar{u} \) satisfies \( \bar{L}u^i \) in \( \Omega \), \( B \bar{u}^i = g \) on \( \partial \Omega \), and is of class \( C^2(\overline{M}) \). That \( u \in C^{2+\alpha}(\overline{M}) \) follows from a standard regularity argument based on Schauder estimates. Since \( \underline{u}_0(x) \leq u^i(x) \leq \underline{u}_0(x) \) for all \( x \in \overline{\Omega} \) and for each \( j = 1, 2, \ldots \), the function \( \bar{u} \) also satisfies \( \underline{u}_0(x) \leq \bar{u}(x) \leq \underline{u}_0(x) \) on \( \overline{\Omega} \).

Finally, let \( u \) be any other solution of the BVP (2.1) satisfying \( v_0 \leq u \leq u_0 \) on \( \overline{\Omega} \). Then for any \( j \), \( u_0 \) is a supersolution and \( u \) is a subsolution of the BVP (3.1), and known results for bounded domains imply the existence of a solution \( U \) of (3.1) such that

\[
\underline{u}_0(x) \leq U(x) \leq \overline{u}_0(x) \quad \text{on } \overline{\Omega}_{a+j}.
\]

But \( u_0 \), by construction, is a maximal solution of the BVP (3.1). Hence \( u(x) \leq U(x) \leq u_0(x) \) on \( \overline{\Omega}_{a+j} \) and, consequently, \( u(x) \leq \bar{u}(x) \leq u_0(x) \) on \( \overline{\Omega} \). Corresponding statements for \( \underline{u}(x) \) follow from similar arguments. This completes the proof of Theorem 3.4.

Corollary 3.5. Assume \( f, g, \) and \( b_0 \) satisfy Assumptions A. Furthermore, assume that \( f(x, 0, 0) \geq 0 \) in \( \Omega \), and \( g(x) \geq 0 \) on \( \partial \Omega \). Then, a
necessary and sufficient condition for the existence of a non-negative solution of (2.1) is the existence of a non-negative supersolution \( u_0 \) of (2.1).

The proof follows easily from Theorem 3.4 by taking \( v_0 \equiv 0 \) on \( \partial \Omega \).

**Corollary 3.6.** Assume \( f, g, \) and \( b_0 \) satisfy Assumptions A. Furthermore, assume that \( f(x, u, p) > 0 \) for all \( x \in \Omega, u > 0 \), and \( p \in \mathbb{R}^n \); and \( g(x) > 0 \) on \( \partial \Omega \) with the strict inequality holding for at least one point \( x \in \partial \Omega \). Then, a necessary and sufficient condition for the existence of a solution \( u \) of (2.1) satisfying \( u(x) > 0 \) in \( \Omega \) is the existence of a non-negative supersolution \( u_0 \) of (2.1).

**Proof.** Let \( u \) be the non-negative solution of (2.1) in \( \Omega \) implied by Corollary 3.5. Since \( Lu = f(x, u, \nabla u) > 0 \) in \( \Omega_{a+i} \) for every positive integer \( i \), and since \( Bu = g > 0 \) on \( \partial \Omega \), and \( u = u_0 > 0 \) on \( \Sigma_{a+i} \) with the strict inequality holding for at least one point \( x \in \partial \Omega_{a+i} \), the maximum principle for elliptic equations implies that \( u(x) > 0 \) throughout \( \Omega_{a+i} \), and since \( i \) is arbitrary, \( u(x) > 0 \) throughout \( \Omega \).

### 4. Existence and Uniqueness Criteria

In this section we derive sufficient conditions on the coefficient \( f \) and the data \( g \) which guarantee the existence of solutions of (2.1) satisfying certain conditions at infinity. Specifically, we derive criteria for the existence of bounded solutions, and solutions tending to zero at infinity. It will be shown that a solution tending to zero at infinity is unique when \( f(x, u, p) \) is non-increasing with respect to \( u \) and satisfies a local Lipschitz condition with respect to \( p \).

Let \( A(x) \) and \( Q(x) \) denote the functions defined by

\[
A(x) = |x|^2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} a_{ii}/Q,
\]

where

\[
Q = \left( \sum_{i,j=1}^{n} a_{ij} x_i x_j \right)/|x|^2.
\]

It is clear that \( 1 < A(x) < \infty \) for all \( x \in \Omega \).

**Lemma 4.1.** Let \( G \) be a bounded subdomain of \( \Omega \). Assume \( f(x, u, p) \) is non-increasing in \( u \) for each \( x \in G, p \in \mathbb{R}^n \), and satisfies a Lipschitz
condition with respect to \( p \) on each compact subset of \( \overline{G} \times R \times R^n \). If \( u, v \) are functions of class \( C^2(\overline{G}) \) satisfying

1. \( Lu \leq f(x, u, \nabla u) \quad \text{in} \ G; \)
2. \( Lv \geq f(x, v, \nabla v) \quad \text{in} \ G; \)
\[ u \leq v + M \quad \text{on} \ \partial G, \]

where \( M \geq 0 \) is a non-negative constant, then
\[ u \leq v + M \quad \text{on} \ \overline{G}. \]

Proof. We consider first the special case where one of the inequalities (1) and (2) above is a strict inequality. It is enough to assume \( M = 0 \) since
\[ L(v + M) = Lv \geq f(x, v, \nabla v) \geq f(x, v + M, \nabla(v + M)). \]

If \( \psi = v - u \geq 0 \) on \( \partial G \) and \( \psi < 0 \) at some point in \( G \), then \( \psi \) has a negative minimum at some point \( x_0 \in G \). Since \( \nabla u(x_0) = \nabla v(x_0) \), we obtain
\[ \nabla \psi(x_0) = f(x, v(x_0), \nabla v(x_0)) - f(x, u(x_0), \nabla u(x_0)) \]
\[ = f(x, v(x_0), \nabla u(x_0)) - f(x, u(x_0), \nabla u(x_0)) \]
\[ \geq 0. \]

But \( L\psi(x_0) \leq 0 \) since \( x_0 \) is a point of minimum of \( \psi \) in \( G \). This contradiction shows that \( u(x) = v(x) = \psi(x) \geq 0 \) on \( \overline{G} \). We consider next the general case. Assume \( u \leq v \) on \( \partial G \) but \( u(x) > v(x) \) for some point \( x \in G \). Then \( \varphi(x) = u(x) - v(x) \) attains its maximum at some point \( x_0 \in G \). Let \( \varepsilon = \max_{x \in G} \varphi(x) \). Let \( k \) be the Lipschitz constant of \( f \) on the set \( \{(x, s, p) : |u(x) - s| \leq 1, |
abla u(x) - p| \leq 1, x \in \overline{G}\} \). Choose \( K > 0 \) large enough such that \( KQ(x) \geq k \) for all \( x \in \overline{G} \). Let \( R > 0 \) be a positive constant such that \( \overline{G} \subset \{x : |x| \leq R\} \). Let \( v_\varepsilon(r) \) be a solution of the equation
\[ v''(r) = (1 + K) v'(r), \quad 0 \leq r \leq R, \quad (4.1) \]
satisfying \( 0 < v'_\varepsilon(r) \leq 1, \) and \( - \text{Minimum} \ [1, \varepsilon/2] \leq v_\varepsilon(r) \leq 0 \) for all \( 0 \leq r \leq R \). Let \( \phi(x) = u(x) + v_\varepsilon(|x|), x \in \overline{G} \). Then \( u(x) - \varepsilon/2 \leq \phi(x) \leq u(x) \) for all \( x \in \overline{G} \), and
\[ L\phi = Lu + Lv_\varepsilon \]
\[ \leq f(x, u, \nabla u) - Q[v''_\varepsilon + v'_\varepsilon(A(x) - 1)/r] \]
\[ \leq f(x, u, \nabla u) - Q(K + 1)v'_\varepsilon \]
\[ \leq f(x, u, \nabla u) - kv'_\varepsilon - Qu'_\varepsilon. \quad (4.2) \]
Since \( k \) is a Lipschitz constant of \( f \), \( f(x, u, \nabla \phi) - f(x, u, \nabla u) \geq -kv' \). We then conclude from (4.2) that

\[
L \phi \leq f(x, u, \nabla \phi) - Qv' < f(x, \phi, \nabla \phi) \tag{4.3}
\]

where the last inequality follows from the monotonicity assumption of \( f \), and the positivity of \( Q(x) \) and \( v'(x) \) on \( G \). Let \( \phi_1(x) = v(x) + \varepsilon/3 \). Then

\[
\phi(x) = u(x) + v(|x|) \leq u(x) \leq \phi_1(x) \quad \text{on } \partial G. \tag{4.4}
\]

Since \( u(x_0) - v(x_0) = \varepsilon \),

\[
\phi(x_0) = v(x_0) + \varepsilon + v_0(|x_0|) \geq v(x_0) + \varepsilon/2 > \phi_1(x_0), \tag{4.5}
\]

where the last inequality follows from (4.1).

The function \( \phi_1 \) also satisfies the inequality

\[
L \phi_1 = L v \geq f(x, v, \nabla v) = f(x, v, \nabla \phi_1) \geq f(x, \phi_1, \nabla \phi_1) \tag{4.6}
\]

by the monotonicity of \( f \).

From (4.3), (4.4), (4.6), and the first part of the proof, we conclude that \( \phi(x) \leq \phi_1(x) \) on \( \bar{G} \). This contradicts (4.5) and completes the proof of Lemma 4.1.

**Theorem 4.2.** Assume the functions \( f \) and \( g \) satisfy conditions \( A, f(x, u, p) \) is non-increasing in \( u \) for each \( x \in \bar{\Omega}, p \in \mathbb{R}^n \), and satisfies a Lipschitz condition with respect to \( p \) on each compact subset of \( \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \). Then the Dirichlet problem

\[
Lu = f(x, u, \nabla u) \quad \text{in } \Omega, \quad Bu = g \quad \text{on } \partial \Omega; \tag{4.7}
\]

\[
\lim_{|x| \to \infty} u(x) = 0.
\]

has a unique solution provided there exists a subsolution \( v_0 \) and a supersolution \( u_0 \) satisfying

\[
v_0(x) \leq u_0(x) \quad \text{on } \bar{\Omega},
\]

\[
\lim_{|x| \to \infty} [u_0(x) - v_0(x)] = 0.
\]

**Proof:** The existence of a solution of (4.7) follows from Theorem 3.4. To prove uniqueness, assume \( u_1, u_2 \) are two such solutions. For any \( \varepsilon > 0 \) there
exists \( R > 0 \) such that \(|u_1(x) - u_2(x)| < \varepsilon\) for all \( x \in \Omega \cap \{ x : |x| \geq R \} \). Let 
\( G = \{ x : |x| \leq R \} \cap \Omega \) and apply Lemma 4.1 to conclude that 
\(|u_1(x) - u_2(x)| < \varepsilon\) for all \( x \in G \), and hence for all \( x \in \Omega \). Since \( \varepsilon > 0 \) is arbitrary, \( u_1 = u_2 \) on \( \overline{\Omega} \).

As a simple application of the above results, consider the BVP

\[-\Delta u = f(x, u, \nabla u), \quad |x| > 1,\]

\[u = g, \quad |x| = 1,\]

where \( f, g \) satisfy Assumptions A and B.

**Corollary 4.3.** The BVP (4.8) has a maximal solution \( \hat{u}(x) \) and a minimal solution \( u(x) \) satisfying

\[0 < u(x) \leq \hat{u}(x) \leq C|x|^{2-n-\varepsilon}, \quad |x| > 1,\]

if the following conditions hold

1. \( f(x, 0, 0) \geq 0, \quad |x| \geq 1; \)
2. \( g(x) > 0, \quad |x| = 1; \)
3. \( \sup_{|x| \geq 1} f(x, C|x|^{2-n-\varepsilon}, C^2(2-n+\varepsilon)^2 |x|^{2(1-n+\varepsilon)}) \leq \varepsilon C(n-2-\varepsilon)|x|^{\varepsilon-n} \)

for all \( |x| \geq 1 \), where \( C = \max_{|x|=1} g(x) \). If \( \varepsilon < n-2 \), and \( f(x, u, p) \) is non-increasing in \( u \) for each \( |x| \geq 1 \), \( p \in \mathbb{R}^n \), and satisfies a Lipschitz condition with respect to \( p \) on each compact subset of \( \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \), then (4.8) has a unique solution \( u(x) \) satisfying

\[0 < u(x) \leq C|x|^{2-n+\varepsilon}.\]

**Proof.** Let \( u_0(x) = C|x|^{2-n+\varepsilon} \). Then condition (3) implies that \( u_0 \) is a supersolution of the BVP 4.8. The conclusions then follow from Theorem 3.4 and 4.2.

The above corollary applies, in particular, to the problem

\[-\Delta u = p(x)u + K|\nabla u|^2, \quad |x| > 1,\]

\[u = g \quad |x| = 1,\]

where \( K \) is a constant, and \( p(x) \) is continuous and of class \( C^\omega[a, b] \) for all \( b \geq a \). In this case conditions (3) of Corollary 4.3 becomes

\[\sup_{|x|=r} p(x) \leq \varepsilon(n-2-\varepsilon)r^2 - KC(n-2-\varepsilon)^2 r^{-n+\varepsilon}.\]
Let $\varepsilon = (n - 2)/2$. Then condition (3) applied to BVP (4.9) becomes

$$\sup_{|x|=r} p(x) \leq \frac{(n - 2)^2}{4} [r^{-2} - KCr^{-(n-2\varepsilon)}].$$

(4.10)

In the linear case $K = 0$, (4.9) is quite sharp in view of the known Hille–Kneser criterion.

5. Dirichlet Problems in Unbounded Domain

For the case when $\Omega$ is not an exterior domain, we require that $\Omega$ allows the following decomposition:

There exists a sequence of bounded domains $\Omega_n$, $n = 1, 2,\ldots$, with boundaries $\partial \Omega_n$ of class $C^{2+\alpha}$ such that

1. $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n = 1, 2,\ldots$, and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$,

2. $x \in \partial \Omega$ and $|x| < n$ implies that $x \in \partial \Omega_n$.

Such domains include cylindrical and conical domains.

Consider the boundary value problem

$$Lu = f(x, u, \nabla u) = 0 \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial \Omega,$$

(5.1)

where $f$, $g$ satisfy the conditions below

(a) $f$ satisfies Assumptions A(a), (b);

(b) $g$ is a real-valued function on $\overline{\Omega}$ of class $C^{2+\alpha}(\overline{\Omega})$.

By replacing the domains $\Omega_{a+j}$ in the proofs of Lemmas 3.1, 3.2, 3.3, by the domains $\Omega_j$, $j = 1, 2,\ldots$, we obtain

THEOREM 5.1. Let $f$, $g$, $\Omega$ satisfy the conditions of Section 5. If there exist a supersolution $u_0$ and a subsolution $v_0$ of the BVP (5.1) such that $v_0 \leq u_0$ on $\overline{\Omega}$, then there exist a maximal solution $\hat{u}$ and a minimal solution $\hat{v}$ of (5.1), $v_0(x) \leq u(x) \leq \hat{u}(x) \leq u_0(x)$ on $\overline{\Omega}$; i.e., if $u$ is any solution of (5.1), with $v_0(x) \leq u(x) \leq u_0(x)$ on $\overline{\Omega}$, then $u(x) \leq u(x) \leq \hat{u}(x)$ on $\overline{\Omega}$.

The proof is very similar to the proof of Theorem 3.4. The details are left to the reader.

Analogues of the results in Section 3 can be easily written down.
REFERENCES


