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Linear Algebra and its Applications 406 (2005) 77–98
LINEAR ALGEBRA AND ITS APPLICATIONS
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A discrete systems approach to cardinal spline Hermite interpolation

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Received 5 March 2004; accepted 6 March 2005
Available online 1 July 2005
Submitted by H. Schneider

Abstract

A cardinal spline Hermite interpolation problem is posed by specifying values, and $m-1$ derivatives, $m \geqslant 1$, at uniformly spaced knots t_k; it may be solved by means of a generalized spline function $w(t)$ (a standard spline function when $m=1$), piecewise a polynomial of degree $n-1=2m+p-1,\ \ p \geqslant 0$, with $w^{(j)}(t)$ continuous across the knots for $j=0,1,2,\ldots,m+p-1$. The problem is studied here for $p>0$ in the context of an $(m+p)$-dimensional system of linear recursion equations satisfied by the values of the m-th through $m+p-1$-st derivatives of $w(t)$ at the knots, whose homogeneous term involves a $p \times p$ matrix $\mathbf{A}$. In the case $m=1$ we relate the characteristic polynomial of $\mathbf{A}$ and certain controllability notions to the standard B-spline and we proceed to show how systems-theoretic ideas can be used to generate systems of basis splines for higher values of m.

AMS classification: 41A05; 41A15; 65D07; 93C55

Keywords: Spline interpolation; Linear recursion equation; Linear difference equation; Discrete linear control

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doi:10.1016/j.laa.2005.03.037

1. A class of discrete linear systems

We consider problems of cardinal Hermite spline interpolation, also described as interpolation problems with “multiple knots”, on an interval $[t_0, t_K] = [t_0, t_0 + T]$, $T > 0$. Such generalized spline interpolation problems have been widely studied; we refer in particular to the work of Schoenberg [14], Micchelli [12] and de Boor and Schoenberg [3] as well as de Boor’s text [2]. Our purpose in this paper is to develop this subject matter in an alternate setting, that of modern linear systems theory, allowing results to be viewed in a different light. Although we are primarily concerned with cardinal splines much of the work in the first part of the paper extends without difficulty to the case of nonuniform knots.

We designate the knots as $t_k,\ k = 0, 1, 2, \dots, K$, with $t_k - t_{k-1} = h = T/K$, $k = 1, 2, \dots, K$, and consider functions $w(t)$ on $[t_0, t_K]$ which reduce to polynomials of degree $\leqslant n - 1$ on each $[t_{k-1}, t_k)$; thus are solutions there of $\frac{d^n w}{dt^n} = 0$. Setting $W_j(t) = w^{(j-1)}(t),\ j = 1, 2, \dots, n$, we have the equivalent first order form

$$\frac{dW}{dt} = \mathbf{M}\, W(t), \quad \mathbf{M}_{j,\ell} = \begin{cases} 1, & \ell = j + 1,\ j = 1, 2, \dots, n - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

We define

$$W_k^+ = W(t_k+) = W(t_k), \quad W_k^- = W(t_k-), \quad k = 0, 1, 2, \dots, K - 1,$$
$$W_K^- = W(t_K);$$

then W_{k-1}^+ serves as the initial state for $W(t)$ on $[t_{k-1}, t_k)$. Thus, on that interval,

$$W(t) = e^{\mathbf{M}(t-t_{k-1})} W_{k-1}^+; \quad W_k^- = e^{\mathbf{M}h} W_{k-1}^+ \equiv \mathbf{E}(h) W_{k-1}^+. \tag{1.2}$$

In (1.2) $\mathbf{E}(h)$ is the upper triangular $n \times n$ matrix with j-row, ℓ-column, entry

$$\mathbf{e}(h)_\ell^j = \frac{h^{\ell-j}}{(\ell - j)!}, \quad 1 \leqslant j \leqslant \ell \leqslant n. \tag{1.3}$$

We suppose, for integral $p \geqslant 1$, that $n = 2m + p$ (the case $p = 0$ will be treated in a separate article). Then

$$\mathbf{E}(h) = \begin{pmatrix} \mathbf{E}_{00}(h) & \mathbf{E}_{01}(h) & \mathbf{E}_{02}(h) \\ \mathbf{O} & \mathbf{E}_{11}(h) & \mathbf{E}_{12}(h) \\ \mathbf{O} & \mathbf{O} & \mathbf{E}_{22}(h) \end{pmatrix}, \tag{1.4}$$

where $\mathbf{E}_{00}(h)$ is $m \times m$, $\mathbf{E}_{01}(h)$ is $m \times p$, $\mathbf{E}_{02}(h)$ is $m \times m, \dots, \mathbf{E}_{11}(h)$ is $p \times p, \dots$, and $\mathbf{O}$ is used generically to indicate the zero matrix of appropriate dimension. In the sequel we will use the simplified notation $\mathbf{E}, \mathbf{E}_{00}, \mathbf{E}_{01}, \dots$, unless we need to refer specifically to the dependence of these matrices on the step length h.

In this article a *generalized spline* is a function $w(t)$, as described prior to (1.1), such that, W_k^j denoting the j-th component of W_k,

$$W_k^{j+} = W_k^{j-} \equiv W_k^j, \quad k = 1, 2, \dots, K,\ j = 1, \dots, n - m = m + p. \tag{1.5}$$

This reduces, of course, to the standard spline definition when $p = 1$. We assume m-dimensional *data vectors* Z_k, $k = 0, 1, 2, \ldots, K$, are given and require

$$W_k^j = Z_k^j, \quad k = 0, 1, 2, \ldots, K, \; j = 1, 2, \ldots, m. \tag{1.6}$$

From a system-theoretic standpoint it is natural, initially, to view the vector $U_k \in E^m$, with components

$$U_{k,j} \equiv \left(W_k^{m+p+j}\right)^+, \quad j = 1, 2, \ldots, m, \tag{1.7}$$

as the "control vector" for the interval $[t_{k-1}, t_k),\; k = 1, 2, \ldots, K$. The system *state* then becomes the reduced vector $\tilde{W}$ consisting of the first $n - m = m + p$ components of W and the two together satisfy the modified system of equations (cf. (1.4))

$$\tilde{W}_k = \begin{pmatrix} \mathbf{E}_{00} & \mathbf{E}_{01} \\ \mathbf{O} & \mathbf{E}_{11} \end{pmatrix} \tilde{W}_{k-1} + \begin{pmatrix} \mathbf{E}_{02} \\ \mathbf{E}_{12} \end{pmatrix} U_{k-1}, \quad k = 1, 2, 3, \ldots \tag{1.8}$$

In this paper we study cardinal Hermite interpolation in the context of this discrete linear control system and another, equivalent, system derived from it. In particular we show that existence and construction of B-splines for solution of such interpolation problems corresponds to existence and construction of "minimal null controls" for (1.8). We also describe a method for solving these general interpolation problems related to the "Green's function" approach developed in [3] and we discuss certain matrix families arising in connection with (1.8).

The first step in this program is to establish the *controllability* (cf. [8,10], e.g.) of (1.8). We recall the standard

Definition 1.1. The system (1.8) is *controllable* on $k = 0, 1, 2, \ldots, K$ if, given an arbitrary terminal state $\hat{W} \in E^{n-m}$, there are control vectors $U_k,\; k = 0, 1, \ldots, K - 1$, such that the solution $\tilde{W}_k$ of that system with $\tilde{W}_0 = 0$ satisfies $\tilde{W}_K = \hat{W}$.

Theorem 1.1. *The system* (1.8) *is controllable on* $k = 0, 1, 2, \ldots, K$ *if* $Km \geqslant n - m$.

Proof. As in (1.2), let $W(t)$ be the n-dimensional vector with components $w_j(t) \equiv \frac{\mathrm{d}^{j-1}w}{\mathrm{d}t^{j-1}},\; j = 1, 2, \ldots, n$. Let $v(t)$ be a solution of $\frac{\mathrm{d}^n v}{\mathrm{d}t^n} = 0$ on $[t_0, t_K]$, i.e., a polynomial of degree $\leqslant n - 1$. We let $V(t)$ be the n-dimensional vector with components $v_j(t) \equiv \frac{\mathrm{d}^{j-1}v}{\mathrm{d}t^{j-1}},\; j = 1, 2, \ldots, n$, and take $\mathbf{Q}$ to be the $n \times n$ matrix with entries $q_{j,\ell} = (-1)^\ell,\; \ell = n - j + 1,\; q_{j,\ell} = 0$ otherwise, $\; j, \ell = 1, 2, \ldots, n$. Then we find that

$$\begin{aligned} &V(t_K)^*\mathbf{Q}W(t_K) - V(t_0)^*\mathbf{Q}W(t_0) \\ &\quad = \sum_{k=1}^{K-1} V(t_k)\mathbf{Q}\left(W(t_k) - W(t_k-)\right) + \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}t} V(t)^*\mathbf{Q}W(t)\,\mathrm{d}t, \end{aligned} \tag{1.9}$$

where $W(t_k-) \equiv \lim_{t\uparrow t_k} W(t)$. The last term vanishes because the integrand is

$$\begin{aligned}
&\sum_{\ell=1}^{n}(-1)^{\ell}\left(w^{(\ell)}(t)v^{(n-\ell)}(t) + w^{(\ell-1)}(t)v^{(n-\ell+1)}(t)\right)\\
&\quad = (-1)^n w^{(n)}(t)v(t) - w(t)v^{(n)}(t) + \sum_{\ell=1}^{n-1}(-1)^{\ell} w^{(\ell)}(t)v^{(n-\ell+1)}(t)\\
&\qquad + \sum_{\ell=2}^{n}(-1)^{\ell} w^{(\ell-1)}(t)v^{(n-\ell+1)}(t),
\end{aligned}$$

$\frac{\mathrm{d}^n w}{\mathrm{d}t^n}(t) \equiv \frac{\mathrm{d}^n v}{\mathrm{d}t^n}(t) \equiv 0$ on each interval $[t_{k-1}, t_k)$ and re-indexing of the last sum shows it to be the negative of the second last. Then (1.9) takes the form

$$\begin{aligned}
&\sum_{j=m+1}^{n} (-1)^j v_j(t_K) w_{n-j+1}(t_K)\\
&\quad = \sum_{j=1}^{m}(-1)^j v_j(t_0) w_{n-j+1}(t_0)\\
&\qquad + \sum_{k=1}^{K-1}\left(\sum_{j=1}^{m}(-1)^j v_j(t_k)\left(w_{n-j+1}(t_k) - w_{n-j+1}(t_k-)\right)\right). \qquad (1.10)
\end{aligned}$$

Let us require $v_j(t_K) = 0,\ j = 1, 2, \ldots, m,\ w_j(t_0) = 0,\ j = 1, 2, \ldots, n-m$, and suppose $(-1)^j v_j(t_K),\ j = m+1, m+2, \ldots, n$, chosen as components of a vector orthogonal to the subspace of all reachable $(n-m)$-dimensional vectors $\tilde{W}(t_K)$ with components $w_j(t_K),\ j = 1, 2, \ldots, n-m$. Then the left hand side of (1.10) is zero for any choice of the vectors $U_0, U_1, \ldots, U_{K-1}$. Since the latter are arbitrary the values $w_j(t_0),\ w_j(t_k) - w_j(t_k-),\ k = 1, 2, \ldots, K-1,\ \ j = n-m+1, n-m+2, \ldots, n$, are arbitrary as well and we conclude $v_j(t_k) = 0,\ k = 0, 1, 2, \ldots, K-1,\ \ j = 1, 2, \ldots, m$. Then the polynomial $v(t)$, of degree $\leqslant n-1$, has a zero of multiplicity m at $t_k,\ k = 0, 1, 2, \ldots, K$. If $(K+1)m \geqslant n$, equivalently $Km \geqslant n-m$, then $v(t) \equiv 0 \Rightarrow v(t_K) = 0$ and we conclude that the achievable vectors $\tilde{W}(t_K)$ cover all of E^{n-m}; the proof is complete. $\square$

Adaptation to the case $\tilde{W}_0 \neq 0$ and/or indices $J, J+1, \ldots, J+K$ is straightforward.

2. A transformed system

In an interpolation problem it is actually the supplied data, the vectors Z_k, which "drive" the system. Our next goal is to replace (1.8) by a transformed system in which this control aspect of the data becomes evident.

Lemma 2.1. *The $m \times m$ matrix $\mathbf{E}_{02}(h)$ in (1.4) and (1.8) is nonsingular for $h \neq 0$.*

Proof. Inspection shows $\mathbf{E}_{02}(h)$ to be the Wronskian matrix for the functions $\rho_j(h) = \frac{h^{n-j}}{(n-j)!},\ j = 1, 2, \ldots, m$. These are linearly independent solutions of the differential equation $\frac{\mathrm{d}^m}{\mathrm{d}h^m}\left(h^{m-n}\rho(h)\right) = 0$ so the Wronskian determinant, $\det(\mathbf{E}_{02}(h))$, must be different from zero everywhere except at $h = 0$. The proof is complete. □

We decompose the $(2m + p)$-dimensional vector W into vector components $Z,\ W$ and U of dimensions $m,\ p$ and m, respectively. The choice of Z for the first of these vector components reflects the notation in (1.6). Then from (1.2)–(1.4) we have

$$\begin{pmatrix} Z_k \\ Y_k \\ U_k \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{00} & \mathbf{E}_{01} & \mathbf{E}_{02} \\ \mathbf{O} & \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{O} & \mathbf{O} & \mathbf{E}_{22} \end{pmatrix} \begin{pmatrix} Z_{k-1} \\ Y_{k-1} \\ U_{k-1} \end{pmatrix}, \quad k = 1, 2, \ldots, K. \tag{2.1}$$

Using Lemma 1.1 we can solve for U_{k-1}:

$$U_{k-1} = \mathbf{E}_{02}^{-1}\left(Z_k - \mathbf{E}_{00}Z_{k-1} - \mathbf{E}_{01}Y_{k-1}\right), \quad k = 1, 2, \ldots, K. \tag{2.2}$$

Then substituting (2.2) into the second equation of (2.1) we have

$$\begin{aligned} Y_k &= \mathbf{E}_{11}Y_{k-1} + \mathbf{E}_{12}\mathbf{E}_{02}^{-1}\left(Z_k - \mathbf{E}_{00}Z_{k-1} - \mathbf{E}_{01}Y_{k-1}\right) \\ &= \left(\mathbf{E}_{11} - \mathbf{E}_{12}\mathbf{E}_{02}^{-1}\mathbf{E}_{01}\right)Y_{k-1} + \mathbf{E}_{12}\mathbf{E}_{02}^{-1}\left(Z_k - \mathbf{E}_{00}Z_{k-1}\right) \\ &\equiv \mathbf{A}Y_{k-1} + \mathbf{B}\left(Z_k - \mathbf{E}_{00}Z_{k-1}\right). \end{aligned} \tag{2.3}$$

When we wish to indicate explicitly the dimensions m and p involved we will append these indices to $\mathbf{A},\ \mathbf{B}$; viz.: $\mathbf{A}_{m,p},\ \mathbf{B}_{m,p}$.

Since the vectors Z_k (cf. (1.6)) are assumed given as data, (2.3) is a linear recursion equation for the vectors Y_k with known inhomogeneous term. Given *any* solution $Y_k,\ k = 0, 1, 2, \ldots, K$, of that system, (2.2) becomes an explicit formula for the vectors $U_{k-1},\ k = 1, 2, \ldots, K$, whose determination, followed by use of (1.2), provides a solution of the interpolation problem. Typically p is a small positive integer and (2.3) is a system of substantially lower dimension than the original. Solutions of the homogeneous counterpart of (2.3) are frequently described in the literature as *null splines*. It would be reasonable to call (2.3) the *coupling equation* for the interpolation problem; when $p = 0$ and (2.3) is vacuous, the problem decouples into individual problems on the intervals $[t_{k-1}, t_k]$.

If we now set $V_{k-1} = Z_k - \mathbf{E}_{00}Z_{k-1},\ k \geqslant 1$, (2.3) becomes

$$\begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{E}_{00} \end{pmatrix} \begin{pmatrix} Y_{k-1} \\ Z_{k-1} \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{I} \end{pmatrix} V_{k-1}, \quad k = 1, 2, \ldots, K, \tag{2.4}$$

and, with the obvious notational modification, (2.2) becomes the “output” relation

$$U_{k-1} = \mathbf{F}Y_{k-1} + \mathbf{E}_{02}^{-1}V_{k-1}, \quad k = 1, 2, \ldots, K. \tag{2.5}$$

It will be recognized that the V_k are zero when the data vectors Z_k are consistent with a polynomial of degree $\leqslant m - 1$.

Proposition 2.1. *The system* (2.4) *is controllable if* $Km \geqslant n - m$.

Proof. The vectors $\tilde{W}_k$ in Theorem 1.1 correspond to the vectors $\binom{Y_k}{Z_k}$ in (2.4). Let initial and terminal states $\binom{Y_0}{Z_0}$ and $\binom{Y_K}{Z_K}$ be chosen for (2.4) and let these be used for $\tilde{W}_0$ and $\tilde{W}_K$ in Theorem 1.1, which then shows the existence of control vectors $U_k,\ k = 0, 1, 2, \ldots, K-1$, "steering" $\tilde{W}_k$ from the first to the second. Then $V_{k-1} = \mathbf{E}_{02}(U_{k-1} - \mathbf{F}Y_{k-1}),\ k = 1, 2, \ldots, K$, solves the same control problem for the system (2.4), completing the proof. □

3. The matrices $\mathbf{A}_{m,p}$

The matrices $\mathbf{A}(h) = \mathbf{A}_{m,p}(h)$ appearing in (2.3) and (2.4) have a long history, having been studied by Greville [6], Schoenberg [14], Lipow and Schoenberg [9] and de Boor and Schoenberg [3], among others. The derivation of $\mathbf{A}(h)$ presented in the previous section is, we believe, comparatively direct. The results of Proposition 3.1 are anticipated in these works but are recapitulated here to make our paper reasonably self-contained and because the proofs given are quite brief.

Proposition 3.1. *The* $p \times p$ *matrix* $\mathbf{A}(h)$ *is similar, for* $h \neq 0$, *to* $\mathbf{A}(1)$ *and to* $\mathbf{A}(h)^{-1}$. *Thus its eigenvalues, but not, in general, its eigenvectors, are independent of* $h \neq 0$. *For each eigenvalue* λ *of* $\mathbf{A}(h)$ *the reciprocal* λ^{-1} *is also an eigenvalue of* $\mathbf{A}(h)$.

Proof. Let $\mathbf{H}_j(h) = \mathrm{diag}(1\ \ h\ \ h^2\ \ \cdots\ \ h^{j-1})$; we will just write $\mathbf{H}(h)$ when $j = n$. It is clear from (1.3) that

$$\mathbf{E}(h) = \mathbf{H}(h)^{-1}\mathbf{E}(1)\mathbf{H}(h). \tag{3.1}$$

Let us block decompose $\mathbf{H}(h)$ as

$$\mathbf{H}(h) = \mathrm{diag}\big(\mathbf{Q}(h), \mathbf{R}(h), \mathbf{S}(h)\big), \tag{3.2}$$

where $\mathbf{Q}(h)$ (not related to $\mathbf{Q}$ in Theorem 1.1) and $\mathbf{S}(h)$ are $m \times m$ and $\mathbf{R}(h)$ is $p \times p$. Substituting (1.4) and (3.2) into (3.1) and computing $\mathbf{A}(h)$ separately from both sides of the resulting equation, we obtain

$$\begin{aligned}\mathbf{A}(h) &= \mathbf{R}(h)^{-1}\mathbf{E}_{11}(1)\mathbf{R}(h)\\ &\quad -\mathbf{R}(h)^{-1}\mathbf{E}_{12}(1)\mathbf{S}(h)\mathbf{S}(h)^{-1}\mathbf{E}_{02}(1)^{-1}\mathbf{Q}(h)\mathbf{Q}(h)^{-1}\mathbf{E}_{01}(1)\mathbf{R}(h)\\ &= \mathbf{R}(h)^{-1}\mathbf{A}(1)\mathbf{R}(h) = \mathbf{H}_p(h)^{-1}\mathbf{A}(1)\mathbf{H}_p(h).\end{aligned} \tag{3.3}$$

and the first stated result follows.

Using the preceding result twice, once with $h \neq 0$ and once with $-h$, we see that

$$\mathbf{A}(h) = \mathbf{R}(h)^{-1}\mathbf{A}(1)\mathbf{R}(h), \quad \mathbf{A}(-h) = \mathbf{R}(-h)^{-1}\mathbf{A}(1)\mathbf{R}(-h) \tag{3.4}$$

and therefore

$$\mathbf{A}(-h) = \mathbf{R}(-h)^{-1}\mathbf{R}(h)\mathbf{A}(h)\mathbf{R}(h)^{-1}\mathbf{R}(-h) = \mathbf{R}(-1)^{-1}\mathbf{A}(h)\mathbf{R}(-1).$$

But the interpolation process works exactly the same way in the reverse direction, i.e., in the direction of decreasing k; the only difference is that h is replaced by $-h$. Thus if we invert (2.1) using (3.1) to obtain the equation expressing Y_{k-1} in terms of Y_k the proof is completed via

$$\mathbf{A}(h)^{-1} = \mathbf{A}(-h) \quad\Rightarrow\quad \mathbf{A}(h)^{-1} = \mathbf{R}(-1)^{-1}\mathbf{A}(h)\mathbf{R}(-1). \qquad \square$$

The matrices $\mathbf{A}$ exhibit a certain nonstandard symmetry already used, indirectly, in Theorem 1.1. Let $\mathbf{P}$ be the $n-1 \times n-1$ matrix (in [5] it is called a "sip" matrix) with k-row and j-column entries $\mathbf{p}_j^k = \delta_{j,n-k}$. Then

$$\mathbf{P} = \begin{pmatrix} \mathbf{O} & \mathbf{O} & \hat{\mathbf{P}} \\ \mathbf{O} & \tilde{\mathbf{P}} & \mathbf{O} \\ \hat{\mathbf{P}} & \mathbf{O} & \mathbf{O} \end{pmatrix} \tag{3.5}$$

with $\hat{\mathbf{P}}$ $m \times m$ and $\tilde{\mathbf{P}}$ $p \times p$. Clearly $\tilde{\mathbf{P}} = \tilde{\mathbf{P}}^* = \tilde{\mathbf{P}}^{-1}$.

Proposition 3.2. *The matrices* $\mathbf{A}\tilde{\mathbf{P}}$ *and* $\tilde{\mathbf{P}}\mathbf{A} = \tilde{\mathbf{P}}^*(\mathbf{A}\tilde{\mathbf{P}})\tilde{\mathbf{P}}$ *are symmetric. Further,* $\mathbf{A}$ *is similar to its adjoint* $\mathbf{A}^*$ *via* $\tilde{\mathbf{P}}$. *If* λ *is an eigenvalue of* $\mathbf{A}$ *with associated eigenvector* Φ *and if* $\Phi^*\mathbf{P}\Phi \neq 0$ *then* λ *must be real. Eigenvectors* Φ *and* Ψ *corresponding to distinct real eigenvalues* λ *and* μ, *respectively, of* $\mathbf{A}$ *satisfy* $\Psi^*\mathbf{P}\Phi = 0$.

Proof. The symmetry of $\mathbf{A}\tilde{\mathbf{P}}$ corresponds to $\mathbf{A}$ being symmetric about its "southwest to northeast" diagonal; i.e., denoting its row m and column ℓ entry by $\mathbf{a}_\ell^m$, $\mathbf{a}_\ell^m = \mathbf{a}_{n-m+1}^{n-\ell+1}$. From the form of $\mathbf{M}$ in (1.1) we see that $\mathbf{M}^* = \mathbf{PMP} = \mathbf{P}^{-1}\mathbf{MP}$, which clearly implies $\mathbf{E}^* = \mathbf{PEP}$ for all h. Use of (1.4) and (3.5) in this equation followed by a block by block analysis shows that

$$\mathbf{A}^* = \left(\mathbf{E}_{11} - \mathbf{E}_{12}\mathbf{E}_{02}^{-1}\mathbf{E}_{01}\right)^* = \tilde{\mathbf{P}}\left(\mathbf{E}_{11} - \mathbf{E}_{12}\mathbf{E}_{02}^{-1}\mathbf{E}_{01}\right)\tilde{\mathbf{P}} = \tilde{\mathbf{P}}^{-1}\mathbf{A}\tilde{\mathbf{P}}$$

from which all of the statements relating $\mathbf{A}$, $\mathbf{A}^*$ and $\tilde{\mathbf{P}}$ follow.

If λ is an eigenvalue of $\mathbf{A}$ with associated eigenvector Φ, we observe that

$$\lambda\Phi^*\tilde{\mathbf{P}}\Phi = \Phi^*\tilde{\mathbf{P}}\mathbf{A}\Phi = \left(\mathbf{A}^*\tilde{\mathbf{P}}\Phi\right)^*\Phi = \left(\tilde{\mathbf{P}}\mathbf{A}\Phi\right)^*\Phi = \overline{\lambda}\left(\tilde{\mathbf{P}}\Phi\right)^*\Phi = \overline{\lambda}\Phi^*\tilde{\mathbf{P}}\Phi. \tag{3.6}$$

If $\Phi^*\tilde{\mathbf{P}}\Phi \neq 0$ we conclude $\lambda = \overline{\lambda}$, i.e., λ is real. The argument showing $\Psi^*\tilde{\mathbf{P}}\Phi = 0$ for $\lambda \neq \mu$ is an easy modification of (3.6). The proof is complete. $\square$

The matrix $\mathbf{A}$ is an example of a matrix which is self-adjoint relative to an indefinite inner product, as developed, e.g., by Gohberg, Lancaster and Rodman in [5]; in this case the indefinite inner product is $[Y_1, Y_2]_{\tilde{\mathbf{P}}} = Y_2^*\tilde{\mathbf{P}}Y_1$.

In the references cited at the beginning of this section $(-1)^m\mathbf{A}_{m,p}$ is identified as an *oscillation matrix* as extensively studied in the treatise of Gantmacher and

Krein [4]. Using this connection more detailed properties of the eigenvalues λ of $\mathbf{A}_{m,p}$ are derived, including results to the effect that the λ are real and simple and include 1 or -1 if and only if p is odd. If a concise way can be found to establish the result $\Phi^*\tilde{\mathbf{P}}\Phi \neq 0$ referred to above, a quick proof of the reality of the eigenvalues λ would result. An apparently more challenging task would be to use new methods to replace the proof of simplicity of the eigenvalues which, in the cited references, depends on the rather intricate development in [4] of the properties of oscillation matrices and the comparably intricate demonstration that $(-1)^m\mathbf{A}_{m,p}$ is such a matrix. We are unable to offer general results in these directions at this writing. We do provide a fairly direct proof for $m = 1$ in the next section. For arbitrary m and $p = 2$ we can offer the following.

Theorem 3.1. *For* $m \geqslant 1,\ p = 2,\ h = 1,$ *we have*

$$\mathbf{A}_{m,2} = (-1)^m \begin{pmatrix} (m+1) & \frac{1}{m+1} \\ m(m+1)(m+2) & (m+1) \end{pmatrix}. \tag{3.7}$$

The characteristic polynomials are $\lambda^2 + (-1)^{m+1}2(m+1)\lambda + 1$ *and the eigenvalues are*

$$\lambda_1 = (-1)^m\big((m+1) + \sqrt{(m+1)^2 - 1}\,\big),$$
$$\lambda_2 = (-1)^m\big((m+1) - \sqrt{(m+1)^2 - 1}\,\big).$$

Proof. The relationship (2.3) must hold when the data z_k are such that the interpolating spline $w(t)$ is an arbitrary polynomial of degree $2m + 1$. Taking $h = 1$, (2.3) becomes, in that case

$$Y_k = \mathbf{A}_{m,2}Y_{k-1} + \mathbf{B}_{m,2}(Z_k - \mathbf{E}_{00}Z_{k-1}). \tag{3.8}$$

We take $t_0 = 0$ and consider the polynomial $w(t) = t^m(1-t)^m(at + b)$ for arbitrary constants $a,\ b$. Computing the vectors $W_0 = \big(Z_0^*\ Y_0^*\ U_0^*\big)^*,\ W_1 = (Z_1^*\ Y_1^*\ U_1^*)^*$ as in Section 2 we find that

$$Y_0 = \begin{pmatrix} m!b \\ (m+1)!(a - mb) \end{pmatrix} = \mathbf{C}_0 \begin{pmatrix} m!a \\ m!b \end{pmatrix}, \quad \mathbf{C}_0 = \begin{pmatrix} 0 & 1 \\ m+1 & -m(m+1) \end{pmatrix}, \tag{3.9}$$

$$Y_1 = (-1)^m\mathbf{C}_1 \begin{pmatrix} m!a \\ m!b \end{pmatrix}, \quad \mathbf{C}_1 = \begin{pmatrix} 1 & 1 \\ (m+1)^2 & m(m+1) \end{pmatrix}. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8) with $k = 1$ and using $Z_0 = Z_1 = 0$ implied by the form of $w(t)$ we have

$$\mathbf{A}_{m,2} = (-1)^m\mathbf{C}_1\mathbf{C}_0^{-1} = (-1)^m \begin{pmatrix} (m+1) & \frac{1}{m+1} \\ m(m+1)(m+2) & (m+1) \end{pmatrix},$$

which is (3.7). This completes the proof. $\square$

4. The matrices $\mathbf{A}_{1,p}$; standard B-splines

Since the standard, $m = 1$, spline interpolation theory is extremely well explored in the literature, the emphasis in this paper necessarily lies with $m > 1$. However, it seems advisable to review some of the standard theory and the matrices $\mathbf{A}_{1,p}$ which arise in that case.

Theorem 4.1. *The eigenvalues of* $\mathbf{A}_{1,p}(h)$, $h \neq 0$, *are real, negative and simple. Further, the eigenvalues of* $\mathbf{A}_{1,p}(h)$ *separate those of* $\mathbf{A}_{1,p+1}(h)$ *and* -1 *is an eigenvalue of* $\mathbf{A}_{1,p}(h)$ *if and only if* p *is odd.*

Proof. From the results of the preceding section, we may without loss of generality carry out the proof for $h = 1$. Suppose, for a complex number λ, we can find a polynomial $w(t)$ of degree $p + 1$, which clearly cannot be identically 0, with the properties

$$w(t + 1) \equiv \lambda w(t) + t^{p+1}, \quad w(0) = 0. \tag{4.1}$$

Taking $t = 0$, (4.1) gives $w(1) = 0$ as well. Defining the vectors $W_k = (\overline{z_k} \;\; Y_k^* \;\; \overline{u_k})^*$ (z_k and u_k are scalar here so we use lower case) as in Section 2 we have $z_0 = z_1 = 0$ and, since $\mathbf{E}_{00} \equiv 1$ for $m = 1$, (2.3) gives $Y_1 = \mathbf{A}_{1,p} Y_0$. However, taking the 1-st through p-th derivatives of (4.1) at $t = 0$ we have $w^{(k)}(1) = \lambda w^{(k)}(0)$, $k = 1, 2, \ldots, p$, which is the same as $Y_1 = \lambda Y_0$. Thus λ is an eigenvalue of $\mathbf{A}_{1,p}$ unless $Y_0 = 0$. If that were the case we would necessarily have $w(t) \equiv a t^{p+1}$ for some constant $a \neq 0$. Then (4.1) would read

$$a(t + 1)^{p+1} \equiv \lambda a t^{p+1} + t^{p+1},$$

which is clearly not possible. We conclude $Y_0 \neq 0$; thus if we can identify distinct numbers λ_k, $k = 1, 2, \ldots, p$, for which $p + 1$-st degree polynomial solutions $w(t) = w(t, \lambda_k)$ of (4.1) exist, then these must be the eigenvalues of $\mathbf{A}_{1,p}$.

Let $Q(\xi)$ be the distribution valued Fourier transform of $w(t)$. Then

$$Q(\xi) = \frac{\beta \delta^{(p+1)}(\xi)}{e^{i\xi} - \lambda} \tag{4.2}$$

for some nonzero scalar β, $\delta^{(k)}(\xi)$ being the k-th distributional derivative of the Dirac delta distribution $\delta(\xi)$. From standard properties of distributions (see, e.g., [15]) (4.2) is a linear combination of the $\delta^{(k)}(\xi)$, $k = 0, 1, 2, \ldots, p + 1$, in which the coefficient of $\delta^{(0)}(\xi) = \delta(\xi)$ is

$$(-1)^{p+1} \frac{d^{p+1}}{d\xi^{p+1}} \left(\frac{1}{e^{i\xi} - \lambda} \right) \Bigg|_{\xi=0}. \tag{4.3}$$

But that coefficient is also $w(0)$. Thus the values of λ we seek are those for which (4.3) is zero.

Setting $x = e^{i\xi}$ we have $dx = ie^{i\xi}\, d\xi = ix d\xi$ and thus $\frac{d}{d\xi} = -ix \frac{d}{dx}$. So we may now seek values of λ for which

$$\phi_{p+1}(1,\lambda) = \phi_{p+1}(x,\lambda)\big|_{x=1} = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p+1}\left(\frac{1}{x-\lambda}\right)\bigg|_{x=1} = 0. \tag{4.4}$$

Let us define, for $j \geqslant 0$,

$$\psi_j(\eta) = \left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^j\left(\frac{1}{1+\eta}\right). \tag{4.5}$$

From the identities

$$\psi_j\left(\frac{1}{\eta}\right) = -\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\psi_{j-1}\left(\frac{1}{\eta}\right), \quad \psi_1\left(\frac{1}{\eta}\right) = \frac{-\eta}{(1+\eta)^2} = \psi_1(\eta), \tag{4.6}$$

we can prove by induction that $\psi_j\left(\frac{1}{\eta}\right) = (-1)^{j-1}\psi_j(\eta),\ j \geqslant 1$. It follows that nonzero, nonunit roots of $\psi_j(\eta),\ j \geqslant 3$, occur in mutually reciprocal pairs and, provided we can establish that all roots of $\psi_j(\eta),\ j \geqslant 1$, are nonnegative and simple, that 1 is a root for j even.

We represent $\psi_j(\eta)$ in the form

$$\psi_j(\eta) = \frac{\eta\gamma_j(\eta)}{(1+\eta)^{j+1}}, \quad j \geqslant 1,$$

wherein $\gamma_j(\eta)$ is a polynomial in η of degree $j-1$. From (4.6) we have $\gamma_1(\eta) = -1$ and we have the readily derived recursion formula (cf. [7], Section 6.5)

$$\gamma_{j+1}(\eta) = (1 - j\eta)\gamma_j(\eta) + \eta(1+\eta)\gamma_j'(\eta), \quad j \geqslant 1. \tag{4.7}$$

From this we see immediately that $\gamma_2(\eta)$ has the single positive root 1. For purpose of induction, let us suppose $\gamma_j(\eta)$ has $j-1$ simple, positive roots. Let the largest of these be $\hat{\eta}$. Rolle's theorem shows that the $j-2$ roots of $\gamma_j'(\eta)$ are all less than $\hat{\eta}$. Thus if $\gamma_j(\eta) = a\eta^{j-1} +$ (lower degree terms) the sign of $\gamma_j'(\hat{\eta})$ must be the same as that of a and $\gamma_j(\eta)$ must have this sign for $\eta - \hat{\eta} > 0$ and small. But (4.7) shows that $\gamma_{j+1}(\eta) = -a\eta^j +$ (lower degree terms) so the ultimate sign of $\gamma_{j+1}(\eta)$ as $\eta \to \infty$ is opposite that of a and we conclude $\gamma_{j+1}(\eta)$ has a root $\tilde{\eta}$ in the interval $(\hat{\eta}, \infty)$. From the earlier result on reciprocal pairs, the least positive root of $\gamma_j(\eta)$ is $\hat{\eta}^{-1}$ and $\gamma_{j+1}(\eta)$ has the root $\tilde{\eta}^{-1}$ in $(0, \hat{\eta}^{-1})$. Since the roots of $\gamma_j(\eta)$ are assumed simple, $\gamma_j'(\eta)$ assumes nonzero values of alternating sign at successive roots of $\gamma_j(\eta)$. Then (4.7) shows that $\gamma_{j+1}(\eta)$ also assumes nonzero values of alternating sign at those points and we conclude γ_{j+1} has a root between each successive pair of roots of $\gamma_j(\eta)$; clearly these must be simple and positive as well and, by induction, we conclude $\gamma_j(\eta)$ has $j-1$ simple, positive roots for all j with the interpolation property just described; $\psi_j(\eta)$ clearly has only $\eta = 0$ as an additional root.

Finally we observe via (4.4) and the transformation $x = -\lambda\eta$ that

$$\psi_{p+1}(\eta) = -\lambda\phi_{p+1}(-\lambda\eta, \lambda) \ \Rightarrow\ \psi_{p+1}\left(\frac{-1}{\lambda}\right) = \phi_{p+1}(1,\lambda)$$

from which it follows that for each of the p positive roots, which we generically indicate by $\frac{-1}{\lambda}$, of $\psi_{p+1}(\eta)$, the corresponding negative value λ is such that

$\phi_{p+1}(1,\lambda)=0$. The result on reciprocal pairs of roots, the interpolation property stated in the theorem, and the fact that $\lambda=-1$ is a root of (4.4) just in case p is odd then all follow from the corresponding results from $\psi_{p+1}(\eta)$ developed above. The proof is complete. □

Using exact arithmetic in Matlab^R, for $p=1,2,3,4$, hence $n=3,4,5,6$, we have calculated the matrices $\mathbf{A}_{1,p}\equiv\mathbf{A}_{1,p}(1)$ with their eigenvalues and characteristic polynomials $\chi(\lambda)$:

$$\mathbf{A}_{1,1}=-1,\quad \lambda=-1,\quad \chi(\lambda)=\lambda+1;$$

$$\mathbf{A}_{1,2}=\begin{pmatrix}-2 & -\frac{1}{2}\\ -6 & -2\end{pmatrix},$$

$$\lambda_1=-2+\sqrt{3},\quad \lambda_2=-2-\sqrt{3},\quad \chi(\lambda)=\lambda^2+4\lambda+1.$$

$$\mathbf{A}_{1,3}=\begin{pmatrix}-3 & -1 & -\frac{1}{6}\\ -12 & -5 & -1\\ -24 & -12 & -3\end{pmatrix},\qquad \begin{aligned}\lambda_1&=-1,\\ \lambda_2&=-5+2\sqrt{6},\\ \lambda_3&=-5-2\sqrt{6},\end{aligned}$$

$$\chi(\lambda)=\lambda^3+11\lambda^2+11\lambda+1.$$

$$\mathbf{A}_{1,4}=\begin{pmatrix}-4 & -\frac{3}{2} & -\frac{1}{3} & -\frac{1}{24}\\ -20 & -9 & -\frac{7}{3} & -\frac{1}{3}\\ -60 & -30 & -9 & -\frac{3}{2}\\ -120 & -60 & -20 & -4\end{pmatrix},$$

$$\lambda_1=-\tfrac{13}{2}+\tfrac{1}{2}\sqrt{105}+\tfrac{1}{2}\sqrt{270-26\sqrt{105}},$$

$$\lambda_2=-\tfrac{13}{2}+\tfrac{1}{2}\sqrt{105}-\tfrac{1}{2}\sqrt{270-26\sqrt{105}},$$

$$\lambda_3=-\tfrac{13}{2}-\tfrac{1}{2}\sqrt{105}+\tfrac{1}{2}\sqrt{270+26\sqrt{105}},$$

$$\lambda_4=-\tfrac{13}{2}-\tfrac{1}{2}\sqrt{105}-\tfrac{1}{2}\sqrt{270+26\sqrt{105}},$$

$$\chi(\lambda)=\lambda^4+26\lambda^3+66\lambda^2+26\lambda+1.$$

Additionally, cf. (2.3), we have the vectors $B_{1,p}(1)$, the $m=1$ versions of the matrices $\mathbf{B}_{m,p}(1)$) (shown transposed to save space),

$$B_{1,1}=(2),\quad B^*_{1,2}=(3\quad 6),$$
$$B^*_{1,3}=(4\quad 12\quad 24),\quad B^*_{1,4}=(5\quad 20\quad 60\quad 120).$$

The following theorem anticipates the more general results of the next section and provides a good example of a proof obtained by system-theoretic methods. The Euler–Frobenius identification is made in [14].

Theorem 4.2. *If the nonzero knot values of the standard B-spline of degree $n-1=p+1$ are normalized so that the first and last of these values are* 1, *these values are the coefficients of the characteristic polynomial of* $\mathbf{A}_{1,p}$, *an Euler–Frobenius polynomial.*

Proof. As in Theorem 4.1 we may work with $h=1$. Let the characteristic polynomial of $\mathbf{A}_{1,p}$ be $\sum_{k=0}^{p}\zeta_{p-k}\lambda^k$. Then the characteristic polynomial of $\left[\begin{smallmatrix}\mathbf{A} & 0\\ 0 & 1\end{smallmatrix}\right]$ is $(\lambda-1)\sum_{k=0}^{p}\zeta_{p-k}\lambda^k \equiv \sum_{k=0}^{p+1}\beta_{p-k}\lambda^k$. We carry out the multiplication and see that $\beta_j=\zeta_{j+1}-\zeta_j,\ \ j=-1,0,1,\ldots,p$. Taking $v_j=\beta_j,\ z_j=\zeta_j$ we have $z_{j+1}=z_j+v_j$, which agrees with the second equation of (2.4), using the lower case because the z_j are scalar when $m=1$. Computing the solution of the spline control system for $Y_{-1}=0,\ z_{-1}=0$ we get, with $\mathscr{B}=(\beta_{-1}\ \ \beta_0\ \ \cdots\ \ \beta_p)^*$,

$$\begin{aligned}\begin{bmatrix}Y_{p+1}\\ z_{p+1}\end{bmatrix} &= \left[\begin{pmatrix}\mathbf{A} & 0\\ 0 & 1\end{pmatrix}^{p+1}\begin{pmatrix}B\\ 1\end{pmatrix},\ldots,\begin{pmatrix}\mathbf{A} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}B\\ 1\end{pmatrix},\begin{pmatrix}B\\ 1\end{pmatrix}\right]\mathscr{B}\\ &= \left[\sum_{k=0}^{p+1}\beta_{p-k}\begin{pmatrix}\mathbf{A} & 0\\ 0 & 1\end{pmatrix}^{k}\right]\begin{pmatrix}B\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned} \tag{4.8}$$

by the Cayley–Hamilton theorem. Continuing forward from $p+1$ and backward from -1 we see that $Y_k=0,\ z_k=0,\ \ k\leqslant -1,\ k\geqslant p+1$. Thus the corresponding nonzero spline $w(t)$ constructed via (2.2), (2.5) and (1.2) has support in $[-1,p+1]$. Since $p+1=n-m$ when $m=1$, Proposition 2.1 and the standard matrix characterization of controllability shows that the matrix multiplying $\mathscr{B}$ in (4.8) is nonsingular if $p+1$ is replaced by p; we conclude there is no nonzero spline with shorter support so $w(t)$ must be the standard B-spline of degree $n-1=p+1$ and the proof is complete. □

5. Hermite interpolation; the matrices $\mathbf{A}_{m,p},\ m\geqslant 2$

When $m\geqslant 2$ the controls V_k and corresponding Z_k are m-dimensional vectors and the relationship to the characteristic polynomials of the matrices $\mathbf{A}_{m,p}$ is no longer as straightforward as in Proposition 5.1. Nevertheless the relationship between minimal null controls and B-splines can be extended to these cases with some modifications.

Let us consider a general discrete linear control system

$$X_k=\mathbf{A}X_{k-1}+\mathbf{B}U_{k-1},\quad X\in E^q,\ U\in E^m. \tag{5.1}$$

For our purposes we may, and do, assume that the columns of $\mathbf{B}$ are linearly independent. Symbols used here are not necessarily related to those signified by the same letter elsewhere in this paper. The coefficient matrices $\mathbf{A},\ \mathbf{B}$ are independent of k. We will assume (5.1) is *controllable*; algebraically that means there is a least positive integer μ such that the "controllability matrix"

$$\mathbf{C}(\mathbf{A},\mathbf{B}) \equiv (\,\mathbf{A}^{\mu}\mathbf{B},\quad \mathbf{A}^{\mu-1}\mathbf{B},\quad \ldots,\quad \mathbf{A}\mathbf{B},\quad \mathbf{B}\,) \tag{5.2}$$

has rank q. The Cayley–Hamilton theorem implies $\mu \leqslant q-1$ if any such μ exists. We will call μ the *controllability index*. It is straightforward to see that the basic control problem posed by selecting an initial state X_0 and a terminal state X_K, for some positive integer K, and then requiring $U_j,\ \ j=0,1,2,\ldots,K-1$, to be chosen so as to steer a solution of (5.1) from X_0 to X_K can be solved for arbitrary $X_0,\ X_K$ if and only if (5.2) has rank q and $K \geqslant \mu$.

Definition 5.1. A sequence $U_j,\ \ j=J,J+1,J+2,\ldots,K-1$, is a (nontrivial) *null control* if at least some of the U_j listed are nonzero and this sequence steers $X_J=0$ to $X_K=0$. The corresponding sequence of states $X_J\ (=0), X_{J+1},\ldots,$ $X_K\ (=0)$ is a *null trajectory*. The two together form a *null pair* of length $K-J+1$. A second null pair consisting of the same vectors as the first but corresponding to indices $\tilde{J}=J+L,\ \tilde{K}=K+L$ for some integer L is *translation equivalent* to the first.

Definition 5.2. A *minimal null pair* is a null pair which cannot be written as a linear combination of null pairs of length shorter than the original.

It is clear that any linear combination of null pairs is also a null pair. Two minimal null pairs of the same length may differ by a null pair of the same, or shorter, length. If $m=1$ and the system (5.1) is controllable, it is easy to see that a null pair is minimal if and only if its length is $q+1$ and that any two minimal null pairs are translation equivalent after comparable normalization. In the standard spline setting of Section 5 this is reflected in the fact that there is really just one B-spline, others being obtained from it by translation. The situation for $m \geqslant 2$ is more complex. However, we have

Theorem 5.1. *Let* (5.1) *be controllable and let* μ *be the controllability index. Then there exists a set* $\mathscr{N}$, *not necessarily unique, of* m *minimal null pairs, each of length* $\leqslant \mu+2$, *such that every null pair is a unique linear combination of the minimal null pairs in* $\mathscr{N}$ *together with their translates.*

Proof. We have already assumed that the columns $B_1, B_2, \ldots, B_m$ of $\mathbf{B}$ are linearly independent. If $m=n$ we stop the process immediately. Otherwise, we form the vectors $\mathbf{A}B_1, \mathbf{A}B_2, \ldots, \mathbf{A}B_m$ and, working backward from $j=m$, we discard any $\mathbf{A}B_j$ which can be written as a linear combination of the others and the B_i;

re-ordering, we can assume we are left with $\mathbf{A}B_j,\ \ j = 1, 2, \ldots, m_1$. It is then clear that the vectors $\mathbf{A}^2 B_j,\ \ j = m_1 + 1, m_1 + 2, \ldots, n,$ will be a linear combination of the vectors $\mathbf{A}^2 B_j,\ \ j = 1, 2, \ldots, m_1$, and the $\mathbf{A}B_\ell,\ \ \ell = 1, 2, \ldots, m_1$. From the $\mathbf{A}^2 B_j,\ \ j = 1, 2, \ldots, m_1$, beginning at $j = m_1$, we discard any that are linear combinations of the others, the $\mathbf{A}B_\ell,\ \ \ell = 1, 2, \ldots, m_1,$ and the $B_\ell,\ \ \ell = 1, 2, \ldots, m$. Continuing in this manner and re-ordering as required at each step, we arrive at non-increasing, nonnegative integers μ_j with $\mu_1 = \mu$ and $\sum_{j=1}^{m} (\mu_j + 1) = q$ together with m sequences of vectors

$$\mathscr{S}_j = \{\, B_j \quad \mathbf{A}B_j \quad \cdots \quad \mathbf{A}^{\mu_j} B_j \,\}, \qquad j = 1, 2, \ldots, m, \tag{5.3}$$

such that $\bigcup_{j=1}^{m} \mathscr{S}_j$ is a basis for E^q.

Using these vectors, a canonical reduction process, due to Brunovsky [1] and also described in [13], can be carried out. Specifically one may construct matrices $\mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3$ of respective dimensions $q \times q,\ \ m \times q,\ \ m \times m$, with $\mathbf{Q}_1$ and $\mathbf{Q}_3$ nonsingular, such that transformations

$$X = \mathbf{Q}_1 \hat{X}, \quad U = \mathbf{Q}_2 \hat{X} + \mathbf{Q}_3 \hat{U}, \tag{5.4}$$

reduce (5.1) to

$$\hat{X}_k = \hat{\mathbf{A}} \hat{X}_{k-1} + \hat{\mathbf{B}} \hat{U}_{k-1}, \tag{5.5}$$

wherein $\hat{\mathbf{A}}$ is block diagonal with diagonal blocks of successive dimensions $\mu_j \times \mu_j,\ \ j = 1, 2, \ldots, m$:

$$\hat{\mathbf{A}}_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{B}} = \begin{pmatrix} \hat{\mathbf{B}}_1 \\ \hat{\mathbf{B}}_2 \\ \vdots \\ \hat{\mathbf{B}}_{m-1} \\ \hat{\mathbf{B}}_m \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{B}}_j = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

the nonzero column of $\hat{\mathbf{B}}_j$ being the j-th. Decomposing $\hat{X}$ in a manner conformable with the indicated decomposition of $\hat{\mathbf{B}}$, (5.5) is equivalent to m lower-dimensional systems, each of dimension $\mu_j + 1$,

$$\hat{X}_{k,j} = \hat{\mathbf{A}}_j \hat{X}_{k-1,j} + \hat{B}_j \hat{u}_{k-1,j}, \quad j = 1, 2, \ldots, m, \tag{5.6}$$

wherein $\hat{B}_j$ is the j-th column of $\hat{\mathbf{B}}_j$ and the $\hat{u}_{k-1,j}$ are the components of $\hat{U}_{k-1}$.

For each of the systems (5.6) the sequence

$$\hat{u}_{0,j} = 1, \quad \hat{u}_{\ell,j} = 0, \quad \ell = 1, 2, \ldots, \mu_j,$$

is readily seen to provide a null control; if we set $\hat{X}_{0,j} = 0$ and use this control then $\hat{X}_{\mu_j,j} = 0$. The control/trajectory pair thus realized is easily seen to be minimal and has length $\mu_j + 2$.

Let $\{\mathscr{X}\}_j \equiv \{\hat{X}_{k,j} \mid k = 0, 1, 2, \ldots, \nu\}$ and $\{\mathscr{U}\} \equiv \{\hat{u}_{k,j} \mid k = 0, 1, 2, \ldots, \nu - 1\}$ be a null pair for (5.6) of length $\nu + 1$. Since the vectors $\hat{\mathbf{A}}^q \hat{B}_j, \;\; q = 0, 1, 2, \ldots, \mu_j - 1$, are linearly independent and $\hat{\mathbf{A}}^q = 0, \;\; q \geqslant \mu_j$, we see that $\hat{u}_{\nu-\mu_j,j}, \ldots, \hat{u}_{\nu-1,j}$ must all be zero. It is then clear that we have the unique relationship

$$\{\mathscr{X}\}_j = \sum_{\ell=0}^{\nu-\mu_j-1} \hat{u}_{\ell,j} \Xi_{j,\ell},$$

where $\Xi_{j,\ell}, \;\; \ell = 0, 1, 2, \ldots, \nu - \mu_j - 1$ is the translated minimal null trajectory sequence

$$\Xi_{j,\ell} = \left\{ 0 \quad \cdots \quad 0 \quad \hat{B}_j \quad \hat{\mathbf{A}}_j \hat{B}_j \quad \cdots \quad \hat{\mathbf{A}}_j^{\mu_j - 1} \hat{B}_j \quad 0 \quad \cdots \quad 0 \right\}.$$

Here $\hat{B}_j$ is in the $(\ell + 1)$-st position. Thus $\{\mathscr{X}\}_j$ is a unique linear combination of translates of $\Xi_{j,0}$. Clearly every null trajectory sequence $\mathscr{X}$ for (5.5) has the form

$$\mathscr{X} = \begin{pmatrix} \mathscr{X}_1 \\ \mathscr{X}_2 \\ \vdots \\ \mathscr{X}_j \end{pmatrix}$$

and thus is a unique linear combination of minimal null trajectory sequences

$$\begin{pmatrix} 0 \\ \vdots \\ \mathscr{X}_{j,\ell} \\ \vdots \\ 0 \end{pmatrix}.$$

The corresponding control sequences bear the same relationships to each other. Reversing (5.4) we have the corresponding result for the original system (5.1) and the proof is complete. $\square$

We define a *B-spline* for the system (2.3) to be a generalized spline $w(t)$ associated via (1.2) with a minimal null trajectory $\left\{ \binom{\hat{Y}_k}{\hat{Z}_k} \right\}$ of that system.

Corollary 5.1. *Every generalized spline $w(t)$ associated via* (1.2) *with a solution of* (2.3) *and vanishing outside a compact interval $[t_J, t_K]$ is a linear combination of B-splines, unique once the sequences $\mathscr{S}_j, \;\; j = 1, 2, \ldots, m$, defined in* (5.3) *have been fixed.*

Remark. We must qualify the uniqueness result as indicated because, for a given system (2.3) there may be several sets of sequences $\mathscr{S}_j$, as defined in (5.3), spanning E^n, leading to different reduced systems (5.5).

Proposition 5.1. *Let* $\binom{Y_k}{Z_k}$, $k = \hat{J}, \hat{J}+1, \ldots, \hat{J}+\hat{K}$, V_k, $k = \hat{J}, \hat{J}+1, \ldots, \hat{J}+\hat{K}-1$, *be an arbitrary solution of* (2.4), *corresponding via* (2.5) *and* (1.2) *to a solution* $w(t)$ *on the interval* $[t_{\hat{J}}, t_{\hat{J}+\hat{K}}]$ *of the Hermite interpolation problem with data vectors* Z_k *and knots* t_k *as indicated. This solution is a restriction of a null pair, which we indicate by the same symbols, with support consisting of an interval* $\left[t_J, t_{J+K}\right]$, *with* $\hat{J} - J$ *and* $J + K - (\hat{J} + \hat{K})$ *both less than or equal to the smallest integer* κ *such that* $\kappa m \geqslant n - m \ (= m + p)$.

Remark. Thus, from Corollary 5.1, each solution $w(t)$ of an interpolation problem on $[t_{\hat{J}}, t_{\hat{J}+\hat{K}}]$, is a linear combination of B-splines whose joint support is, in general, a larger interval $[t_J, t_{J+K}]$ whose size is restricted as indicated in the proposition.

Proof. The controllability result of Proposition 2.1 shows that, whatever the state $\binom{Y_{\hat{J}}}{Z_{\hat{J}}}$, there will be $J \leqslant \hat{J}$ with $\hat{J} - J$ restricted as indicated, and controls V_k, $k = J, J+1, \ldots, \hat{J}-1$, steering the zero state at $k = J$ to $\binom{Y_{\hat{J}}}{Z_{\hat{J}}}$ at $k = \hat{J}$. Since the properties of the interpolation control system (2.4) are invariant under reversal of the order of the indices, there are also control vectors steering $\binom{Y_{\hat{J}+\hat{K}}}{Z_{\hat{J}+\hat{K}}}$ to the zero state at $k = J + K$ as indicated and the proof is complete. □

The Brunovsky reduction cited in Theorem 5.1, while useful theoretically, is rather ungainly in practice, as may be seen from its recapitulation in [13]. Fortunately it is generally not necessary to go through the details of this procedure in order to generate B-splines for a given pair m, p when these are of moderate size.

Example 1. We consider the case $m = 2$, $p = 2$, fifth degree generalized spline Hermite interpolation. Again it will be sufficient to work with $h = 1$, for which we have, in that case,

$$\begin{pmatrix} y_{k,1} \\ y_{k,2} \end{pmatrix} = \begin{pmatrix} 3 & \frac{1}{3} \\ 24 & 3 \end{pmatrix} \begin{pmatrix} y_{k-1,1} \\ y_{k-1,2} \end{pmatrix} + \begin{pmatrix} -20 & 8 \\ -120 & 36 \end{pmatrix} \begin{pmatrix} z_{k,1} \\ z_{k,2} \end{pmatrix} + \begin{pmatrix} 20 & 12 \\ 120 & 84 \end{pmatrix} \begin{pmatrix} z_{k-1,1} \\ z_{k-1,2} \end{pmatrix}. \tag{5.7}$$

Corresponding to (2.5) we set

$$\begin{pmatrix} v_{k-1,1} \\ v_{k-1,2} \end{pmatrix} = \begin{pmatrix} z_{k,1} \\ z_{k,2} \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_{k-1,1} \\ z_{k-1,2} \end{pmatrix} \tag{5.8}$$

and we have the control system (2.4),

$$\begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{2,2} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix} \begin{pmatrix} Y_{k-1} \\ Z_{k-1} \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{2,2} \\ \mathbf{I}_2 \end{pmatrix} V_{k-1},$$

or, more specifically,

$$\begin{pmatrix} y_{k,1} \\ y_{k,2} \\ z_{k,1} \\ z_{k,2} \end{pmatrix} = \begin{pmatrix} 3 & \frac{1}{3} & 0 & 0 \\ 24 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{k-1,1} \\ y_{k-1,2} \\ z_{k-1,1} \\ z_{k-1,2} \end{pmatrix} + \begin{pmatrix} -20 & 8 \\ -120 & 36 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_{k-1,1} \\ v_{k-1,2} \end{pmatrix}. \tag{5.9}$$

Applying the standard test for controllability, the 4×4 and 4×6 matrices

$$\left[\begin{pmatrix} \mathbf{A}_{2,2} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{2,2} \\ \mathbf{I} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{B}_{2,2} \\ \mathbf{I} \end{pmatrix} \right] \tag{5.10}$$

and

$$\left[\begin{pmatrix} \mathbf{A}_{2,2} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix}^2 \begin{pmatrix} \mathbf{B}_{2,2} \\ \mathbf{I} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{A}_{2,2} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{2,2} \\ \mathbf{I} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{B}_{2,2} \\ \mathbf{I} \end{pmatrix} \right], \tag{5.11}$$

respectively, are each seen to have rank 4. The nonsingularity of the 4×4 matrix in (5.10) confirms that (5.9) is a controllable system. On the other hand the fact that the 4×6 matrix in (5.11) continues (obviously) to have rank 4 implies the existence of a two-dimensional null space which may be seen to be spanned by the vectors

$$V = \begin{pmatrix} 1 \\ 2.5 \\ -2.5 \\ -5 \\ 1.5 \\ 2.5 \end{pmatrix} = \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \quad \text{and} \quad \hat{V} = \begin{pmatrix} .8 \\ 1 \\ -2.6 \\ 0 \\ -.2 \\ -1 \end{pmatrix} = \begin{pmatrix} \hat{V}_0 \\ \hat{V}_1 \\ \hat{V}_2 \end{pmatrix}.$$

Once we have the two sets of vectors V_0, V_1 and V_2 we can use (5.8) and (5.7) with $Z_0 = 0,\ Y_0 = 0$ to generate corresponding sets $Z_k,\ Y_k,\ k = 1, 2, 3$, and we can verify $Z_3 = 0,\ Y_3 = 0$ in each case. Then we generate the vectors $U_k,\ k = 0, 1, 2$, from the corresponding version of (2.5) and, finally, generate the corresponding splines using (1.2). The resulting B-splines, $b_1(t)$ and $b_2(t)$, are shown in Fig. 1.

Example 1 worked out in the very "clean" manner which, in fact, is typical for cases wherein p is an integer multiple of m. When this is not the case some additional complications occur, the most notable of which is the occurrence of B-splines whose supports have different lengths. This feature is developed in the next example.

Example 2. We consider the case $m = 2,\ p = 3$, Hermite interpolation with generalized splines of degree $2m + p - 1 = 6$. Here (2.4) is five-dimensional, taking the form (for $h = 1$)

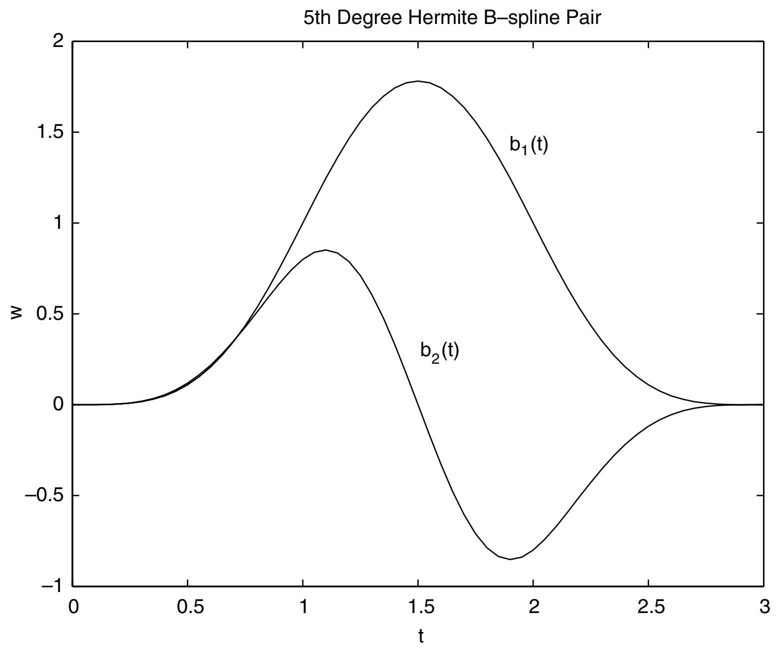

Fig. 1.

$$\begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{2,3} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix} \begin{pmatrix} Y_{k-1} \\ Z_{k-1} \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{2,3} \\ \mathbf{I}_2 \end{pmatrix} V_{k-1}$$

with matrices $\mathbf{E}_{00,2}$ and $\mathbf{I}_2$ as in Example 1 and

$$\mathbf{A}_{2,3} = \begin{pmatrix} 6 & 1 & 1/12 \\ 60 & 11 & 1 \\ 300 & 60 & 6 \end{pmatrix}, \quad \mathbf{B}_{2,3} = \begin{pmatrix} -30 & 10 \\ -240 & 60 \\ -1080 & 240 \end{pmatrix}.$$

When we form the controllability matrices in this case we find that the 5×6 matrix

$$\left[\begin{pmatrix} \mathbf{A}_{2,3} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix}^2 \begin{pmatrix} \mathbf{B}_{2,3} \\ \mathbf{I}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{A}_{2,3} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{2,3} \\ \mathbf{I} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{B}_{2,3} \\ \mathbf{I}_2 \end{pmatrix} \right], \tag{5.12}$$

has rank 5. The one-dimensional null space is spanned by the six-dimensional vector V whose components, listed two at a time, are

$$V_{1,0} = \begin{pmatrix} 7/24 \\ 1 \end{pmatrix}, \quad V_{1,1} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad V_{1,2} = \begin{pmatrix} 17/24 \\ 1 \end{pmatrix}. \tag{5.13}$$

Going through the process already described in Example 1, this leads to null pair of length 4 and a corresponding B-spline, $b_1(t)$ with support of length 3, shown in Fig. 2. Next we augment (5.12), adjoining

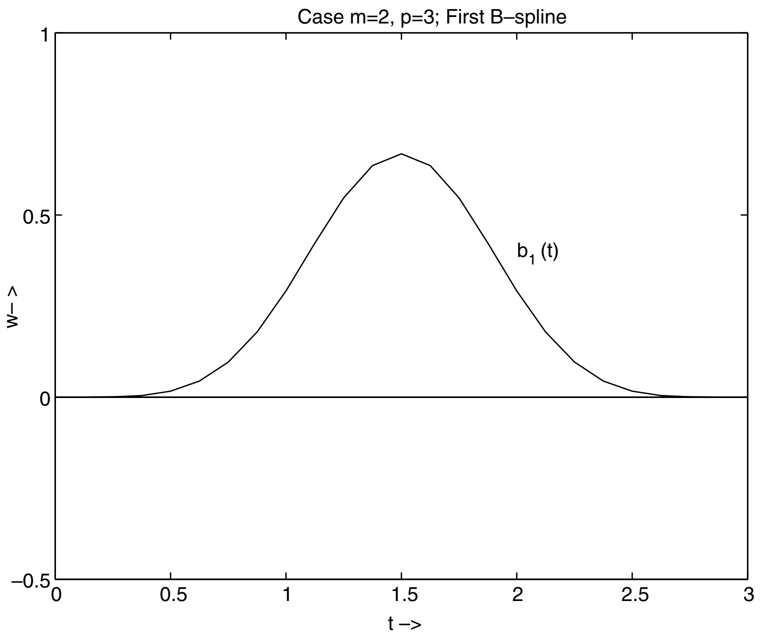

Fig. 2.

$$\begin{pmatrix} \mathbf{A}_{2,3} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_{00,2} \end{pmatrix}^3 \begin{pmatrix} \mathbf{B}_{2,3} \\ \mathbf{I}_2 \end{pmatrix}$$

at the left to produce a 5×8 matrix which, in addition to the obvious extension of (5.13), has null vectors with two-dimensional component vectors

$$V_{2,0} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad V_{2,1} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}, \quad V_{2,2} = \begin{pmatrix} 0 \\ -23 \end{pmatrix}, \quad V_{2,3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

$$V_{3,0} = \begin{pmatrix} 31/24 \\ 2 \end{pmatrix}, \quad V_{3,1} = \begin{pmatrix} -18 \\ -3 \end{pmatrix}, \quad V_{3,2} = \begin{pmatrix} 401/24 \\ 0 \end{pmatrix}, \quad V_{3,3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These yield, by the process already described, B-splines $b_2(t), b_3(t)$, shown in Fig. 3.

These, by appearance, are almost linearly dependent. Let us define $b_{1+}(t)$ to be the B-spline obtained by shifting $b_1(t)$ one unit to the right. Using the Gram-Schmidt process we adjust both $b_2(t)$ and $b_3(t)$ so as to be orthogonal to both $b_1(t)$ and $b_{1+}(t)$; we designate the new splines by $\beta_2(t)$ and $\beta_3(t)$. These, shown in Fig. 4, are, in fact, linearly dependent.

We can select any one of these; say we redefine $b_2(t) = \beta_2(t)$. Then the new set $b_1(t), b_2(t)$, together with their translates, form a complete set of B-splines for the case $m = 2, \ p = 3$. These are shown, with $b_2(t)$ re-normalized, in Fig. 5. (These

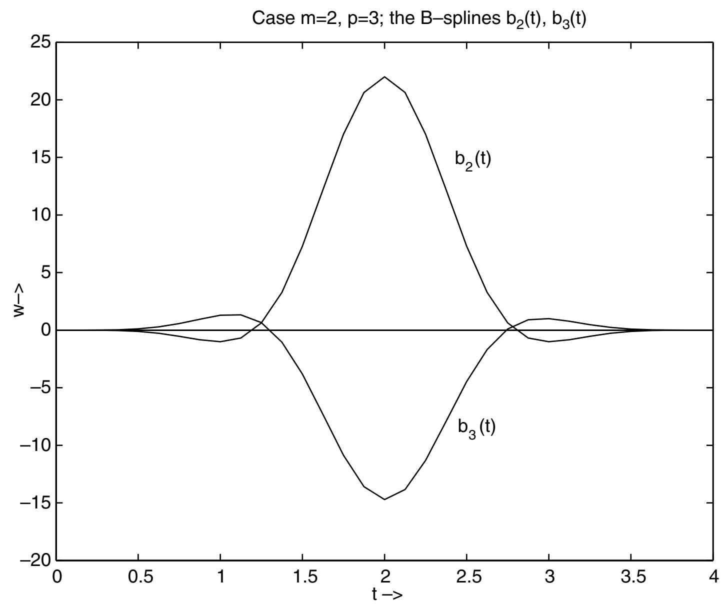
Case m=2, p=3; the B–splines $b_2(t)$, $b_3(t)$
$b_2(t)$
$b_3(t)$
w–>
t –>
25
20
15
10
5
0
−5
−10
−15
−20
0
0.5
1
1.5
2
2.5
3
3.5
4

Fig. 3.

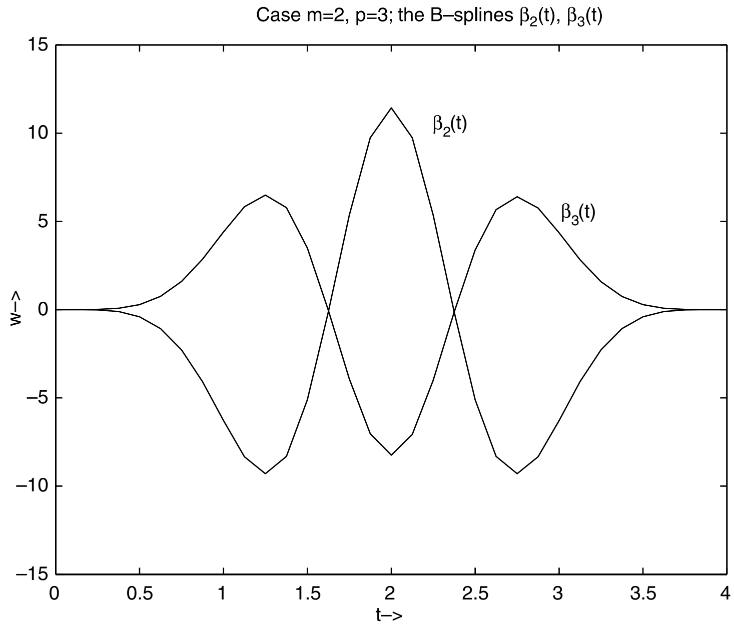
Case m=2, p=3; the B–splines $\beta_2(t)$, $\beta_3(t)$
$\beta_2(t)$
$\beta_3(t)$
w–>
t–>
15
10
5
0
−5
−10
−15
0
0.5
1
1.5
2
2.5
3
3.5
4

Fig. 4.

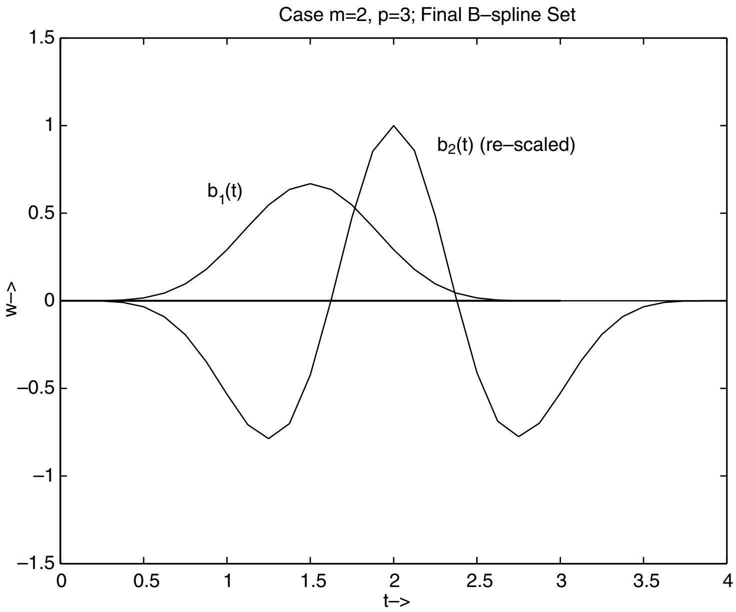

Fig. 5.

plots were done with limited resolution; $b_2(t)$ is smoother than it appears in Figs. 3–5. The support of $b_1(t)$ extends from $t = 0$ to $t = 3$, though this also is not particularly evident from the figures.) A pair of B-splines each of whose support has the same length, i.e., 4, can be obtained as $b_2(t) \pm b_1(t)$. These might well be preferable in computations.

These results, and the method, may be compared with Schoenberg's development of cardinal Hermite B-splines in [14] and with the work of Merz [11].

Acknowledgment

The authors wish to thank Professors Carl de Boor and Hans Schneider of the University of Wisconsin – Madison and Professor Olga Holtz of the Technical University of Berlin for very helpful discussions of the subject matter.

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