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Characterization of the existence of a pure-strategy Nash equilibrium ☆

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ABSTRACT

In this work, we provide a necessary and sufficient condition for the existence of a pure-strategy Nash equilibrium for non-cooperative games in topological spaces.

1. Introduction

In mathematical economics, the main problem in investigating various kinds of economic models is showing the existence of an equilibrium, and already, a number of equilibrium existence results in economic models have been investigated by many authors (e.g., see, [1–7]).

The purpose of this work is to present a theorem that completely characterizes the existence of a pure-strategy Nash equilibrium for non-cooperative games in topological spaces. We do so by introducing the $\mathcal{C}$-quasiconcavity condition which unifies the diagonal transfer quasiconcavity (weaker than the quasiconcavity) due to Baye et al. [5] and the $\mathcal{C}$-concavity (weaker than concavity) due to Kim and Lee [6].

For the remainder of this section we give some definitions and notations.

Throughout this work, all topological spaces are assumed to be Hausdorff.

Let A be a subset of a topological space X. We denote by $\mathrm{cl}_X A$ the closure of A in X. Let Δ_n be the standard n-dimensional simplex in $\mathbb{R}^{n+1}$. If A is a subset of a vector space, we denote by $\mathrm{co}A$ the convex hull of A.

Let I be a finite set of players. A *non-cooperative game* is a family of ordered tuples $\Gamma = (X_i, u_i)$ where the non-empty set X_i is the ith player's pure strategy space, and $u_i : X = \prod_{i\in I} X_i \to \mathbb{R}$ is the ith player's payoff function. The set X is the Cartesian product of the individual strategy spaces. Denote by X_{-i} the product $\prod_{i\in I\setminus\{i\}} X_i$. Denote by x_i and x_{-i} an element of X_i and X_{-i}, respectively. Denote an arbitrary point of X by $x = (x_i, x_{-i})$, with x_i in X_i and x_{-i} in X_{-i}. Moreover, (x_i, z_{-i}) denotes the point y in X with $y_i = x_i$ and $y_{-i} = z_{-i}$. A point $x^* \in X$ is said to be a *pure-strategy Nash equilibrium* for Γ if $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$ for all $x_i \in X_i$ and for all $i \in I$.

2. Characterization of a pure-strategy Nash equilibrium

Definition 1. Let X be a topological space, and $A, Y \subseteq X$. A function $f : X \times Y \to \mathbb{R}$ is called *$\mathcal{C}$-quasiconcave* on A if, for any finite subset $\{x^0, x^1, \ldots, x^n\}$ of A, there exists a continuous mapping $\phi_n : \Delta_n \to Y$ such that

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$f(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) \geq \min\{f(x^i, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) : i \in J\}$ for all $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, where $J = \{i \in \{0, 1, \ldots, n\} : \lambda_i \neq 0\}$.

For the $\mathcal{C}$-quasiconcavity, we have the following two propositions that show that the $\mathcal{C}$-quasiconcavity unifies the diagonal transfer quasiconcavity (weaker than the quasiconcavity) due to Baye et al. [5] and the $\mathcal{C}$-concavity (weaker than concavity) due to Kim and Lee [6].

Proposition 1. *Let X be a convex subset of a topological vector space. Let us have $\emptyset \neq A \subseteq X$, C a non-empty convex subset of X, and $f : X \times C \to \mathbb{R}$ a function. If f is diagonally transfer quasiconcave on A,*[1] *then f is $\mathcal{C}$-quasiconcave on A.*

Proof. Let $\{x^0, x^1, \ldots, x^n\}$ be a finite subset of A. Since f is diagonally transfer quasiconcave on A, there exists a finite subset $\{y^0, y^1, \ldots, y^n\}$ of C such that for any subset $\{y^{k_0}, y^{k_1}, \ldots, y^{k_s}\} \subseteq \{y^0, y^1, \ldots, y^n\}$, $0 \leq s \leq n$, and any $y^* \in \mathrm{co}\{y^{k_0}, y^{k_1}, \ldots, y^{k_s}\}$, we have $\min_{0\leq l\leq s} f(x^{k_l}, y^*) \leq f(y^*, y^*)$. Now we define the mapping $\phi_n : \Delta_n \to C$ as follows:

$$\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n) = \lambda_0 y_0 + \lambda_1 y_1 + \cdots + \lambda_n y_n,$$

for all $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$.

Obviously, ϕ_n is continuous. Let $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$ and $J = \{i \in \{0, 1, \ldots, n\} : \lambda_i \neq 0\}$. Then

$$\sum_{j\in J} \lambda_j y^j \in \mathrm{co}\{y^j : j \in J\} \text{ and } \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n) = \sum_{i=0}^{n} \lambda_i y^i = \sum_{j\in J} \lambda_j y^j.$$

Consequently,

$$\begin{aligned} f(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) &= f\left(\sum_{j\in J} \lambda_j y^j, \sum_{j\in J} \lambda_j y^j\right) \\ &\geq \min\left\{f\left(x^j, \sum_{j\in J} \lambda_j y^j\right) : j \in J\right\} \\ &= \min\{f(x^j, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) : j \in J\}. \end{aligned}$$

This completes the proof. □

Proposition 2. *Let X be a topological space, and $f : X \times X \to \mathbb{R}$ a function. If f is $\mathcal{C}$- concave on X,*[2] *then f is $\mathcal{C}$-quasiconcave on X.*

Proof. Let $\{x^0, x^1, \ldots, x^n\}$ be a finite subset of X. Since f is $\mathcal{C}$-concave on X, there exists a continuous mapping $\phi_n : \Delta_n \to X$ such that

$$f(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), y) \geq \lambda_0 f(x^0, y) + \lambda_1 fU(x^1, y) + \cdots + \lambda_n f(x^n, y),$$

for all $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$ and all $y \in X$. In particular,

$$f(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) \geq \sum_{i=0}^{n} \lambda_i f(x^i, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)).$$

Let $J = \{i \in \{0, 1, \ldots, n\} : \lambda_i \neq 0\}$. Then

$$\begin{aligned} f(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) &\geq \sum_{i=0}^{n} \lambda_i f(x^i, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) \\ &= \sum_{j\in J} \lambda_j f(x^j, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) \\ &\geq \min\{f(x^j, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) : j \in J\} \sum_{j\in J} \lambda_j \\ &= \min\{f(x^j, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) : j \in J\}. \end{aligned}$$

This completes the proof. □

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[1] Diagonal transfer quasiconcavity, due to Baye et al. [5], requires that for any finite subset $\{x^0, x^1, \ldots, x^n\}$ of A, there exists a finite subset $\{y^0, y^1, \ldots, y^n\}$ of C such that for any subset $\{y^{k_0}, y^{k_1}, \ldots, y^{k_s}\} \subseteq \{y^0, y^1, \ldots, y^n\}$, $0 \leq s \leq n$, and any $y^* \in co\{y^{k_0}, y^{k_1}, \ldots, y^{k_s}\}$, one has $\min_{0\leq l\leq s} f(x^{k_l}, y^*) \leq f(y^*, y^*)$. This is a weaker requirement than quasiconcavity and the diagonal quasiconcavity due to Zhou and Chen [11].

[2] $\mathcal{C}$-concavity, due to Kim and Lee [6], requires that for any finite subset $\{x^0, x^1, \ldots, x^n\}$ of X, there exists a continuous mapping $\phi_n : \Delta_n \to X$ such that $f(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), y) \geq \sum_{i=0}^{n} \lambda_i f(x^i, y)$ for all $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$ and all $y \in X$. This is a weaker requirement than concavity and the CF-concavity due to Forgö [9].

Let $\Gamma = (X_i, u_i)$ be a non-cooperative game. Following the method introduced by Nikaido and Isoda [8], the aggregate function $U : X \times X$ is given by

$$U(x, y) = \sum_{i \in I} u_i(x_i, y_{-i}),$$

for any $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X = \prod_{i=1}^{n} X_i$.

Then we shall need the following:

Lemma 1 (*See Proposition 1 of [7]*). *Let $\Gamma = (X_i, u_i)$ be a non-cooperative game, and $\bar{x} \in X$. Then $\bar{x}$ is a pure-strategy Nash equilibrium of Γ if and only if $U(\bar{x}, \bar{x}) \geq U(x, \bar{x})$ for all $x \in X$.*

The following theorem states our main result.

Theorem 1. *Let Γ be a non-cooperative game, and $U : X \times X \to \mathbb{R}$ be the aggregate function. Then Γ has a pure-strategy Nash equilibrium if and only if there exists a non-empty compact subset C of X such that the following hold:*

(i) *C has the fixed point property[3];*

(ii) *the restricted mapping $U|_{X \times C} : X \times C \to \mathbb{R}$ is diagonally transfer continuous on C[4] and is $\mathcal{C}$- quasiconcave on X.*

Proof. *Necessity*. Suppose that the game Γ has a pure-strategy Nash equilibrium $x^* \in X$. Let $C = \{x^*\}$. Obviously, C is compact, and (i) is satisfied. The restricted mapping $U|_{X \times C}$ clearly is diagonally transfer continuous on C. We want to show that $U|_{X \times C}$ is $\mathcal{C}$-quasiconcave. Let $\{x^0, x^1, \ldots, x^n\}$ be a finite subset of X. Now we define the mapping $\phi_n : \Delta_n \to C$ by

$$\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n) = x^*,$$

for all $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$. Obviously, ϕ_n is continuous. Let $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$. By Lemma 1, we have

$$U(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) = U(x^*, x^*) \geq U(x, x^*),$$

for any $x \in X$. In particular, if we put $J = \{i \in \{0, 1, \ldots, n\} : \lambda_i \neq 0\}$, then

$$U(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) = U(x^*, x^*) \geq U(x^i, x^*)$$

for all $i \in J$, and thus

$$U(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) \geq \min\{U(x^i, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) : i \in J\}.$$

Sufficiency. Let C be a compact subset of X satisfying (i) and (ii). We show that Γ has a pure-strategy Nash equilibrium. For typographical reasons, we use H to denote the mapping $U|_{X \times C}$. For each $x \in X$, let

$$G(x) = \{y \in C : H(x, y) \leq H(y, y)\}.$$

We first prove $\cap_{x \in X} \text{cl}_C G(x) = \cap_{x \in X} G(x)$. It is clear that $\cap_{x \in X} \text{cl}_C G(x) \supseteq \cap_{x \in X} G(x)$. So we only need to show $\cap_{x \in X} \text{cl}_C G(x) \subseteq \cap_{x \in X} G(x)$. Let $y \in (C \setminus \cap_{x \in X} G(x))$. Then there is an $x \in X$ such that $y \notin G(x)$, i.e., $H(x, y) > H(y, y)$. By the diagonal transfer continuity of H, there exist some $x' \in X$ and some neighbourhood $N(y)$ of y in C such that $H(x', z) > H(z, z)$ for all $z \in N(y)$. Thus $y \notin \text{cl}_C G(x')$.

Now we show that the family $\{\text{cl}_C G(x) : x \in X\}$ has the finite intersection property.

Suppose, by way of contradiction, that $\{\text{cl}_C G(x) : x \in X\}$ does not have the finite intersection property, i.e., there exists some finite subset $\{x^0, x^1, \ldots, x^n\}$ of X such that $\cap_{i=0}^{n} \text{cl}_C G(x^i) = \emptyset$. Then $\cup_{i=0}^{n} (C \setminus \text{cl}_C G(x^i)) = C$. Since C is compact, there is a partition of unity $\{\alpha_i : i = 0, 1, \ldots, n\}$ subordinate to $\{C \setminus \text{cl}_C G(x^i) : i = 0, 1, \ldots, n\}$, i.e., for each $i = 0, 1, \ldots, n$, there exists a continuous function $\alpha_i : C \to [0, 1]$ such that (1) $\alpha_i^{-1}(0, 1] \subseteq C \setminus \text{cl}_C G(x^i)$; (2) for each $x \in C$, $\sum_{i=0}^{n} \alpha_i(x) = 1$.

Since H is $\mathcal{C}$-quasiconcave in X, there exists a continuous mapping $\phi_n : \Delta_n \to C$ such that

$$H(\phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n), \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) \geq \min\{H(x^j, \phi_n(\lambda_0, \lambda_1, \ldots, \lambda_n)) : j \in J\}, \quad (3)$$

for all $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, where $J = \{i \in \{0, 1, \ldots, n\} : \lambda_i \neq 0\}$.

Now consider the map $\psi : C \to C$, defined by

$$\psi(x) = \phi_n(\alpha_0(x), \alpha_1(x), \ldots, \alpha_n(x)), \quad \text{for each } x \in C.$$

Since ϕ_n and all α_i are continuous, ψ also is continuous. By the condition (i), there exists an element $\bar{x}$ of C such that $\psi(\bar{x}) = \bar{x}$, and thus $\phi_n(\alpha_0(\bar{x}), \alpha_1(\bar{x}), \ldots, \alpha_n(\bar{x})) = \bar{x}$.

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3 The fixed point property, due to Granas and Dugundji [10], requires that every continuous mapping $f : C \to C$ has a fixed point.

4 Diagonal transfer continuity, due to Baye et al. [5], requires that for every $(x, y) \in X \times C$, $U(x, y) > U(y, y)$ implies that there exist some point $x' \in X$ and some neighbourhood $N(y)$ of y in C such that $U(x', z) > U(z, z)$ for all $z \in N(y)$. This is a weaker requirement than continuity in $X \times C$.

Let $J = \{i \in \{0, 1, \dots, n\} : \alpha_i(\bar{x}) \neq 0\}$. Then $J \neq \emptyset$ by (2). By (1), for any $j \in J$, we have $\bar{x} \in \alpha_j^{-1}(0, 1] \subseteq C \setminus \mathrm{cl}_C G(x^j)$, and therefore, $\bar{x} \notin G(x^j)$, and thus $H(x^j, \bar{x}) > H(\bar{x}, \bar{x})$. Therefore,

$$\min\{H(x^j, \bar{x}) : j \in J\} > H(\bar{x}, \bar{x}).$$

Combining this fact and (3), we have

$$\begin{aligned} H(\bar{x}, \bar{x}) &= H(\psi(\bar{x}), \psi(\bar{x})) \\ &= H(\phi_n(\alpha_0(\bar{x}), \alpha_1(\bar{x}), \dots, \alpha_n(\bar{x})), \phi_n(\alpha_0(\bar{x}), \alpha_1(\bar{x}), \dots, \alpha_n(\bar{x}))) \\ &\geq \min\{H(x^j, \phi_n(\alpha_0(\bar{x}), \alpha_1(\bar{x}), \dots, \alpha_n(\bar{x}))) : j \in J\} \\ &= \min\{H(x^j, \bar{x}) : j \in J\} > H(\bar{x}, \bar{x}). \end{aligned}$$

This is a contradiction. Since C is compact, $\cap\{\mathrm{cl}_C G(x) : x \in X\} \neq \emptyset$. Pick out an element $x^* \in \cap\{\mathrm{cl}_C G(x) : x \in X\}$. Then by the previous arguments, we have $x^* \in \cap\{G(x) : x \in X\}$. It is easy to see that x^* is a pure-strategy Nash equilibrium of Γ. □

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