Periodic and subharmonic solutions of a class of subquadratic second-order Hamiltonian systems ✤

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Received 12 June 2005
Available online 19 June 2006
Submitted by J. Mawhin

Abstract

Some solvability conditions of periodic and subharmonic solutions are obtained for a class of subquadratic nonautonomous second-order Hamiltonian systems by the minimax methods in critical point theory.

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Keywords: Periodic solution; Subharmonic solutions; Saddle Point Theorem; Condition (C); Subquadratic; Hamiltonian systems

1. Introduction and main results

Consider the second-order Hamiltonian systems

$$\ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in \mathbb{R},$$

where $F: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is $T$-periodic in its first variable with $T > 0$ and satisfies the following assumption:

✩ Supported by National Natural Science Foundation of China (No. 10471113) and by the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE, PR China.

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doi:10.1016/j.jmaa.2006.05.064
(A) \( F(t, x) \) is measurable in \( t \) for all \( x \in \mathbb{R}^N \), continuously differentiable in \( x \) for a.e. \( t \in [0, T] \), and there exist \( a \in C(R^+, R^+) \) and \( b \in L^1(0, T; R^+) \) such that

\[
\left| F(t, x) \right| \leq a(|x|)b(t), \quad \left| \nabla F(t, x) \right| \leq a(|x|)b(t)
\]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \).

It has been proved that problem (1) has infinitely distinct subharmonic solutions under suitable conditions (see [1,3,6–11,13–15,18,19]). After Rabinowitz [14] consider the subquadratic potential second-order Hamiltonian systems, [1,6] consider the superquadratic second-order Hamiltonian systems with a changing sign potential. The convex potentials (see [3,8,19]), the even potentials (see [18]), the subquadratic potential (see [9–11,13,15]) were also considered. In 1980, Rabinowitz [14] obtained the following result.

**Theorem A.** [14] Suppose that \( F \in C^1(R \times \mathbb{R}^N, \mathbb{R}) \) is \( T \)-periodic in its first variable with \( T > 0 \) and satisfies the following conditions:

1. **(F1)** There exist \( 1 < \mu < 2 \) and \( L > 0 \) such that for all \( |x| \geq L \) and \( t \in [0, T] \),

\[
0 < x \cdot \nabla F(t, x) \leq \mu F(t, x).
\]

2. **(F2)** There exist \( a_1, a_2 > 0 \) and \( s \in (1, \mu) \) such that for all \( x \in \mathbb{R}^N \) and \( t \in [0, T] \),

\[
F(t, x) \geq a_1|x|^s - a_2.
\]

Then problem (1) possesses an unbounded sequence of solutions \( u_k \) with \( u_k \) having period \( kT \) for every positive integer \( k \).

Recently, Tang and Wu [17] have obtained the existence of periodic solution for problem (1) under the subquadratic condition: there exist \( 0 < \mu < 2 \) and \( L > 0 \) such that

\[
x \cdot \nabla F(t, x) \leq \mu F(t, x)
\]

for all \( |x| \geq L \) and \( t \in [0, T] \) and the coercive condition

\[
F(t, x) \to +\infty
\]

as \( |x| \to \infty \) uniformly for a.e. \( t \in [0, T] \).

Motivated by the results of [14,17], in this paper, we will obtain the existence of subharmonic solutions for problem (1) under the subquadratic condition. A contribution in this direction is [5], where the authors use the subquadratic condition to study the quasilinear boundary-value problems by variational method.

Furthermore, under the subquadratic condition, periodic solution of problem (1) is obtained. Let us point out that, for the subquadratic condition, periodic solution has been studied in [12,13,16,17]. Our main results are the following theorems:

**Theorem 1.** Suppose that \( F \) satisfies assumption (A) and the following conditions:

\[
\lim_{|x| \to \infty} \left[ x \cdot \nabla F(t, x) - 2F(t, x) \right] = -\infty \quad \text{uniformly for a.e. } t \in [0, T],
\]

\[
\lim_{|x| \to \infty} \frac{F(t, x)}{|x|^2} = 0 \quad \text{uniformly for a.e. } t \in [0, T].
\]
Then problem (1) has \( kT \)-periodic solution \( u_k \in H^1_{kT} \) for every positive integer \( k \) such that \( \|u\|_\infty \to \infty \) as \( k \to \infty \), where

\[ H^1_{kT} = \left\{ u : [0, kT] \mapsto \mathbb{R}^N \mid u \text{ is absolutely continuous}, \right. \]
\[ \left. u(0) = u(kT) \text{ and } \dot{u} \in L^2(0, kT; \mathbb{R}^N) \right\} \]

is a Hilbert space with the norm

\[ \|u\| = \left( \int_0^{kT} |u|^2 \, dt + \int_0^{kT} |\dot{u}|^2 \, dt \right)^{1/2} \text{ for } u \in H^1_{kT}. \]

**Remark 1.** Theorem 1 greatly generalizes Theorem A (see [14]). Obviously, conditions (F1) and (F2) are more stronger than those of (2) and (3). What is more, there are functions \( F \) satisfying our Theorem 1 and not satisfying the corresponding results in [1,3,6–11,13–15,18,19]. For example, let

\[ F(t, x) = \left( 1 + |x|^2 \right) \frac{1}{2} \ln \left( 1 + |x|^2 \right) + (x, e(t)) \]

for all \( x \in \mathbb{R}^N \) and \( t \in \mathbb{R} \), where \( e \) is \( T \)-periodic and \( e \in L^\infty(\mathbb{R}; \mathbb{R}^N) \). For elliptic equation, the corresponding condition (2) is due to D.G. Costa and C.A. Magalhaes [4].

**Theorem 2.** Suppose that \( F \) satisfies assumption (A) and the following conditions:

\[ \lim_{|x| \to \infty} \left[ x \cdot \nabla F(t, x) - 2F(t, x) \right] = -\infty \text{ uniformly for a.e. } t \in [0, T], \]

\[ 0 \leq \liminf_{|x| \to \infty} \frac{F(t, x)}{|x|^2} \leq \limsup_{|x| \to \infty} \frac{F(t, x)}{|x|^2} < \frac{1}{2} \omega^2 \text{ uniformly for a.e. } t \in [0, T], \]

where \( \omega = 2\pi / T \). Then problem (1) has a \( T \)-periodic solution \( u \in H^1_T \).

**Remark 2.** Theorem 2 generalizes Theorem 1 in [17]. There are functions \( F \) satisfying Theorem 2 and not satisfying the corresponding results in [5,13,16,17]. For example, let

\[ F(t, x) = \left( 1 + |x|^2 \right) \frac{1}{2} \ln \left( 1 + |x|^2 \right) + \frac{1}{2} \lambda |x|^2 + (x, e(t)) \]

for all \( x \in \mathbb{R}^N \) and \( t \in [0, T] \), where \( 0 \leq \lambda < \omega^2 \), \( e \in L^\infty(0, T; \mathbb{R}^N) \).

2. **Proofs of theorems**

Let \( k \) be a positive integer. For \( u \in H^1_{kT} \), let

\[ \tilde{u} = \frac{1}{kT} \int_0^{kT} u(t) \, dt, \quad \tilde{u}(t) = u(t) - \tilde{u}, \]

and

\[ \tilde{H}^1_{kT} = \left\{ u \in H^1_{kT} \mid \tilde{u} = 0 \right\}. \]
It is easy to know that $\tilde{H}^1_{kT}$ is a subset of $H^1_{kT}$, and $H^1_{kT} = \tilde{H}^1_{kT} + R^N$. Then one has

$$\int_0^{kT} |\tilde{u}(t)|^2 \leq \frac{k^2 T^2}{4\pi^2} \int_0^{kT} |\dot{u}(t)|^2 \, dt. \quad (6)$$

It follows from assumption (A) that the functional $\varphi_k$ given by

$$\varphi_k(u) = \frac{1}{2} \int_0^{kT} |\dot{u}(t)|^2 \, dt - \int_0^{kT} F(t, u(t)) \, dt$$

is continuously differentiable on $H^1_{kT}$ (see [13]). Moreover, one has

$$\langle \varphi'_k(u), v \rangle = \int_0^{kT} (\dot{u}, \dot{v}) \, dt - \int_0^{kT} (\nabla F(t, u), v) \, dt$$

for $u, v \in H^1_{kT}$, where $(\cdot, \cdot)$ and $|\cdot|$ are the usual inner product and norm of $R^N$. It is well known that the $kT$-periodic solutions of problem (1) correspond to the critical points of the functional $\varphi_k$.

In order to prove our theorems, we need the following result.

**Lemma 1.** Suppose (A), (2) and (3) hold. Then function $\varphi_k$ satisfies condition (C), i.e., for every constant $c$ and sequence $\{u_n\} \subset H^1_{kT}$, $\{u_n\}$ has a convergent subsequence if $\varphi_k(u_n) \to c$ and $(1 + |u_n|)\|\varphi'_k(u_n)\| \to 0$ as $n \to \infty$. This condition is due to G. Cerami [2].

**Proof.** Assume that $\{u_n\} \subset H^1_{kT}$ is a (C) sequence of $\varphi_k$, that is, $\varphi_k(u_n) \to c$ and $(1 + |u_n|)\|\varphi'_k(u_n)\| \to 0$ as $n \to \infty$. Then one has

$$\lim_{n \to \infty} \left[ \langle \varphi'_k(u_n), u_n \rangle - 2\varphi_k(u_n) \right] = -2c.$$

More precisely, we have

$$\lim_{n \to \infty} \int_0^{kT} \left[ (\nabla F(t, u_n), u_n) - 2F(t, u_n) \right] \, dt = 2c. \quad (7)$$

Now we prove $\{u_n\}$ is bounded by contradiction. If $\{u_n\}$ is unbounded, without loss of generality, we may assume that

$$\|u_n\| \to \infty \quad \text{as} \quad n \to \infty.$$ 

Put $z_n = \frac{u_n}{\|u_n\|}$, we have $\|z_n\| = 1$. Going to a subsequence if necessary, we may assume that:

$z_n \rightharpoonup z$ weakly in $H^1_{kT}$, $z_n \to z$ strongly in $L^2(0, kT)$ and $z_n(t) \to z(t)$ for a.e. $t \in [0, kT]$.

By (3) and the $T$-periodicity of $F(t, x)$ in the first variable, we get, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(t, x) \leq \varepsilon |x|^2$$

for all $|x| \geq \delta$ and a.e. $t \in R$.

Without loss of generality, we may assume that the function $b$ in assumption (A) is $T$-periodic and assumption (A) holds for all $t \in R$ by the $T$-periodicity of $F(t, x)$ in the first variable.
By assumption (A), it is easy to know that
\[ |F(t, x)| \leq \max_{s \in [0, \delta]} a(s)b(t) \]
for all \( |x| \leq \delta \) and a.e. \( t \in R \). Hence, we obtain
\[ F(t, x) \leq \varepsilon |x|^2 + \max_{s \in [0, \delta]} a(s)b(t) \]
for all \( x \in R^N \) and a.e. \( t \in R \).

By (8), we have
\[
\phi_k(u_n) = \frac{1}{2} \int_0^{kT} |\dot{u}_n|^2 dt - \int_0^{kT} F(t, u_n) dt
\]
\[
= \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \int_0^{kT} |u_n|^2 dt - \int_0^{kT} F(t, u_n) dt
\]
\[ \geq \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \int_0^{kT} |u_n|^2 dt - \varepsilon \int_0^{kT} |u_n|^2 dt - \max_{s \in [0, \delta]} a(s) \int_0^{kT} b(t) dt
\]
\[ = \frac{1}{2} \|u_n\|^2 - \left( \frac{1}{2} + \varepsilon \right) \int_0^{kT} |u_n|^2 dt - c_1 k,
\]
where \( c_1 = \max_{s \in [0, \delta]} a(s) \int_0^{T} b(t) dt \). Therefore, one obtains
\[ \frac{\phi_k(u_n)}{\|u_n\|^2} \geq \frac{1}{2} - \left( \frac{1}{2} + \varepsilon \right) \frac{\int_0^{kT} |u_n|^2 dt}{\|u_n\|^2} - \frac{c_1 k}{\|u_n\|^2}
\[ \geq \frac{1}{2} - \left( \frac{1}{2} + \varepsilon \right) \frac{\|z_n\|^2_{L^2}}{\|u_n\|^2} - \frac{c_1 k}{\|u_n\|^2}.
\]
Passing to the limit in the inequality, by using \( \phi_k(u_n) \to c \) as \( n \to \infty \), we obtain
\[ \frac{1}{2} - \left( \frac{1}{2} + \varepsilon \right) \|z\|^2_{L^2} \leq 0
\]
which implies that \( z \neq 0 \).

Now by (2), there exists \( \delta_0 > 0 \) such that
\[ (\nabla F(t, x), x) - 2F(t, x) \leq 0
\]
for all \( |x| \geq \delta_0 \) and a.e. \( t \in [0, T] \), and by assumption (A) we have
\[ (\nabla F(t, x), x) - 2F(t, x) \leq c_2 b(t)
\]
for all \( |x| \leq \delta_0 \) and a.e. \( t \in [0, T] \), where \( c_2 = (2 + \delta_0) \max_{[0, \delta_0]} a(s) \). So we get
\[ (\nabla F(t, x), x) - 2F(t, x) \leq c_2 b(t)
\]
for all \( x \in R^N \) and a.e. \( t \in R \) by the \( T \)-periodicity of \( F(t, x) \) in the first variable. Hence, we get
\[ kT \int_0^1 \left[ (\nabla F(t,u_n), u_n) - 2F(t,u_n) \right] dt \]

\[ = \int_{\{t: z(t) \neq 0\}} \left[ (\nabla F(t,u_n), u_n) - 2F(t,u_n) \right] dt \]

\[ + \int_{[0,kT] \setminus \{t: z(t) \neq 0\}} \left[ (\nabla F(t,u_n), u_n) - 2F(t,u_n) \right] dt \]

\[ \leq \int_{\{t: z(t) \neq 0\}} \left[ (\nabla F(t,u_n), u_n) - 2F(t,u_n) \right] dt + \int_{[0,kT] \setminus \{t: z(t) \neq 0\}} c_2 b(t) dt. \]

An application of Fatou’s lemma yields

\[ kT \int_0^1 \left[ (\nabla F(t,u_n), u_n) - 2F(t,u_n) \right] dt \to -\infty \quad \text{as} \quad n \to \infty, \]

which is a contradiction to (7). In a similar way to Proposition 4.3 in [13], we can prove that \{u_n\} has a convergent subsequence. Thus, the proof of Lemma 1 is complete. \( \square \)

Now we give the proof of the main results.

**Proof of Theorem 1.** From Lemma 1 we obtain that \( \varphi_k \in C^1(\tilde{H}_{kT}^1, R) \) satisfies condition (C). We know that a deformation lemma can be proved with the weaker condition (C) replacing the usual condition (PS), and the Saddle Point Theorem holds true under condition (C). Set \( e_k(t) = k(\cos k^{-1} \omega t)x_0 \) for all \( t \in R \) and some \( x_0 \in R^N \) with \( |x_0| = 1 \). By the Saddle Point Theorem (see Theorem 4.6 in [13]), we only need to prove

\[ (l_1) \quad \varphi_k(u) \to +\infty \text{ as } \|u\| \to \infty \text{ in } \tilde{H}_{kT}^1, \]

which implies that \( \inf_{\tilde{H}_{kT}^1} \varphi_k > -\infty \).

\[ (l_2) \quad \varphi_k(x + e_k) \to -\infty \text{ as } |x| \to \infty \text{ in } R^N. \]

We first prove \((l_1)\). By (6) and (8), we have

\[ \varphi_k(u) = \frac{1}{2} kT \int_0^T |\dot{u}|^2 dt - \int_0^T F(t,u) dt \geq \frac{1}{2} kT \int_0^T |\dot{u}|^2 dt - \epsilon \int_0^T |u|^2 dt - \max_{s \in [0,\delta]} a(s)b(t) dt \]

\[ \geq \left( \frac{1}{2} - \epsilon k^2 \omega^2 \right) kT \int_0^T |\dot{u}|^2 dt - k \max_{s \in [0,\delta]} a(s) \int_0^T b(t) dt \]

for all \( u \in \tilde{H}_{kT}^1 \). Let \( \epsilon = k^{-2} \omega^2 / 4 \), we get \( \varphi_k(u) \to +\infty \) as \( \|u\| \to \infty \).
By (2) and the $T$-periodicity of $F(t, x)$ in the first variable, we obtain, for every $\beta > 0$, there exists $\delta_1 > 0$ such that

$$x \cdot \nabla F(t, x) - 2F(t, x) \leq -2\beta \quad \text{for all } |x| \geq \delta_1 \text{ and a.e. } t \in R.$$  

(9)

Let $s \geq 1$, using (9) and integrating the relation

$$d ds \left[ \frac{F(t, sx)}{s^2} \right] = \frac{s x \cdot \nabla F(t, sx) - 2F(t, sx)}{s^3} \leq -\frac{2\beta}{s^3}$$

over an interval $[1, S] \subset [1, \infty)$, we get

$$\frac{F(t, Sx)}{S^2} - \frac{F(t, x)}{x^2} \leq \beta \left[ \frac{1}{S^2} - 1 \right].$$

Therefore, since $\lim_{S \to \infty} \frac{F(t, Sx)}{S^2} = 0$ by (3), we obtain

$$F(t, x) \geq \beta$$

(10)

for all $|x| \geq \delta_1$ and a.e. $t \in R$. That is,

$$F(t, x) \to \infty \quad \text{as } |x| \to \infty.$$ 

Then also by assumption (A), there is a constant $c_3 > 0$ such that

$$-F(t, x) \leq c_3 b(t)$$

(11)

for all $x \in R^N$ and a.e. $t \in R$. Since $e_k(t) = k (\cos k^{-1} \omega t)x_0$, we have

$$\dot{e}_k(t) = -\omega \left( \sin k^{-1} \omega t \right)x_0$$

for all $t \in R$, which implies that

$$\|\dot{e}_k\|_{L^2(0, kT; R^N)}^2 = kT \omega^2 / 2.$$

Hence, one has

$$\varphi_k(x + e_k) = kT \omega^2 / 4 - \int_0^{kT} F(t, x + e_k) dt$$

for all $x \in R^N$. It follows from (10) that

$$\varphi_k(x + e_k) = kT \omega^2 / 4 - \int_0^{kT} F(t, x + e_k) dt \leq kT \omega^2 / 4 - \beta kT$$

for all $|x| \geq \delta_1 + k$, which implies that

$$\varphi_k(x + e_k) \to -\infty \quad \text{as } |x| \to \infty,$$

by the arbitrariness of $\beta$. So there exists a critical point $u_k \in H^1_{kT}$ for $\varphi_k$ such that

$$-\infty < \inf_{H^1_{kT}} \varphi_k \leq \varphi_k(u_k) \leq \sup_{R^N + e_k} \varphi_k.$$

For fixed $x \in R^N$, set

$$A_k = \{ t \in [0, kT] \mid |x + k (\cos k^{-1} \omega t)x_0| \leq \delta_1 \}.$$
Then we have
\[ \text{meas } A_k \leq kT/2 \]
for all large \( k \), which follows from the fact that
\[ \bigcup_{j=0}^{k-1} (jT + A) \subset [0, kT] \setminus A_k, \]
where \( A = [0, \frac{T}{8}] \cup [\frac{3T}{8}, \frac{5T}{8}] \cup [\frac{7T}{8}, T] \).

By (10) and (11), we get
\[ k^{-1} \varphi_k(x + e_k) = \frac{T\omega^2}{4} - k^{-1} \int_0^{kT} F(t, x + e_k) \, dt \]
\[ = \frac{T\omega^2}{4} - k^{-1} \int_{A_k} F(t, x + e_k) \, dt - k^{-1} \int_{[0,kT]\setminus A_k} F(t, x + e_k) \, dt \]
\[ \leq \frac{T\omega^2}{4} + k^{-1} \int_{A_k} c_3 b(t) \, dt - k^{-1} \int_{[0,kT]\setminus A_k} \beta \, dt \]
\[ \leq \frac{T\omega^2}{4} + c_3 \int_0^T b(t) \, dt - \beta T/2 \]
for every \( x \in \mathbb{R}^N \) and all large \( k \). Hence, one has
\[ \sup_{x \in \mathbb{R}^N} k^{-1} \varphi_k(x + e_k) \leq \frac{T\omega^2}{4} + c_3 \int_0^T b(t) \, dt - \beta T/2 \]
for all large \( k \), which implies that
\[ \limsup_{k \to \infty} \sup_{x \in \mathbb{R}^N} k^{-1} \varphi_k(x + e_k) \leq \frac{T\omega^2}{4} + c_3 \int_0^T b(t) \, dt - \beta T/2. \]

By the arbitrariness of \( \beta \), we obtain
\[ \limsup_{k \to \infty} \sup_{x \in \mathbb{R}^N} k^{-1} \varphi_k(x + e_k) = -\infty \]
which follows that
\[ \limsup_{k \to \infty} k^{-1} \varphi_k(u_k) = -\infty. \]  \hspace{1cm} (12)

Now we prove that \( \|u_k\|_\infty \to \infty \) as \( k \to \infty \). If not, going to a subsequence if necessary, we may assume that
\[ \|u_k\|_\infty \leq c_4 \]
for all \( k \in \mathbb{N} \) and some positive constant \( c_4 \). Hence, we have
\[
\begin{align*}
&k^{-1} \varphi_k(u_k) \geq -k^{-1} \int_{0}^{kT} F(t, u_k(t)) \, dt \\
&\geq -k^{-1} \max_{0 \leq s \leq c_4} a(s) \int_{0}^{kT} b(t) \, dt \\
&= -\max_{0 \leq s \leq c_4} a(s) \int_{0}^{T} b(t) \, dt.
\end{align*}
\]

It follows that
\[
\liminf_{k \to \infty} k^{-1} \varphi_k(u_k) > -\infty
\]
which is a contradiction to (12). Therefore Theorem 1 holds. \(\square\)

**Proof of Theorem 2.** Let \(\tilde{H}_T^1 = \{u \in H_T^1 \mid \bar{u} = 0\}\). Then \(H_T^1 = \tilde{H}_T^1 + R^N\) and \(R^N \neq \{0\}\) is finite-dimensional. It follows from assumption (A) that the functional \(\varphi\) given by

\[
\varphi(u) = \frac{1}{2} \int_{0}^{T} \left[|\dot{u}(t)|^2 - F(t, u(t))\right] \, dt
\]

is continuously differentiable on \(H_T^1\).

Let \(\varepsilon_1 = (\omega^2/2 - \limsup_{|x| \to \infty} \frac{F(t, x)}{|x|^2})/2 > 0\), there exists \(\delta_2 > 0\) such that

\[
F(t, x) \leq \frac{\omega^2 - \varepsilon_1}{2} |x|^2 \quad \text{for all } |x| \geq \delta_2 \text{ and a.e. } t \in [0, T].
\]

Let \(c_4 = \max_{s \in [0, \delta_2]} a(s)\). By assumption (A), we know

\[
|F(t, x)| \leq c_4 b(t) \quad \text{for all } |x| \leq \delta_2 \text{ and a.e. } t \in [0, T].
\]

Hence, we obtain

\[
F(t, x) \leq \frac{\omega^2 - \varepsilon_1}{2} |x|^2 + c_4 b(t)
\]

for all \(x \in R^N\) and a.e. \(t \in [0, T]\).

Moreover, in a way similar to the proof of Theorem 1, it shows that conditions (4) and (5) imply

\[
F(t, x) \to +\infty \quad \text{as } |x| \to \infty \text{ uniformly for a.e. } t \in [0, T].
\]

In a way similar to the proof of Lemma 1, we can easily prove that \(\varphi\) satisfies condition (C) by (13). As the same as in the proof of Theorem 1, we only need to prove

\[
(l_3) \ \varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty \text{ in } \tilde{H}_T^1, \text{ which implies that} \inf_{\tilde{H}_T^1} \varphi > -\infty,
\]

\[
(l_4) \ \varphi(x) \to -\infty \quad \text{as } |x| \to \infty \text{ in } R^N.
\]

By (13), \((l_3)\) holds as the same as the proof of Theorem 1 and \((l_4)\) follows directly from (14). Hence, Theorem 2 holds. \(\square\)
Acknowledgment

The authors thank the referee for valuable suggestions.

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