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Γ^+-STRUCTURES—I: A FREE GROUP FUNCTOR FOR STABLE HOMOTOPY THEORY

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§0. INTRODUCTION

DURING the 1950's a number of ideas were developed relating topological groups and monoids to loop spaces. James [8] showed that the free monoid on a pointed connected space serves as a model for the loop space of the suspension of the space. Then Milnor [16] and Kan [9] showed that by using the free group the restriction to connected spaces is overcome. Milnor also showed [18] that a topological group is a loop space, and Kan [9] (also Milnor [17]) showed that a loop space is always homotopy equivalent to a topological group. Later, Dold and Lashof [5] showed that here "group" could be replaced by "monoid with monoid of components a group".

We have here been working in an appropriate topological category (see note at the end of §4 and remarks at the beginning of §2). We shall continue to do so.

This is the first of three papers in which we present the analogous theory for infinite loop spaces. We define a functor Γ^+ from pointed spaces to topological monoids whose rôle is analogous to that of the free monoid functor in the above. In particular, corresponding to James' theorem we have

THEOREM A. *If X is a pointed connected space of the homotopy type of a CW-complex, then Γ^+X is naturally homotopy equivalent to $\Omega^\infty\Sigma^\infty X$.*

Here $\Omega^\infty\Sigma^\infty X$ means $\varinjlim_n \Omega^n\Sigma^n X$ where the maps of the directed system are given by

$$\Omega^n i_{\Sigma^n X} : \Omega^n\Sigma^n X \longrightarrow \Omega^n\Omega\Sigma\Sigma^n X = \Omega^{n+1}\Sigma^{n+1} X$$

where $i_X : X \to \Omega\Sigma X$ is the adjoint of the identity map of the suspension $\Sigma X \to \Sigma X$.

Again, the connectivity restriction may be avoided by using the corresponding group functor. Thus we define a functor Γ from pointed spaces to topological groups whose rôle is analogous to that of the free group. Corresponding to the Milnor–Kan theorem we obtain

THEOREM B. *If X is a pointed space of the homotopy type of a CW-complex, then ΓX is naturally homotopy equivalent to $\Omega^\infty\Sigma^\infty X$.*

The free monoid and free group on X appear naturally as a submonoid of Γ^+X and subgroup of ΓX, respectively.

The proof of Theorem A has been outlined in [2]. A proof of Theorem B is given here. Theorem A follows from Theorem B since the natural inclusion $\Gamma^+X \to \Gamma X$ is a homotopy equivalence if X is connected (Corollary 5.4).

In the second paper we shall give results involving Γ^+ and infinite loop spaces analogous to those of Dold and Lashof on monoids and loop spaces. In the third paper we prove that $\Omega^\infty\Sigma^\infty X$ is stably homotopy equivalent to a bouquet of spaces. Most of the material in these papers is in [7].

The basic idea in what follows is implicit in the work of Dyer and Lashof on homology operations on infinite loop spaces [6]. This work may be thought of as an appendix to theirs.

Milgram [15] has also given a model for $\Omega^\infty\Sigma^\infty X$ but we have not considered the relationship between his model and ΓX. Also, the related work of May and Segal must be mentioned. Segal has obtained (as [23; Proposition 3.5]) Theorem B in the case of $X = S^0$ (the most interesting case) by quite different means. May's work is perhaps closer to ours. His original paper ([13]) was restricted to connected spaces, but also dealt with nth loop spaces for all nonnegative integers n. But he has now removed [14] the connectivity condition in the case of infinite loop spaces. Thus he has obtained a result [14; Theorem 3.7] which is equivalent to Theorem B. However, May's approach and ours are at opposite extremes: he works in as general a setting as possible, whereas we work with one particular example of his "E_∞-operads" and we show [3] that this is sufficient.

The plan of this paper is as follows. We work in the category of simplical sets. The reasons for this are referred to in §2 which also contains a few well-known results which we need later. This is preceded by §1 containing notation in finite set theory which is used in subsequent definitions. In §3 the functor Γ^+ is defined and discussed. The functor Γ is defined and Theorem B proved in §4, subject to certain lemmata. These lemmata are proved in the last three sections using a theorem on the homology of the universal group of a monoid which is given in §5. Lemma 7.1 proved in §8 may be of interest in its own right.

§1. PRELIMINARIES: THE FINITE PERMUTATION GROUPS

The basic construction (of Γ^+X for a pointed space X) involves the finite permutation groups. In order to deal with this we need some notation which we present in this section. The results stated are all trivial and are stated explicitly simply for clarity.

Definition 1.1. For n a positive integer we write S_n for the group of permutations of the set $\mathbf{n} = \{1, 2, \ldots, n\}$ written on the left, i.e.

$$(\sigma.\tau)(i) = \sigma(\tau(i)) \quad \text{for} \quad \sigma, \tau \in S_n,\ i \in \mathbf{n}.$$

When necessary, we regard S_0 as the trivial group.

Definition 1.2. For m and n positive integers, we write C^n_m for the set of strictly monotonically increasing maps $\mathbf{m} \to \mathbf{n}$: thus C^n_n has only one element and, if $m > n$, C^n_m is the empty set. We note that C^n_m may be identified canonically with the set of subsets of $\mathbf{n}$ of order m, i.e. given $\alpha \in C^n_m$, this corresponds to $\{\alpha(1), \ldots, \alpha(m)\} \subset \mathbf{n}$.

Definition 1.3. $\alpha \in C^n_m$ induces a group monomorphism $\alpha_* : S_m \to S_n$ defined by

$$\alpha_*(\sigma)(\alpha(i)) = \alpha(\sigma(i)),$$
$$\alpha_*(\sigma)(j) \quad = j,$$

for $i \in \mathbf{m}$, $j \in \mathbf{n} - \alpha(\mathbf{m})$, $\sigma \in S_m$.

PROPOSITION 1.4. *Suppose* $\alpha \in C^n_m$, $\beta \in C^p_u$. *Then*

$$(\beta.\,\alpha)_* = \beta_*.\,\alpha_*: S_m \to S_p.$$

Definition 1.5. Suppose $\sigma \in S_n$ and $\alpha \in C^n_m$. Then the image of the map $\sigma.\,\alpha: \mathbf{m} \to \mathbf{n}$ is a subset of $\mathbf{n}$ of order m and so, by the identification of Definition 1.2, corresponds to an element of C^n_m which we write $\sigma_*(\alpha)$. In general $\sigma_*(\alpha) \neq \sigma.\alpha$ since $\sigma.\alpha \notin C^n_m$. However, there is a unique map $\alpha^*(\sigma) \in S_m$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{m} \arrow[r, "\alpha"] \arrow[d, "\alpha^*(\sigma)"'] & \mathbf{n} \arrow[d, "\sigma"] \\
\mathbf{m} \arrow[r, "\sigma_*(\alpha)"] & \mathbf{n}
\end{tikzcd}$$

Thus we have defined a *reduction map* $\alpha^*: S_n \to S_m$ for $\alpha \in C^n_m$.

PROPOSITION 1.6. *Suppose* $\alpha \in C^n_m$, $\beta \in C^p_n$. *Then*

$$(\beta.\alpha)^* = \alpha^*.\beta^*: S_p \to S_m.$$

α^* is not in general a homomorphism, rather the following.

PROPOSITION 1.7. *Suppose* $\alpha \in C^n_m$, $\sigma \in S_n$, $\tau \in S_n$. *Then*

$$\alpha^*(\sigma.\tau) = (\tau_*(\alpha))^*(\sigma).\alpha^*(\tau) \in S_m.$$

COROLLARY 1.8. *Suppose* $\alpha \in C^n_m$, $\sigma \in S_n$, $\nu \in S_m$. *Then*

$$\alpha^*(\sigma.\alpha_*(\nu)) = \alpha^*(\sigma).\nu.$$

This corollary simply says that if we define a right action of S_m on S_n by composing the product in S_n with $(1 \times \alpha_*)$: $S_n \times S_m \to S_n \times S_n$, then α^* is a right S_m-map.

Example 1.9. Let ρ: $\mathbf{n} - 1 \to \mathbf{n}$ be the inclusion map $\rho(i) = i$ for all $i \in \mathbf{n} - 1$. Then we shall think of the monomorphism ρ_* as an inclusion and omit the symbol, i.e. write $\rho_*(\sigma) = \sigma \in S_n$ for $\sigma \in S_{n-1}$, thus $S_{n-1} \subset S_n$. We shall use the notation R: $S_n \to S_{n-1}$ for ρ^*.

Any element of C^n_m can be expressed as the composition of an element of S_n with *n-m* maps like ρ. Thus, although for convenience we use the general reduction maps in our definitions, for computations it is sufficient to know R.

PROPOSITION 1.10. R: $S_n \to S_{n-1}$ *is characterised by*

(a) *it is a right* S_{n-1}*-map*, i.e. $R(\sigma.\tau) = R(\sigma).\tau$ *for* $\sigma \in S_n$, $\tau \in S_{n-1} \subset S_n$.

(b) $R(\tau_{k,n}) = 1$ *for* $1 \leqslant k \leqslant n$, *where* $\tau_{k,n} = (k, k+1, \ldots, n-1, n) \in S_n$.

Finally, in this section, we give some notation involving the action of S_n on sets.

Definition 1.11. Given a set X, the group S_n acts on the right of the n-fold cartesian product X^n by

$$(x_1, x_2, \ldots, x_n).\ \sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for $x_i \in X$, $\sigma \in S_n$.

Similarly, $\alpha \in C_m^n$ induces a map $\alpha^*\colon X^n \to X^m$ by

$$\alpha^*(x_1, \ldots, x_n) = (x_{\alpha(1)}, \ldots, x_{\alpha(n)}),$$

for $x_i \in X$. Suppose further that X has a base point, $*$. Then we say that α is *entire* for $(x_1, \ldots, x_n)$ if the coordinates it omits are the base point, i.e. $x_i = *$ if $i \in \mathbf{n} - \alpha(\mathbf{m})$. In particular, $\rho\colon \mathbf{n} - 1 \to \mathbf{n}$ is entire for points of $(X^{n-1} \times \{*\} \subset X^n$ and $R(x_1, \ldots, x_{n-1}, *) = (x_1, \ldots, x_{n-1})$.

These definitions are related as follows (cf. Proposition 1.7).

PROPOSITION 1.12. *Suppose $\alpha \in C_m^n$, $\sigma \in S_n$, $\mathbf{x} \in X^n$. Then*

$$\alpha^*(\mathbf{x}.\sigma) = (\sigma_*(\alpha))^*(\mathbf{x}).\alpha^*(\sigma),$$

and α is entire for $\mathbf{x}.\sigma$ if and only if $\sigma_*(\alpha)$ is entire for $\mathbf{x}$.

§2. PRELIMINARIES: THE SIMPLICIAL CATEGORY

In this section we collect together a few well-known ideas which we shall need.

Firstly, let us clarify our category of operations. We shall work in the categories of pointed simplicial sets, simplicial monoids and simplicial groups. The basic ideas are collected together conveniently in [12]. We shall follow the notations of that book for face and degeneracy maps. By a pointed simplicial set we mean a simplicial pointed set, i.e. for each non-negative integer n the set of n-simplices has a base point and these are preserved by the face and degeneracy maps. We denote all these base points by the same symbol, $*$, and refer to "the base point" of the simplicial set.

This choice of categories is for simplicity. When working in the category of pointed topological spaces of the homotopy type of a CW-complex, consideration has always to be given as to whether a given construction remains within the category. For example products of CW-complexes must be given the compactly generated topology. These considerations are avoided by working with simplicial sets. The results obtained are easily translated into the topological category by using the realisation and singular complex functors and the equivalence of categories they induce (see [19], [12; 16.6]). In fact, in an attempt to avoid excessive pedantry, we shall sometimes use topological language (space, subspace) when referring to simplicial objects.

The following result will be useful.

LEMMA 2.1. *Suppose X is a pointed simplicial set having additional pointed structure maps $\sigma\colon X_n \to X_{n+1}$ for all $n \geqslant 0$ such that*

$\sigma.\sigma = s_0.\sigma\colon X_n \to X_{n+2}$ *for* $n \geqslant 0$,
$\sigma.s_i = s_{i+1}.\sigma\colon X_n \to X_{n+2}$ *for* $n \geqslant 0,\ 0 \leqslant i \leqslant n$,
$\partial_0.\sigma = 1\colon X_n \to X_n$ *for* $n \geqslant 0$,
$\partial_i.\sigma = \sigma.\partial_{i-1}\colon X_n \to X_n$ *for* $n \geqslant 1,\ 1 \leqslant i \leqslant n+1$.

Then the discrete pointed simplicial set $\hat{X}$, with $\hat{X}_n = \partial_1\sigma(X_0)$, all structure maps equal to the identity map and the same base point as X, is a deformation retract of X.

Proof. In the simplicial set category the unit interval, I, has two vertices, 0 and 1, and one non-degenerate 1-simplex, e, with $\partial_1 e = 0$, $\partial_0 e = 1$. We define a map $h\colon X \times I \to X$ by

$$\begin{aligned} h(x, 0) &= s_0{}^n\partial_1\sigma\partial_0{}^n(x),\\ h(x, 1) &= x,\\ h(x, s_0{}^t s_1{}^{n-t-1}e) &= \sigma^{t+1}\,\partial^{t+1}(x),\ 0 \leqslant t \leqslant n-1, \end{aligned}$$

for $x \in x_n$. Here 0 and 1 denote degeneracies of the vertices. It is mechanical to check that this is a pointed simplicial map. The inclusion $i\colon \hat{X} \to X$ is given by $i(x) = s_0{}^n(x)$ for $x \in \hat{X}_n = \partial_1\sigma(X_0) \subset X_0$. Thus if $r\colon X \to \hat{X}$ is given by $r(x) = \partial_1\sigma\partial_0{}^n(x)$ for $x \in X_n$, $r.i = 1\colon \hat{X} \to \hat{X}$ and h provides a pointed homotopy between $i.r$ and $1\colon X \to X$ showing that r is a pointed deformation retraction.

Note. The conditions on the maps σ require that they behave like extra face maps s_{-1}.

Next we recall MacLane's definition for the classifying space of a group.

Definition 2.2. Given a discrete group G, we define a simplicial group WG by

$$(WG)_n = G^{n+1} = \{\langle g_0, g_1, \ldots, g_n\rangle \mid g_i \in G\}$$

$$\partial_i\langle g_0, \ldots, g_n\rangle = \langle g_0, \ldots, \hat{g}_i, \ldots, g_n\rangle \quad \text{(i.e. omit } g_i\text{)}$$

$$s_i\langle g_0, \ldots, g_n\rangle = \langle g_0, \ldots, g_i, g_i, \ldots, g_n\rangle \quad \text{(i.e. repeat } g_i\text{)}$$

for $n \geqslant 0$, $0 \leqslant i \leqslant n$. The usual direct sum product in G^{n+1} makes WG a simplicial group. G acts freely on WG by

$$\langle g_0, \ldots, g_n\rangle\cdot g = \langle g_0\cdot g, \ldots, g_k\cdot g\rangle$$

for g_i, $g \in G$. It is straightforward to check that the necessary relations hold.

PROPOSITION 2.3. *The pointed simplicial set WG is contractible.*

Proof The identity of WG serves as a base point. For each $n \geqslant 0$ we define a pointed map $\sigma\colon (WG)_n \to (WG)_{n+1}$ by

$$\sigma\langle g_0, \ldots, g_n\rangle = \langle 1, g_0, \ldots, g_n\rangle$$

where 1 is the unit of G. These maps satisfy the relations of Lemma 2.1 hence giving the result, since $\partial_1\sigma(WG)_0 = \{\langle 1\rangle\}$.

Thus $\overline{W}G$ obtained from WG by factoring out by the action of G is a classifying space for G, i.e. an Eilenberg–MacLane space $K(G, 1)$.

We note that the discrete space G occurs naturally as a subspace of WG, namely as the set of vertices and their degeneracies.

Note 2.4. In fact the above definition gives a functor W from pointed sets to pointed simplicial sets which respects products, i.e. $W(A_1 \times A_2) = WA_1 \times WA_2$ for sets A_1, A_2. So a map $f\colon A_1 \times \ldots \times A_n \to A$ gives rise to a map $Wf\colon WA_1 \times \ldots \times WA_n \to WA$ which we shall also denote by f when there is no risk of confusion. In particular, given $\alpha \in C_m^n$ we have maps $\alpha_*\colon WS_m \to WS_n$ and $\alpha^*\colon WS_n \to WS_m$ coming from those of Definitions 1.3 and 1.5 and the results of §1 imply corresponding results for these maps.

Finally we consider the homotopy theory of simplicial groups.

PROPOSITION 2.5. (J. C. Moore [12; 17.3 and 17.4]) *Suppose that G is a simplicial group. Let* $\bar{G}_n = \bigcap_{i=1}^{n} ker\,[\partial_i : G_n \to G_{n-1}]$ *for* $n \geqslant 1$, $\bar{G}_0 = G_0$. *Then*

(i) $\partial_0(\bar{G}_n) \subset \bar{G}_{n-1}$,
(ii) $\partial_0(\bar{G}_{n+1}) \subset ker\,[\partial_0 : \bar{G}_n \to \bar{G}_{n-1}]$,
(iii) $\partial_0(\bar{G}_n)$ *is a normal subgroup of* $\bar{G}_{n-1}$,

for $n \geqslant 1$, *i.e.* $(\bar{G}, \partial_0)$ *is a non-abelian chain complex. There is a natural isomorphism* $H_q(\bar{G}) \cong \pi_q(G)$ *for all* $q \geqslant 0$.

COROLLARY 2.6. *A short exact sequence of simplicial groups*

$$1 \to G' \to G \to G'' \to 1$$

gives rise to a natural long exact sequence of homotopy groups

$$\dots \to \pi_{i+1}(G'') \to \pi_i(G') \to \pi_i(G) \to \pi_i(G'') \to \\ \pi_{i-1}(G') \to \dots \to \pi_0(G') \to \pi_0(G) \to \pi_0(G'') \to 1.$$

Proof. This is simply the homology long exact sequence of a short exact sequence of chain complexes.

We shall use this corollary frequently.

§3. THE FREE MONOID FUNCTOR Γ^+

In this section we define a functor Γ^+ from the category of pointed simplicial sets to the category of simplicial monoids. We show that, for all simplicial sets X, $\Gamma^+ X$ is a free monoid.

Definition 3.1. Suppose X is a pointed simplicial set. Then the following relations generate an equivalence relation on the disjoint union $\mathscr{U}(X) = \coprod_{n \geqslant 0} WS_n \times X^n$.

$$\text{(a)}\ (w, \mathbf{x}) \sim (w.\sigma, \mathbf{x}.\sigma),$$
$$\text{(b)}\ (w, \mathbf{x}) \sim (\alpha^*(w), \alpha^*(\mathbf{x})),$$

for $w \in WS_n$, $\mathbf{x} \in X^n$, $\sigma \in S_n$, $\alpha \in C^n_m$ which is entire for $\mathbf{x}$, $n \geqslant m \geqslant 0$. On factoring out by this equivalence relation, which respects the face and degeneracy maps, we obtain a simplicial set which we write $\Gamma^+ X$. We denote the equivalence class of $(w, \mathbf{x}) \in WS_n \times X^n$ as $[w, \mathbf{x}] \in \Gamma^+ X$. Clearly $(1, \varnothing) \in WS_0 \times X^0$ provides a canonical base point $[1, \varnothing] \in \Gamma^+ X$ which we write 1 (in anticipation of the monoid structure).

Γ^+ is a functor from pointed simplicial sets to pointed simplicial sets, for if $f: X \to Y$ is a pointed simplicial map, then $\Gamma^+ f: \Gamma^+ X \to \Gamma^+ Y$ given by

$$\Gamma^+ f[w, x_1, \dots, x_n] = [w, f(x_1), \dots, f(x_n)],$$

$w \in WS_n$, $x_i \in X$, $n \geqslant 0$, is a well-defined pointed simplicial map with the required properties.

Remarks 3.2. By the remarks in Example 1.9 we may replace (b) in the above definition by

$$(b')(w, x_1, \ldots, x_{n-1}, *) \sim (R(w), x_1, \ldots, x_{n-1}),$$

for $w \in WS_n$, $x_i \in X$, $n \leqslant 1$.

We have remarked in §2 that the discrete space S_n occurs naturally as a subspace of WS_n, and so the disjoint union $\coprod_{n\geqslant 0} S_n \times X^n$ as a subspace of $\mathscr{U}(X)$. The image of $\coprod_{n\geqslant 0} S_n \times X^n$ in Γ^+X is simply the free monoid on the pointed space X which we write F^+X, i.e. the reduced product space X_∞ of James [8]. This may be thought of as motivating the equivalance relation of Definition 3.1.

We now define two natural transformations involving Γ^+ leading to the definition of a Γ^+-structure on a space.

Definition 3.3. Given a pointed space X, there is a natural embedding $\iota_X: X \to \Gamma^+X$ given by $\iota_X(x) = [1,x]$ for all $x \in X$. Here $1 \in S_1 \subset WS_1$.

When we refer to "the pair (Γ^+X, X)" we mean X embedded in Γ^+X by ι_X.

Secondly, we recall that Γ^+X is intended to serve as a model for $\Omega^\infty\Sigma^\infty x$. There is a natural map $\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty X \to \Omega^\infty\Sigma^\infty X$ defined using the evaluation $\Sigma\Omega A \to A$. There is an obvious candidate as a model for this map which we now give. To motivate the details we remark that the obvious map $F^+F^+X \to F^+X$ (of which the following is an extension) provides a model for $\Omega\Sigma\Omega\Sigma X \to \Omega\Sigma X$ in the theory of James.

Definition 3.4. Let k and n be non-negative integers. For integers i such that $1 \leqslant i \leqslant k$ we define $\lambda_i : \mathbf{n} \to \mathbf{kn}$ by

$$\lambda_i(j) = (i-1)n + j$$

for $j \in \mathbf{n}$, i.e. λ_i maps $\mathbf{n}$ to the ith block of n elements of $\mathbf{kn}$. By Definition 1.3 and Note 2.4 this gives rise to a homomorphism

$$(\lambda_i)_* : WS_n \to WS_{kn}$$

which we also denote by λ_i with no risk of confusion.

Also, given $\sigma \in S_k$ we define $\mu(\sigma) \in S_{kn}$ as the element which permutes k blocks of n elements by σ, i.e.

$$\mu(\sigma)((i-1)n + j) = (\sigma(i) - 1)n + j,$$

for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$. This defines a homomorphism $S_k \to S_{kn}$ and so a homomorphism

$$\mu: WS_k \to WS_{kn}.$$

With this notation we may define a map

$$\hbar_X : \coprod_{k\geqslant 0} \coprod_{n\geqslant 0} WS_k \times (WS_n \times X^n)^k \to \mathscr{U}(X).$$

Suppose $(w, \boldsymbol{\alpha}) = (w, \alpha_1, \ldots, \alpha_k) \in WS_k \times (WS_n \times X^n)^k$ where $\alpha_i = (w_i, \mathbf{x}^i) \in WS_n \times X^n$. Then we define $\hbar_X(w, \boldsymbol{\alpha}) \in WS_{kn} \times X^{kn}$ by

$$\hbar_X(w, \boldsymbol{\alpha}) = (\mu(w).\lambda_1(w_1)\ldots\lambda_k(w_k), \mathbf{x}^1, \ldots, \mathbf{x}^k)$$

where we identify $(X^n)^k$ with X^{kn} in the obvious way.

PROPOSITION 3.5. *$\hbar_X$ induces a natural map $h_X : \Gamma^+\Gamma^+X \to \Gamma^+X$.*

Proof. We omit the details of the proof which is straightforward and tedious (they are given in [7; Proposition 1.4.6]). It simply consists of the verification of the following.

(a) Given $w \in WS_k$, $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}' \in (WS_n \times X^n)^k$ such that $\alpha_i \sim \alpha_i'$ by 3.1(a) for $1 \leqslant i \leqslant k$, then $\hbar_X(w, \boldsymbol{\alpha}) \sim \hbar_X(w, \boldsymbol{\alpha}')$ by 3.1(a).

(b) Given $w \in WS_k$, $\boldsymbol{\alpha} \in (WS_n \times X^n)^k$, $\boldsymbol{\alpha}' \in (WS_m \times X^m)^k$ such that $\alpha_i \sim \alpha_i'$ by 3.1(b) for $1 \leqslant i \leqslant k$, then $\hbar_X(w, \boldsymbol{\alpha}) \sim \hbar_X(w, \boldsymbol{\alpha}')$ by 3.1(b).
Hence $\hbar_X$ induces a map $\mathscr{U}(\Gamma^+X) \to \Gamma^+X$.

(c) Given $(w, \boldsymbol{\alpha}) \in WS_k \times (WS_n \times X^n)^k$, $\sigma \in S_k$, then $\hbar_X(w.\sigma, \boldsymbol{\alpha}.\sigma) \sim \hbar_X(w, \boldsymbol{\alpha})$ by 3.1(a).

(d) Given $(w, \boldsymbol{\alpha})$ as in (c) and $\beta \in C_j^k$ which is entire for the image of $\boldsymbol{\alpha}$ in $(\Gamma^+X)^k$ (i.e. if $i \in \mathbf{k} - \beta(\mathbf{j})$, then $\mathbf{x}^i = (*, \ldots, *) \in X^n$), then $\hbar_X(\beta^*(w), \beta^*(\boldsymbol{\alpha})) \sim \hbar_X(w, \boldsymbol{\alpha})$ by 3.1(b).
Thus, from Definition 3.1, a map $h_X : \Gamma^+\Gamma^+X \to \Gamma^+X$ is induced. Its naturality for maps of X follows from that of $\hbar_X$.

It is in checking the details of this sort of argument that the simple results of §1 are useful. We notice also that the use of 3.1(b) rather than 3.2(b′) makes the details less cumbersome.

The natural transformation h has very pleasant properties.

PROPOSITION 3.6. *Given a pointed space X, the following diagrams are commutative*

$$\begin{tikzcd} \Gamma^+X \arrow[r, "\Gamma^+\iota_X"] \arrow[dr, "\iota"'] & \Gamma^+\Gamma^+X \arrow[d, "h_X"] & \Gamma^+X \arrow[l, "\iota_{\Gamma^+X}"'] \arrow[dl, "\iota"] \\ & \Gamma^+X & \end{tikzcd} \qquad \begin{tikzcd} \Gamma^+\Gamma^+\Gamma^+X \arrow[r, "h_{\Gamma^+X}"] \arrow[d, "\Gamma^+h_X"'] & \Gamma^+\Gamma^+X \arrow[d, "h_X"] \\ \Gamma^+\Gamma^+X \arrow[r, "h_X"] & \Gamma^+X. \end{tikzcd}$$

This simply says that (Γ^+, ι, h) forms a triple (as defined in [4]). The name monad has also been used for such objects ([13]).

Definition 3.7. A Γ^+*-structure* on a pointed space X is a pointed map $f : \Gamma^+X \to X$ such that the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "1"'] & \Gamma^+X \arrow[d, "f"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \Gamma^+\Gamma^+X \arrow[r, "h_X"] \arrow[d, "\Gamma^+f"'] & \Gamma^+X \arrow[d, "f"] \\ \Gamma^+X \arrow[r, "f"] & X. \end{tikzcd}$$

In the categorical language of [4] this says that X is a (Γ^+, ι, h)-algebra.

Example 3.8. $h_X : \Gamma^+\Gamma^+X \to \Gamma^+X$ provides Γ^+X with a Γ^+-structure by Proposition 3.6.

We have remarked (3.2) that F^+X, the free monoid on X, sits inside Γ^+X in a natural way. Further, on restriction h_X provides a map $F^+F^+X \to F^+X$. So if X has a Γ^+-structure, $: \Gamma^+X \to X$, the restriction $F| : F^+X \to X$ provides it with an F^+-structure (with the obvious meaning). However, it is clear that the condition that X has and F^+-structure is just the same as the condition that it have the structure of a simplicial monoid; the map

$F^+X \to X$ provides the product, the commuting triangle says that the base point acts as a unit and the commuting square that the product is associative.

The extension of a monoid structure $F^+X \to X$ on X to a Γ^+-structure implies that X is homotopy abelian in a very strong sense. In fact it has the structure of an infinite loop space (see [3]). Only a small part of the Γ^+-structure is needed to prove the following.

PROPOSITION 3.9. *If a pointed simplicial set has a Γ^+-structure, then it is a homotopy abelian simplicial monoid with base point as unit.*

Proof. Let $f\colon \Gamma^+ X \to X$ be a Γ^+-structure on X. Then the product on X is given by $x.y = f([1, x, y])$ for $x, y \in X$, where $1 \in S_2 \subset WS_2$. As we have remarked, the diagrams of Definition 3.7 ensure that this product is associative and has the base point as unit. The required homotopy is given by

$$I \times X^2 \xrightarrow{\alpha \times 1} WS_2 \times X^2 \subset \mathscr{U}(X) \longrightarrow \Gamma^+ X \xrightarrow{f} X$$

where $\alpha\colon I \to WS_2$ is given by $\alpha(e) = \langle 1, (1, 2)\rangle \in (WS_2)_1$ for the non-degenerate $e \in I_1$.

COROLLARY 3.10. *Γ^+X is a homotopy abelian monoid with $1 = [1, \varnothing]$ as unit. The naturality of h implies that Γ^+ is a functor to simplicial monoids.*

Finally, in our consideration of the structure of Γ^+X, we prove the following.

PROPOSITION 3.11. *Γ^+X is a free monoid.*

Proof. Let $\iota_m : \mathbf{n} \to \mathbf{m+n}$ be the map given by $i_m(i) = m + i$ for $i \in \mathbf{n}$. Then, by Definition 1.3 and Note 2.4, we have a map

$$(\mathrm{i}_m)_* : WS_n \longrightarrow WS_{m+n}.$$

Now, using Relation 3.1(b) enough times we see that any element of Γ^+X may be written in the form $[w, \mathbf{x}]$ with $w \in WS_n$, $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$, $x_i \neq *$ for $1 \leqslant \mathrm{i} \leqslant n$, $n \geqslant 0$. Then $[w, \mathbf{x}]$ is irreducible if and only if there does not exist $\sigma \in S_n$ so that $w.\sigma \in WS_r . (\mathrm{i}_r)_* WS_{n-r} \subset WS_n$ for some r, $1 \leqslant r \leqslant n-1$. To show that Γ^+X is free we must show that each element of Γ^+X can be written in a unique way as a product of irreducible elements. This follows (by induction) from and is equivalent to the following lemma.

LEMMA 3.12. *Given $\alpha, \beta, \xi, \eta \in \Gamma^+X$ with α, β irreducible, if $\alpha.\xi = \beta.\eta$ then $\alpha = \beta$ and $\xi = \eta$.*

Proof. Suppose $\alpha = [v, \mathbf{x}]$, $(v, \mathbf{x}) \in WS_m \times X^m$, $\beta = [v', \mathbf{x}']$, $(v', \mathbf{x}') \in WS_n \times X^n$, $\xi = [w, \mathbf{y}]$, $(w, \mathbf{y}) \in WS_p \times X^p$, $\eta = [w', \mathbf{y}']$, $(w', \mathbf{y}') \in WS_q \times X^q$, all in the form described above with no coordinates the base point. Then, since $\alpha.\xi = \beta.\eta$, there is a $\sigma \in S_{m+p}$ such that

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{x}', \mathbf{y}').\sigma$$

$$v.(i_m)_* w = (v'.(i_n)_* w').\sigma,$$

by 3.1(a). Since no coordinate is the base point, 3.1(b) cannot arise. We consider two cases.

Suppose $m = n$. Then $\sigma \in S_m . (i_m)_* S_p \subset S_{m+p}$. For, if not, given $\tau \in S_m . (i_m)_* S_p$, it is clear that $\tau.\sigma \notin S_m . (i_m)_* S_p$. So, by Note 2.4, $v.(i_m)_* w = (v'.(i_n)_* w').\sigma \notin WS_m.(i_m)_* WS_p$ which is a contradiction.

So suppose $\sigma = \sigma_1.(i_m)_* \sigma_2$, $\sigma_1 \in S_m$, $\sigma_2 \in S_p$. Then

$$v = v'.\sigma_1 \quad \text{and} \quad \mathbf{x} = \mathbf{x}'.\sigma_1$$
$$w = w'.\sigma_2 \quad \text{and} \quad \mathbf{y} = \mathbf{y}'.\sigma_2$$

Thus, by 3.1(a), $\alpha = \beta$ and $\xi = \eta$.

On the other hand, suppose $m \neq n$. Without loss of generality we may suppose $m < n$. σ: $\mathbf{m} + \mathbf{p} \to \mathbf{n} + \mathbf{q}$ is a one-to-one map. Suppose that for some $i \in \mathbf{m} \subset \mathbf{m} + \mathbf{n}$, $\sigma(i) > n$. If $\tau \in S_n.(i_n)_* S_q$, $\tau\sigma(i) > n > m$, i.e. $\tau.\sigma \notin S_m.(i_m)_* S_p$. Hence $(v'.(i_n)_* w').\sigma \notin WS_m.(i_m)_* WS_p$ which is a contradiction. Hence $\sigma(\mathbf{m}) \subset \mathbf{n} \subset \mathbf{n} + \mathbf{q}$.

Thus we may define $\bar{\sigma} \in S_n \subset S_{n+q}$ by

$$\bar{\sigma}(i) = \sigma(i) \quad \text{for} \quad 1 \leqslant i \leqslant m,$$
$$\bar{\sigma}(i) = \text{and } k \in \mathbf{n} \text{ such that } k \notin \bar{\sigma}(\mathbf{i-1}) \text{ for } m < i \leqslant n.$$

Then $(v'.(i_n)_* w').\bar{\sigma} \in WS_m.(i_m)_* WS_p$. So $v'.\bar{\sigma} \in WS_m.(i_m)_* WS_{n-m} \subset WS_n$, which contradicts the irreducibility of β, showing that this case is impossible.

Thus we have proved Lemma 3.12 and so Proposition 3.11.

We conclude this section by identifying the monoid of components of $\Gamma^+ X$.

PROPOSITION 3.13. *Given a pointed space* X, $\pi_0(\Gamma^+ X) = \mathbb{Z}^+ \pi_0(X)$, *the free abelian monoid on the pointed set* $\pi_0(X)$.

Proof. Since $(WS_n)_0 = S_n$, the vertices $(\Gamma^+ X)_0 = F^+ X_0$, the free monoid on the set of vertices of X. Now, given a simplicial set A, $\pi_0(A) = A_0/\sim$, where $\sim$ is the equivalence relation generated by $\partial_0 a \sim \partial_1 a$ for $a \in A_1$. Any 1-simplex $\xi \in \Gamma^+ X$ may be written $\xi = [\langle 1, \sigma\rangle, x_1, \ldots, x_n]$, $\sigma \in S_n$, $x_i \in X_1$. Then $\partial_0 \xi = [\sigma, \partial_0 x_1, \ldots, \partial_0 x_n] = [1, \partial_0 x_{\sigma^{-1}(1)}, \ldots, \partial_0 x_{\sigma^{-1}(n)}]$, and $\partial_1 \xi = [1, \partial_1 x_1, \ldots, \partial_1 x_n]$. Thus we see that for $\pi_0(\Gamma^+ X)$ the order of the coordinates does not matter and so the result is as stated.

§4. THE FREE GROUP FUNCTOR Γ

Proposition 3.13 tells us that in general $\Gamma^+ X$ cannot be a model for $\Omega^\infty \Sigma^\infty X$, just as $F^+ X$ is not for $\Omega\Sigma X$. In this section we define a free group functor Γ and show, using results to be proved in subsequent sections, that ΓX and $\Omega^\infty \Sigma^\infty X$ are homotopy equivalent.

Definition 4.1. Given a monoid M by the *universal group* of M is meant a group UM which is universal with respect to homomorphisms from M to groups, i.e. there is a natural monoid homomorphism u: $M \to UM$ such that, given a monoid homomorphism f: $M \to G$ to a group G, there is a unique group homomorphism F : $UM \to G$ such that $F.u = f$: $M \to G$.

The following result guarantees the existence of the universal group. It is unique up to isomorphism for the usual categorical reasons.

PROPOSITION 4.2. *Given a monoid* M, *let* FM *be the free group on the pointed (by the unit) set* M *and let* N *be the normal subgroup of* FM *generated by elements of the form* $m_1.m_2.m_3^{-1}$ *where* $m_i \in M$ *and* $m_1.m_2 = m_3$ *in* M. *Then the composite*

$$M \to FM \to FM/N$$

of the quotient map with the inclusion as generators is a monoid homomorphism satisfying the universal condition.

The proof is straightforward. It should be noted that the inclusion $M \to FM$ is not a homomorphism.

By taking the universal group of the monoids and monoid homomorphisms involved we may define the universal simplicial group UM of a simplicial monoid M.

Definition 4.3. Given a pointed simplicial set X we define ΓX to be the universal simplicial group of the free simplicial monoid $\Gamma^+ X$.

Of course, Γ is a functor from pointed simplicial sets to simplicial groups. There is a natural embedding

$$\iota_X : X \to \Gamma X$$

obtained by composing the universal group map $u: \Gamma^+ X \to \Gamma X$ with the natural embedding $\iota_X : X \to \Gamma^+ X$. This is an embedding for, as $\Gamma^+ X$ is a free monoid, ΓX is a free group on the same generators and so u is an embedding. When we refer to "the pair $(\Gamma X, X)$" we mean X embedded in ΓX by ι_X.

Definition 4.4. Given pointed spaces X and Y, there is a natural map

$$\psi: (\Gamma X) \times Y \to \Gamma(X \times Y)$$

given by

$$\begin{aligned} &\psi([w, x_1, \ldots, x_n], y) = [w, (x_1, y), \ldots, (x_n, y)] \\ &\psi(\xi . \eta, y) = \psi(\xi, y) . \psi(\eta, y) \\ &\psi(\xi^{-1}, y) = \psi(\xi, y)^{-1} \end{aligned}$$

for $w \in WS_n$, $x_i \in X$, $y \in Y$, ξ, $\eta \in \Gamma X$.

Similarly we may define a natural map

$$\psi': (\Gamma X) \wedge Y \to \Gamma(X \wedge Y).$$

In particular we have maps

$$C\Gamma X = (\Gamma X) \wedge I \to \Gamma(CX)$$
$$\Sigma\Gamma X = (\Gamma X) \wedge S^1 \to \Gamma(\Sigma X)$$

and so, on taking adjoints

$$\beta_X : \Gamma X \to P\Gamma CX$$
$$\gamma_X : \Gamma X \to \Omega\Gamma\Sigma X.$$

Here I and S^1 are simplicial set models for the unit interval and 1-sphere. A possibility for I, in which we take the vertex 0 as base point, is given in the proof of Lemma 2.1 and the corresponding S^1 is obtained by identifying the two vertices. For future use we define a map

$$c: I \wedge I \to I$$

by $(s_0 e, s_1 e) \mapsto s_0 e$, $(s_1 e, s_0 e) \mapsto s_0 e$. This gives a natural transformation

$$c_X : CCX \to CX.$$

In fact we replace I, S^1 and c by their associated Kan complexes $Ex^\infty(I)$, $Ex^\infty(S^1)$ and $Ex^\infty(c)$ (see [10]) but still use the notations I, S^1 and c. This means that, for example, $\Omega X = X^{S^1}$ does give a simplicial model for the loops on the pointed simplicial set X. We define the cone, suspension, path space and loop space functors in this way, rather than using the usual economical simplicial definitions, so that the natural transformations β and γ are defined. They are clearly homomorphisms of simplicial groups.

PROPOSITION 4.5. *Γ is a homotopy functor, i.e. if f and g: $X \to Y$ are homotopic pointed maps, then so are Γf and Γg: $\Gamma X \to \Gamma Y$. Thus if X and Y are homotopy equivalent pointed Kan complexes, so are ΓX and ΓY.*

Proof. This follows from the existence of a map $(\Gamma X) \times I \to \Gamma(X \times I)$ by the above definition.

In §7 we shall prove the following:

LEMMA 4.6. *Given a pointed space X, the natural simplicial group homomorphism γ_X : $\Gamma X \to \Omega\Gamma\Sigma X$ is a homotopy equivalence.*

Given this, it is not difficult to obtain our theorem. For each integer $n \geqslant 0$ there is a natural homomorphism

$$\Omega^n \gamma_{\Sigma^n X} : \Omega^n \Gamma \Sigma^n X \to \Omega^{n+1} \Gamma \Sigma^{n+1} X.$$

Using these maps we define the direct limit $\varinjlim_n \Omega^n \Gamma \Sigma^n X$ which we write $\Omega^\infty \Gamma \Sigma^\infty X$. But, by the lemma, each map of the directed system is a (weak) homotopy equivalence and so the same is true of the natural homomorphism $\Gamma X \to \Omega^\infty \Gamma \Sigma^\infty X$, using J. H. C. Whitehead's theorem. Thus we have the following.

COROLLARY 4.7. *Given a pointed space X, the natural simplicial group homorphism $\Gamma X \to \Omega^\infty \Gamma \Sigma^\infty X$ is a homotopy equivalence.*

This is halfway to the theorem. Now the adjoint of the identity map $\Sigma X \to \Sigma X$ gives a natural inclusion i_X : $X \to \Omega\Sigma X$ such that $\gamma_X . \iota_X = \Omega \iota_{\Gamma X} . i_X$: $X \to \Omega\Gamma\Sigma X$. Hence, for each $n \geqslant 0$,

$$(\Omega^n \gamma_{\Sigma^n X}, \Omega^n i_{\Sigma^n X}) \colon (\Omega^n \Gamma \Sigma^n X, \quad \Omega^n \Sigma^n X) \to (\Omega^{n+1} \Gamma \Sigma^{n+1} X, \quad \Omega^{n+1} \Sigma^{n+1} X)$$

is a map of pairs, thus giving $(\Omega^\infty \Gamma \Sigma^\infty X, \Omega^\infty \Sigma^\infty X)$ as direct limit. However, in §6 we shall prove the following.

LEMMA 4.8. *If X is an $(n-1)$-connected pointed space and $n > 1$, then the pair $(\Gamma X, X)$ is $(2n-1)$-connected.*

Hence, if $n > 1$, the pair $(\Gamma \Sigma^n X, \Sigma^n X)$ is $(2n-1)$-connected. It follows that the pair $(\Omega^n \Gamma \Sigma^n X, \Omega^n \Sigma^n X)$ is $(n-1)$-connected, and so the pair $(\Omega^\infty \Gamma \Sigma^\infty X, \Omega^\infty \Sigma^\infty X)$ is ∞-connected. Thus, using J. H. C. Whitehead's theorem, we have proved the following:

COROLLARY 4.9. *If X is a pointed Kan complex, the natural inclusion map $\Omega^\infty \Sigma^\infty X \to \Omega^\infty \Sigma \Gamma^\infty X$ is a homotopy equivalence.*

Taking Corollaries 4.7 and 4.9 together we obtain our approximation theorem.

THEOREM 4.10. *If X is a pointed Kan complex then ΓX and $\Omega^\infty\Sigma^\infty X$ are naturally homotopy equivalent.*

$X = S^0$, the 0- sphere, is an important example. Then $\Gamma^+ S^0 = \coprod_{n\geq 0} \overline{W}S_n$. The monoid structure is given by

$$\overline{W}S_m \times \overline{W}S_n \to \overline{W}S_{m+n}$$

coming from the obvious inclusion $S_m \times S_n \to S_{m+n}$. This leads quickly to the following result (see [21] and [4; Theorem 4.1]).

COROLLARY 4.11. *There is a map*

$$\omega\colon\ \overline{W}S_\infty \to (\Omega^\infty S^\infty)_0$$

which induces an isomorphism of integral homology.

Here $\overline{W}S_\infty = \varinjlim_n \overline{W}S_n$ under the inclusion maps of Example 1.9 and $(\Omega^\infty S^\infty)_0$ is the connected component of the base point.

The remainder of this paper is devoted to the proofs of Lemmata 4.6 and 4.8.

A note on the topological version

Of course, Theorem 4.10 is true in the category of compactly generated topological spaces of the homotopy type of a CW-complex [20], [25]. In this category products are given the compactly generated topology and the phrases "topological monoid" and "topological group" adopt the corresponding meanings. Given a pointed space X in this category, a topological monoid $\Gamma^+ X$ and a topological group ΓX may be defined just as in the simplicial version above using the realizations of the simplicial sets WS_n. The topological version of the theorem then follows from the simplicial version using the singular complex functor, since Γ is a homotopy functor.

§5. THE HOMOLOGY OF THE UNIVERSAL GROUP OF THE MONOID

This section contains the statement of a theorem of Quillen's which is used subsequently in the proofs of Lemma to 4.6 and 4.8. This theorem includes the theorems of [4] and [7] as special cases.

Throughout this section, M denotes a simplicial monoid and $u\colon M \to UM$ the universal group homomorphism. The problem is to determine the homology of UM knowing that of M.

The product on M induces a product on $\pi_0(M)$ so that it forms a monoid with the component of the identity of M as identity. Similarly, $\pi_0(UM)$ is a group. Further, u_* : $\pi_0(M) \to \pi_0(UM)$ is a monoid homomorphism and so by the universal property has a unique natural extension

$$U\pi_0(M) \to \pi_0(UM).$$

PROPOSITION 5.1. *This map is a group isomorphism.*

The proof is straightforward. We identify these two groups by this isomorphism.

Suppose now that k is a commutative ring. Then, for any space X, there is a natural inclusion

$$\pi_0(X) \subset k\pi_0(X) \cong H_0(X; k) \subset H_*(X; k).$$

$H_*(M; k)$ is a $k\pi_0(M)$ - module and we may define a natural map

$$\theta_M : H_*(M; k) \otimes_{k\pi_0(M)} kU\pi_0(M) \to H_*(UM; k)$$

by $\theta_M(a, \lambda\xi) = \lambda u_*(a).\xi$ for $a \in H_*(M)$, $\lambda \in k$ and $\xi \in U\pi_0(M) \cong \pi_0(UM)$. θ_M composed with the natural inclusion of $H_*(M)$ in $H_*(M) \otimes kU\pi_0(M)$ is the map induced by u, the universal group homomorphism.

Quillen has proved the following [22].

THEOREM 5.2. *If* (a) *M is free, and* (b) *$\pi_0(M)$ is in the centre of the ring $H_*(M;k)$, then θ_M is a ring isomorphism.*

We shall use the following implication of this theorem.

COROLLARY 5.3. *Suppose $f: M_1 \to M_2$ is a homomorphism of simplicial monoids satisfying* (a) *and* (b) *of the theorem. Then if f induces an isomorphism of homology rings with coefficients in some commutative ring k, so does the simplicial group map $Uf: UM_1 \to UM_2$.*

This follows by the naturality of θ.

The theorem also shows that $\Gamma^+ X$ is equivalent to $\Omega^\infty \Sigma^\infty X$ for a connected pointed space X as follows.

COROLLARY 5.4. *If X is a connected pointed space, the inclusion*

$$\Gamma^+ X \to \Gamma X$$

is a homotopy equivalence.

Proof. By Proposition 3.13, $\Gamma^+ X$ is connected. So, by the theorem, the natural inclusion induces an isomorphism of homology groups. Hence, by the bar spectral sequence, the natural inclusion

$$\overline{W}\Gamma^+ X \to \overline{W}\Gamma X$$

of classifying spaces induces an isomorphism of the homology groups of simply connected spaces and so is a homotopy equivalence. The result follows on taking loop spaces (using [5]).

§6. A FILTRATION OF $\Gamma^+ X$: THE PROOF OF LEMMA 4.8

Definition 6.1. For each integer $m \geqslant 0$, we write $\Gamma_m^+ X$ for the subspace of $\Gamma^+ X$ which is the image of $\coprod_{n=0}^{m} WS_n \times X^n \subset \mathcal{U}(X)$ under the identification map of Definition 3.1. Then, if $m \leqslant n$, there is a natural inclusion $\Gamma_m^+ X \subset \Gamma_n^+ X$ and $\Gamma^+ X = \varinjlim_m \Gamma_m^+ X$.

We refer to the filtration

$$\{1\} = \Gamma_0^+ X \subset \iota_X(X) = \Gamma_1^+ X \subset \ldots \subset \Gamma_m^+ X \subset \ldots \subset \Gamma^+ X$$

as the filtration of $\Gamma^+ X$ *by word length*, by analogy with the corresponding filtration of $F^+ X$.

For each $m \geqslant 1$ the quotient space $\Gamma_m^+ X/\Gamma_{m-1}^+ X$, which we write $\mathscr{D}_m X$, is known as the *m-adic construction*. The usual definition of $\mathscr{D}_m X$ is as the identification space obtained by factoring out by the obvious free action of S_m on $WS_m \times X^{(m)}/WS_m \times \{*\}$, where $X^{(m)}$ denotes the m-fold smash product of X with itself. This is clearly homeomorphic to the above definition. These constructions have been useful in homotopy computations.

The filtration of $\Gamma^+ X$ by word length gives rise to a spectral sequence E_{**}^* converging to the homology of $\Gamma^+ X$ (e.g. [24; p.469]). This spectral sequence has

$$E_{p,q}^1 = H_{p+q}(\Gamma_p^+ X,\ \Gamma_{p-1}^+ X) = \tilde{H}_{p+q}(\mathscr{D}_p X) \quad \text{for } p > 0.$$

But if X is $(n-1)$-connected, $\mathscr{D}_p X$ is clearly $(np-1)$-connected. Hence the convergence of the spectral sequence implies the following result.

PROPOSITION 6.2. *If X is an $(n-1)$-connected pointed Kan complex, then $H_i(\Gamma^+ X, X) = 0$ for $i < 2n$.*

Now $\Gamma^+ X$ is free (Proposition 3.11) and homotopy abelian (Corollary 3.10) and so Theorem 5.2 applies giving the following result.

PROPOSITION 6.3. *If X is an $(n-1)$-connected pointed Kan complex, then $H_i(\Gamma X, X) = 0$ for $i < 2n$.*

Lemma 4.8 follows from this when $n > 1$ by the relative hurewicz isomorphism theorem. To show that ΓX is 1-connected (which we need to apply the hurewicz theorem) we need only to replace X by its minimal complex which has only one (degenerate) 1-simplex [12; Lemma 9.2], so that the same is true of ΓX. We can do this since Γ is homotopy invariant (Proposition 4.5).

§7. THE PROOF OF LEMMA 4.6

This follows easily from the following result which is proved in the next section.

LEMMA 7.1. *Let T be a functor from the category of pointed simplicial sets to the category of simplicial groups which is given in each dimension (Definition 7.2). Suppose that T has the property that, for any two pointed spaces A_1 and A_2, the homomorphism*

$$Ti_1 \times Ti_2 : TA_1 \times TA_2 \to T(A_1 \vee A_2)$$

is a homotopy equivalence, where the i_j $(j = 1,2)$ are the inclusion maps. Then, if (A,B) is a pair of pointed spaces and q: $A \to A/B$ denotes the map identifying B to the base point, the natural map

$$TB \to \ker\,[Tq : TA \to T(A/B)]$$

is a homotopy equivalence.

This says that if T turns trivial cofibrations into fibrations up to homotopy then it does the same to all cofibrations.

Definition 7.2. A functor T from simplicial $\mathscr{C}_1$-objects to simplicial $\mathscr{C}_2$-objects is *given in each dimension* if there are functors $T_n : \mathscr{C}_1 \to \mathscr{C}_2$ for each integer $n \geqslant 0$ such that

$$(TX)_n = T_n(X_n), \ (Tf)_n = T_n(f_n)$$

for simplicial $\mathscr{C}_1$-objects X and simplicial $\mathscr{C}_1$-maps f.

The functor Γ clearly has this property. Further,

LEMMA 7.3. *Given pointed spaces A_1 and A_2, the homomorphism of simplicial groups*

$$\Gamma i_1 \times \Gamma i_2 : \Gamma A_1 \times \Gamma A_2 \to \Gamma(A_1 \vee A_2)$$

is a homotopy equivalance, where the i_j are the inclusion maps.

So Lemma 7.1 applies to the functor Γ.

COROLLARY 7.4. *If (A,B) is a pair of pointed spaces and q: $A \to A/B$ denotes the map identifying B to the base point, then the natural map*

$$\Gamma B \to \ker\,[\Gamma q\colon TA \to T(A/B)]$$

is a homotopy equivalence.

This is just what is needed to prove Lemma 4.6. For consider the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \ker(\Gamma q) \arrow[r] \arrow[dd, dashed, "\alpha_X"] & \Gamma CX \arrow[r, "\Gamma q"] \arrow[dd, "\beta_X"] & \Gamma\Sigma X \arrow[r] \arrow[dd, "1"] & 1 \\
\Gamma X \arrow[ur] \arrow[dr, "\gamma_X"'] & & & & \\
1 \arrow[r] & \Omega\Gamma\Sigma X \arrow[r] & P\Gamma\Sigma X \arrow[r] & \Gamma\Sigma X \arrow[r] & 1.
\end{tikzcd}$$

Here q: $CX \to \Sigma X$ is the map collapsing the base $X \subset CX$ of the cone to the base point, the bottom row is the obvious short exact sequence of simplicial groups, and the natural map β_X is

$$(P\Gamma q).(P\Gamma c_X).\beta_{CX} : \Gamma CX \to P\Gamma CCX \to P\Gamma CX \to P\Gamma\Sigma X$$

where β and c are the natural transformations of Definition 4.4. Then, since both rows are exact, α_X is induced such that the triangle commutes. By Corollary 2.6 and J. H. C. Whitehead's theorem, α_X is a homotopy equivalence. But $\Gamma X \to \ker\,(\Gamma q)$ is a homotopy equivalence by Corollary 7.4. Hence so is γ_X.

So the proof of Lemma 4.6 is complete (apart from 7.1) when we have proved Lemma 7.3. This follows from a corresponding result for monoids, namely

LEMMA 7.5. *The homomorphism of monoids*

$$\Gamma^+ i_1 \times \Gamma^+ i_2 : \Gamma^+ A_1 \times \Gamma^+ A_2 \to \Gamma^+(A_1 \vee A_2)$$

induces an isomorphism of homology groups.

For Proposition 3.11 and Corollary 3.10 ensure that conditions (a) and (b) of §5 are satisfied for the monoids involved. Since $U(\Gamma^+ A_1 \times \Gamma^+ A_2) = \Gamma A_1 \times \Gamma A_2$, Corollary 5.3

ensures that $\Gamma i_1 \times \Gamma i_2$ of Lemma 7.3 induces an isomorphism of homology groups. But by J. H. C. Whitehead's theorem such a homomorphism of simplicial groups is a homotopy equivalence as required.

Proof of Lemma 7.5. The filtration of Γ^+ by word length was described in Definition 6.1. If $\Gamma^+ A_1 \times \Gamma^+ A_2$ is filtered by the corresponding product filtration then $(\Gamma^+ i_1) \times (\Gamma^+ i_2)$ is filtration preserving and so induces a map of quotient spaces

$$\bigvee_{p=0}^{N} (\mathscr{D}_p A_1) \wedge (\mathscr{D}_{n-p} A_2) \to \mathscr{D}_n (A_1 \vee A_2)$$

for each $n \geqslant 0$, where we write $\mathscr{D}_0 A_i = S^0$.

PROPOSITION 7.6. *For each $n \geqslant 0$ this map is a homotopy equivalence.*

Hence the map $(\Gamma^+ i_1) \times (\Gamma^+ i_2)$ induces an isomorphism of the E^1-terms of the homology spectral sequences of $\Gamma^+ A_1 \times \Gamma^+ A_2$ and $\Gamma^+(A_1 \vee A_2)$ with respect to these filtrations. Since the filtrations are bounded below the spectral sequences converge and so Lemma 7.5 follows.

Proof of Proposition 7.6. We write X_p for the bouquet of $\binom{n}{p}$ copies of $A_1{}^{(p)} \wedge A_2{}^{(n-p)}$ in $(A_1 \vee A_2)^{(n)}$ so that

$$(A_1 \vee A_2)^{(n)} = \bigvee_{p=0}^{n} X_p .$$

The subspaces X_p are invariant under the action of S_n and so

$$\mathscr{D}_n(A_1 \vee A_2) = WS_n \ltimes_{S_n} \Big(\bigvee_{p=0}^{n} X_p \Big) \cong \bigvee_{p=0}^{n} (WS_n \ltimes_{S_n} X_p).$$

The map of the proposition restricts to an embedding of $\mathscr{D}_p A_1 \wedge \mathscr{D}_{n-p} A_2$ in $WS_n \ltimes_{S_n} X_p$. So the proposition is proved when a deformation

$$r\colon WS_n \ltimes_{S_n} X_p \to W(S_p \times S_{n-p}) \ltimes_{S_p \times S_{n-p}} (A_1{}^{(p)} \wedge A_2{}^{(n-p)}) \cong \mathscr{D}_p A_1 \wedge \mathscr{D}_{n-p} A_2$$

is given.

The inclusion $S_p \times S_{n-p} \to S_n$ induces an equivariant $(S_p \times S_{n-p})$-map $W(S_{p_n} \times S_{n-p}) \to WS_n$. On factoring out by the action we obtain a homotopy equivalence of two Eilenberg–Maclane spaces $K(S_p \times S_{n-p}, 1)$ and so there is an equivariant deformation retract $d\colon WS_n \to W(S_p \times S_{n-p})$.

Suppose $\xi = [w, a_1, \ldots, a_p, \ b_{p+1}, \ldots, b_n] \in WS_n \ltimes_{S_n} X_p$, where $w \in WS_n$, $a_i \in A$, $b_j \in B$. Then we define $r(\xi) = [d(w), a_1, \ldots, b_n]$. This is well-defined and is the required deformation retract, thus proving Proposition 7.6 and so Lemma 7.3.

§8. A GENERALISATION OF A THEOREM OF KAN AND WHITEHEAD

In this section we prove Lemma 7.1. On the way we obtain a long exact sequence associated with a group valued functor which has previously been obtained by Kan and Whitehead in the abelian case.

To begin with, suppose T is a functor from the category of pointed sets to the category of groups. Let (A,B) be a pair of pointed sets and write $i : B \to A$ for the inclusion map. Then we define a pointed simplicial set $L = L(A,B)$ as follows.

For $n \geqslant 0$, $L_n = A \vee B_1 \vee \ldots \vee B_n$ where $B_i = B$.

$\partial_0 : L_n \to L_{n-1}\,(n \geqslant 1)$ is given by

$$\begin{aligned} \partial_0 \mid A &= 1_A : A && \to && A \subset L_{n-1}, \\ \partial_0 \mid B_1 &= i : B_1 && \to && A \subset L_{n-1}, \\ \partial_0 \mid B_j &= 1_B : B_j && \to && B_{j-1} \subset L_{n-1}, \quad 1 < j \leqslant n. \end{aligned}$$

$\partial_i : L_n \to L_{n-1}\,(n \geqslant 1,\ 0 < i < n)$ is given by

$$\begin{aligned} \partial_i \mid A &= 1_A : A && \to && A \subset L_{n-1}, \\ \partial_i \mid B_j &= 1_B : B_j && \to && B_j \subset L_{n-1}, \quad 1 \leqslant j \leqslant i, \\ \partial_i \mid B_j &= 1_B : B_j && \to && B_{j-1} \subset L_{n-1}, \quad i < j \leqslant n. \end{aligned}$$

$\partial_n : L_n \to L_{n-1}\,(n \geqslant 1)$ is given by

$$\begin{aligned} \partial_n \mid A &= 1_A : A && \to && A \subset L_{n-1}, \\ \partial_n \mid B_j &= 1_B : B_j && \to && B_j \subset L_{n-1}, \quad 1 \leqslant j < n, \\ \partial_n \mid B_n &= * : B_n && \to && \{*\} \subset L_{n-1}. \end{aligned}$$

$s_i : L_n \to L_{n+1}\,(n \geqslant 0)$ is given by

$$\begin{aligned} s_i \mid A &= 1_A : A && \to && A \subset L_{n-1}, \\ s_i \mid B_j &= 1_B : B_j && \to && B_j \subset L_{n+1}, \quad 1 \leqslant j \leqslant i, \\ s_i \mid B_j &= 1_B : B_j && \to && B_{j+1} \subset L_{n+1}, \quad i < j \leqslant n. \end{aligned}$$

It is tedious but straightforward to check that these face and degeneracy maps make L a simplicial set.

From $L(A,B)$ we obtain a simplicial group $K = K(A,B)$ by composing L, thought of as a contravariant functor from the simplicial category Δ (e.g. [12; 5.6]) to the category of pointed sets, with the functor T, i.e. $K_n = T(L_n)$ and similarly for the face and degeneracy maps which are also written ∂_i and s_i in K.

Now, for $n \geqslant 0$, we write $M_n = M_n(A,B)$ for $\bigcap_{i=1}^{n} \ker\,[\partial_i : K_n \to K_{n-1}]$. In particular, $M_0 = K_0 = T(A)$. It is a simple implication of the relations between degeneracy maps that $\partial_0(M_n) \subset M_{n-1}$. The first step in the proof of Lemma 7.1 is

LEMMA 8.1. *There is a natural long exact sequence of groups*

$$\ldots \xrightarrow{\partial_0} M_n(A,B) \xrightarrow{\partial_0} M_{n-1}(A,B) \xrightarrow{\partial_0} \ldots \xrightarrow{\partial_0} M_0(A,B) = T(A)$$
$$\xrightarrow{Tq} T(A/B) \to 1,$$

where $q \colon A \to A/B$ is the map identifying B to the base point.

In the case of T a functor from pointed sets to abelian groups this is the exact sequence of Kan and Whitehead [11; Proposition 16.1].

Proof. We have defined the natural sequence and it remains to check exactness. In L we may define further maps $\alpha\colon L_n \to L_{n+1}$ for all $n \geqslant 0$ as follows.

$$\alpha | A - B = 1_{A-B} : A - B \to A - B \subset A \subset L_{n+1},$$
$$\alpha | B(\subset A) = 1_B : B \to B_1 \subset L_{n-1},$$
$$\alpha | B_j = 1_B : B_j \to B_{j+1} \subset L_{n+1}.$$

So there are maps $T(\alpha)$: $K_n \to K_{n+1}$ for all $n \geqslant 0$ and it is again straightforward to check that these maps satisfy the conditions of the additional structure maps σ in Lemma 2.1. Thus the map $r: K \to \hat{K}$, where $\hat{K}_n = \partial_1 T(\alpha) K_0$, given by $r(k) = \partial_1 T(\alpha)\partial_0{}^n(k)$, for $k \in K_n$, is a deformation retract. But $\partial_1 T(\alpha) K_0 = \partial_1 T(\alpha) T(L_0) = T(\partial_1 \alpha L_0) = T(qA) = T(A/B)$ and r is given by $r(k) = T(q)\partial_0{}^n(k)$ for $k \in K_n$.

However, Proposition 2.5 provides a means of computing the homotopy groups of a simplicial group. For $G = K$, $\bar{G}_n = M_n$ and for $G = \hat{K}$, $\bar{G}_n = 1$ for $n > 0$ and $\bar{G}_0 = \hat{K}_0 = T(A/B)$. By the naturality of Proposition 2.5 the map r gives rise to a map of chain complexes

$$\begin{array}{ccccccccccc}
\xrightarrow{\partial_0} & M_n & \xrightarrow{\partial_0} & M_{n-1} & \xrightarrow{\partial_0} & \cdots & \xrightarrow{\partial_0} & M_0 = T(A) & \longrightarrow & 1 \\
 & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle T(q)} & & \\
\longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & T(A/B) & \longrightarrow & 1
\end{array}$$

and since r is a deformation retract, this induces an isomorphism of homology groups proving Lemma 8.1.

Now we turn to the situation of Lemma 7.1. Suppose T is given by T_m in dimension m. Then the above lemma applies in each dimension m to the functor T_m and the pair of pointed sets (A_m, B_m) to give a long exact sequence. By naturality these fit together to give

COROLLARY 8.2. *There is a natural long exact sequence of simplicial groups*

$$\cdots \xrightarrow{\partial_0} M_n(A,B) \xrightarrow{\partial_0} M_{n-1}(A,B) \xrightarrow{\partial_0} \cdots$$
$$\cdots \xrightarrow{\partial_0} M_0(A,B) = T(A) \xrightarrow{Tq} T(A/B) \longrightarrow 1.$$

Now we consider the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & T(B) \arrow[r, "i_2"] \arrow[d, dashed, "\mu(A,B)"'] \arrow[dr, "T(i_2)"] & T(A) \times T(B) \arrow[r, "p_1"] \arrow[d, "T(i_1) \times T(i_2)"] & T(A) \arrow[r] \arrow[d, "1"] & 1 \\
1 \arrow[r] & M_1(A,B) \arrow[r] & T(A \vee B) \arrow[r, "T(p_1)"] & T(A) \arrow[r] & 1
\end{tikzcd}$$

where p_i denotes projections onto the ith factor and i_i injection as the ith factor in all cases. The two rows are short exact sequences of simplicial groups (the bottom one by the definition of M_1) and $T(p_1)$. $T(i_2) = T(*) = *$: $T(B) \to T(A)$. Hence $T(i_2)$ may be factored through $M_1(A,B)$ by a unique map $\mu(A,B)$: $T(B) \to M_1(A,B)$.

PROPOSITION 8.3. *The natural map* $\mu(A,B)$: $T(B) \to M_1(A,B)$ *is a homotopy equivalence if* T *has the property assumed in Lemma* 7.1.

This follows by Corollary 2.6., the five lemma and J. H. C. Whitehead's theorem.

Returning to the long exact sequence of Corollary 8.2, let C_0 be the kernel of $Tq : M_0 = T(A) \to T(A/B)$. Then Lemma 7.1 follows quickly from

PROPOSITION 8.4. *The natural map $M_1 \to C_0$ is a homotopy equivalence.*

For consider the following diagram.

$$\begin{tikzcd}
 & TB \arrow[dd, "\mu(A,B)"'] \arrow[drr, dashed, "\nu"] \arrow[drrr, bend left, "Ti"] & & & & & \\
 & & 1 \arrow[r] & C_0 \arrow[r] & T(A) \arrow[r, "Tq"] & T(A/B) \arrow[r] & 1. \\
M_2 \arrow[r, "\partial_0"] & M_1(A,B) \arrow[urr, dashed, "\lambda"] \arrow[urrr, bend right, "\partial_0"'] & & & & &
\end{tikzcd}$$

Here the bottom row is the sequence of Lemma 8.1. Clearly $\partial_0 : M_1(A,B) \to T(A)$ factors through C_0 by a map λ and T_i factors through C_0 by a map ν. Our aim is to prove ν to be a homotopy equivalence.

$\partial_0.\mu(A,B) = \partial_0.T(i_2)$: $T(B) \to T(A \vee B) \to T(A)$ by the definition of μ, and $\partial_0.T(i_2) = Ti$ by the definition of ∂_0. Hence, since $C_0 \to TA$ is a monomorphism, $\lambda.\mu = \nu$. But λ and μ are homotopy equivalences (by Propositions 8.4 and 8.3) and thus ν is as required.

It remains to give the

Proof of Proposition 8.4. Firstly we prove that $M_n(A,B)$ is contractible for all $n \geqslant 2$ and all pairs (A, B). The proof is by induction on n.

Consider the pair $(A \vee B_0, B)$ where $B_0 \cong B$ and the inclusion is given by $i:B \to A \subset A \vee B_0$. Then for $n \geqslant 1$ there is a natural isomorphism

$$L_n(A \vee B_0, B) = A \vee B_0 \vee B_1 \vee \ldots \vee B_n \cong A \vee B_1 \vee \ldots \vee B_{n+1} = L_{n+1}(A,B),$$

with $B_i \leftrightarrow B_{i+1}$ for $0 \leqslant i \leqslant n$, and thus a natural isomorphism

$$K_n(A \vee B_0, B) \cong K_{n+1}(A,B).$$

Hence $M_n(A \vee B_0, B) \cong \bigcap_{i=2}^{n+1} \ker\,[\partial_i : K_{n+1}(A,B) \to K_n(A,B)]$.

$\partial_1 : \bigcap_{i=2}^{n+1} \ker\,[\partial_i : K_{n+1}(A,B) \to K_n(A,B)] \to M_n(A,B)$ is an epimorphism, for $s_0 \mid M_n(A,B)$ provides a right inverse. The kernel of this map is $M_{n+1}(A,B)$ by its definition. Thus we have a natural short exact sequence of simplicial groups

$$1 \to M_{n+1}(A,B) \to M_n(A \vee B_0, B) \to M_n(A,B) \to 1. \tag{8.5}$$

Now consider the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & M_2(A,B) \arrow[r] & M_1(A \vee B_0, B) \arrow[r] & M_1(A,B) \arrow[r] & 1. \\
 & & T(B) \arrow[u, "\mu(A \vee B_0,B)"] \arrow[ur, "\mu(A,B)"'] & &
\end{tikzcd}$$

The triangle is commutative from the definition of its maps. Hence, since both maps μ are homotopy equivalences by Propositions 8.3, so is the other map of the triangle. Thus, using Corollary 2.6, $M_2(A,B)$ is contractible (for all pairs (A,B)—in particular the pair

$(A \vee B_0, B))$. Then the short exact sequences 8.5 provide the inductive steps required.

Now, for $i > 0$, let C_i be the kernel of the map $\hat{c}_0 : M_i \to M_{i-1}$. Then the long exact sequence of Corollary 8.2 breaks up into short exact sequences.

$$1 \to C_i \to M_i \to C_{i-1} \to 1, \quad \text{for} \quad i > 0. \tag{8.6}$$

Since M_i is contractible for all $i > 1$, if C_i is m-connected then C_{i-1} is $(m+1)$-connected, and so C_1 is $(m+i-1)$-connected, again using Corollary 2.6. But for $m = -1$, this is true for all $i > 1$, i.e. C_1 is $(i-2)$-connected for all i. The result follows on applying Corollary 2.6 to 8.6 for $i = 1$, by J. H. C. Whitehead's theorem.

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