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Structure of finite groups under certain arithmetical conditions on class sizes

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Abstract

Let G be a finite solvable group. We assume that the set of conjugacy class sizes of G is $\{1, m, n, mk\}$ with m and n coprime positive integers greater than 1 and k a divisor of n. Then we obtain several properties on the structure of G.

Keywords: Group theory; Conjugacy classes; Solvable groups

1. Introduction

There has been considerable interest in studying the structure of a finite group given only the set of conjugacy class sizes. If a group G has exactly two class sizes, $\{1, m\}$, then N. Itô shows in [11] that G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. When a group has exactly three class sizes there also exist several results. For instance, Itô shows in [12] that such groups are solvable by appealing to the Feit–Thompson theorem and some deep classification theorems of M. Suzuki. A.R. Camina obtains in [7] some properties when the class sizes are $\{1, p^a, p^a q^b\}$ for distinct primes p and q. It was first proved in [10] and later reproved in [6] that if the conjugacy class sizes of G are $\{1, m, n\}$ with $(m, n) = 1$, and $m, n > 1$, then $G/Z(G)$ is a Frobenius group (G is then

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called a quasi-Frobenius group) and the inverse image in G of the kernel and a complement are abelian.

In general, groups with four conjugacy class sizes may be not solvable or even may be simple. In 1970 [13], Itô shows that $SL(2, 2^m)$ for $m \geqslant 2$ are the only simple groups with four class sizes. In 1987, E. Fisman and Z. Arad used the classification of the finite simple groups to show that groups with two classes of coprime size cannot be simple [8], and of course, they are not necessarily solvable. On the other hand, Camina proves in [7] that if the class sizes of G are $\{1, p^a, q^b, p^a q^b\}$, with p and q two distinct primes, then G is nilpotent. Notice that the hypotheses of Camina's theorem imply the solvability of G just by using Burnside's $p^a q^b$-theorem. The authors extend this result in [4] and [5], showing that when the set of conjugacy class sizes of a group G is $\{1, m, n, mn\}$, with m and n arbitrary positive integers such that $(m, n) = 1$, then G is nilpotent, $m = p^a$ and $n = q^b$ for two primes p and q.

In this paper we analyze the structure of a solvable group with four conjugacy class sizes, $\{1, m, n, mk\}$, where m and n are coprime integers greater than 1 and k is any divisor of n. When $k = 1$ or $k = n$ then the structure of G is completely determined as we have explained above. So we shall study the case $1 < k < n$. We prove the following

Theorem A. *Let G be a solvable group whose conjugacy class sizes are $\{1, m, n, mk\}$, where $m, n > 1$ are coprime integers and $1 < k < n$ is a divisor of n. Let π be the set of primes dividing m and let K and H be a Hall π-subgroup and π-complement of G, respectively. Then K is abelian, $k = q^a$ for some prime q and $H = QA$, with A an abelian subgroup and Q a Sylow q-subgroup of G. Furthermore, one of the following statements holds*:

(1) *If $m > n$, then $K \trianglelefteq G$, $H = Q \times A$, $\mathbf{O}_{\pi'}(G) \subseteq Z(G)$ and G is a quasi-Frobenius group.*
(2) *If $n > m$, then $\mathbf{O}_{\pi}(G) \subseteq Z(G)$ and $\mathbf{O}_{\pi',\pi}(G)$ is a quasi-Frobenius group. Moreover,*
  (2.1) *If $n = q^r$, then $H = Q \times A$ and $A \subseteq Z(G)$.*
  (2.2) *If n is not a prime-power, then $A \nsubseteq Z(G)$ and*
    (a) *either $H = Q \times A$ is normal in G (and consequently, G is quasi-Frobenius),*
    (b) *or both Q and KQ are normal in G and Q is abelian.*

We remark that in order to prove this theorem we make use of some results concerning local information of the group given the class sizes of π-elements for an arbitrary set π of primes. For instance, we shall need Theorem C of [3] and the main result of [2].

The following examples show that each one of the types described in Theorem A can occur. Let us consider the quaternion group $Q = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ a^b = a^{-1}\rangle$ of order 8 and let $A = \langle c\rangle \cong \mathbb{Z}_3$, both acting on $K = \langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_{13} \times \mathbb{Z}_{13}$ as follows:

$$x^a = y^8, \qquad y^a = x^8, \qquad x^b = y, \qquad y^b = x^{12}, \qquad x^c = x^3, \qquad y^c = y^3.$$

It is easy to see that $Q \times A$ acts Frobeniusly on K. If we consider the semidirect product $G = [K](Q \times A)$, then one can check that the conjugacy class sizes of G are $\{1, 169, 338, 24\}$. This provides an example of a group of type (1) with $m = 169$, $n = 24$, $k = 2$ and $\pi = \{13\}$.

The symmetric group on four letters is an example of group of type (2.1). The conjugacy class sizes of S_4 are $\{1, 3, 8, 6\}$, and we take $m = 3$, $n = 8$, $k = 2$ and $\pi = \{3\}$.

Now we are going to construct a group satisfying the conditions described in (2.2)(a). Let us consider again the quaternion group of order 8, $Q = \langle a, b\rangle$ and let $A = \langle c\rangle \cong \mathbb{Z}_7$. We define the action of a cyclic group $K = \langle x\rangle$ of order 3 on both groups in the natural way:

$$a^x = ab, \qquad b^x = a, \qquad c^x = c^2$$

and define $G = [Q \times A]K$. The conjugacy class sizes of G are $\{1, 3, 6, 28\}$. In this case, $m = 3$, $n = 28$, $k = 2$, $\pi = \{3\}$ and G has a normal π-complement which factorizes as described in (2.2)(a). We can proceed in this way constructing other groups of this type, just changing the group A, in such a way that the set of primes dividing n is as large as wanted.

Finally the affine semilinear group $G = \Gamma(2^3)$ of order 168 (see for instance p. 147 of [9]) is an example of group satisfying the conditions given in (2.2)(b). The conjugacy class sizes of G are $\{1, 7, 24, 28\}$. In this case, we take $m = 7$, $n = 24$, $k = 4$ and $\pi = \{7\}$, and G possesses an abelian normal Sylow 2-subgroup, Q, and an abelian Hall π-subgroup, K, such that KQ is also normal in G. Notice that this is the only case in which the π-complements of G do not factorize as $Q \times A$.

We will denote by x^G the conjugacy class of x in G and we call $|x^G|$ the index of x in G. The rest of the notation is standard.

2. Preliminary results

We shall need the following elementary results on conjugacy classes of π-elements where π is an arbitrary set of primes.

Lemma 1. *Let G be a π-separable group.*

(a) *The conjugacy class size of any π'-element of G is a π-number if and only if G has abelian Hall π'-subgroups.*
(b) *The conjugacy class size of every π-element of G is a π-number if and only if $G = H \times K$, where H and K are a Hall π-subgroup and a π-complement of G, respectively.*
(c) *If $x \in G$ and $|x^G|$ is a π-number, then $x \in \mathbf{O}_{\pi,\pi'}(G)$.*

Proof. (a) Suppose that the class size of any π'-element is a π-number and work by induction on $|G|$ so as to prove that any π-complement of G is abelian. Assume first that $\mathbf{O}_\pi(G) > 1$ and write $\overline{G} = G/\mathbf{O}_\pi(G)$. As the hypotheses are inherited by factor groups, we get that $\overline{G}$ has abelian π-complements and then so has G.

Assume now that $\mathbf{O}_\pi(G) = 1$ and thus $\mathbf{O}_{\pi'}(G) > 1$. By hypothesis, for any π'-element $g \in G$ there exists a π-complement H of G such that $g \in H \subseteq C_G(g)$. Then

$$g \in C_G(H) \subseteq C_G\big(\mathbf{O}_{\pi'}(G)\big) \subseteq \mathbf{O}_{\pi'}(G).$$

It follows that G has a normal π-complement and moreover, that it is abelian. The converse direction is obvious.

(b) is exactly Lemma 8 of [3] and (c) is exactly Theorem C of [3]. □

We shall use the following result due to Itô, which characterizes the structure of those groups which possess only two conjugacy class sizes.

Theorem 2. *Suppose that* 1 *and* $m > 1$ *are the only lengths of conjugacy classes of a group* G. *Then* $G = P \times A$, *where* $P \in \mathrm{Syl}_p(G)$ *and* A *is abelian. In particular, then* m *is a power of* p.

Proof. See Theorem 33.6 of [9]. □

The authors obtained in [2] the following generalization of Itô's theorem for p-regular conjugacy classes in p-solvable groups.

Theorem 3. *Suppose that* G *is a finite* p*-solvable group and that* $\{1, m\}$ *are the* p*-regular conjugacy class sizes of* G. *Then* $m = p^a q^b$, *with* q *a prime distinct from* p *and* $a, b \geqslant 0$. *If* $b = 0$ *the* G *has abelian* p*-complement. If* $b \neq 0$, *then* $G = PQ \times A$, *with* $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ *and* $A \subseteq Z(G)$. *Furthermore, if* $a = 0$ *then* $G = P \times Q \times A$.

Proof. This is exactly Theorem A of [2]. □

We shall also make use of the classic Thompson's $A \times B$-lemma.

Theorem 4. *Let* AB *be a finite group represented as a group of automorphisms of a* p*-group* G *with* $[A, B] = 1 = [A, C_G(B)]$, B *a* p*-group and* $A = \mathbf{O}^p(A)$. *Then* $[A, G] = 1$.

Proof. See for instance 24.2 of [1]. □

The following result is a generalization for π-separable groups of a classic theorem of Burnside which asserts that if a group G possesses a Sylow p-subgroup such that $N_G(P) = C_G(P)$, then G is p-nilpotent.

Theorem 5. *Let* G *be a* π*-separable group and suppose that* H *is a Hall* π*-subgroup of* G *such that* $N_G(H) = C_G(H)$. *Then* G *has a normal* π*-complement.*

Proof. It is sufficient for instance to rewrite the proof of 17.9 of [9] (the above mentioned Burnside's theorem). □

3. Proof

The proof of Theorem A has been divided into 19 steps. In Step 7, we shall show that the number k of the statement of the theorem may be equal to q^a or $q^t r^l$ for some primes q and r, but from Step 14 to 19 we shall prove that the second case cannot occur.

Proof of Theorem A

Step 1. There are no π-elements of index m. Consequently, there exist π'-elements of index m.

Proof. Suppose that there exists a π-element x of index m. Then $x \in K$, where K is a Hall π-subgroup of G. If y is a π'-element of $C_G(x)$, then $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$, and so the index of y in $C_G(x)$ is 1 or k. By Lemma 1(b), we may write $C_G(x) = H \times K_x$, with H a π-complement of G and K_x a π-subgroup.

Now suppose that there exists a π'-element y of index n. Replacing y by some conjugated we can assume that $K \subseteq C_G(y)$. Hence $x \in C_G(y)$, so $C_G(xy) = C_G(x) \cap C_G(y)$ and accordingly, the index of xy in G is divisible by $|G : C_G(y)| = n$ and $|G : C_G(x)| = m$, a contradiction. Therefore, every π'-element of G has index $1, m$ or mk. Now, choose $z \in G$ of index n with $K \subseteq C_G(z)$. We consider the decomposition $z = z_\pi z_{\pi'}$, with z_π and $z_{\pi'}$ the π-part and π'-part of z, respectively. We have $C_G(z) \subseteq C_G(z_{\pi'})$, so by the above property $z_{\pi'} \in Z(G)$ and $|z_\pi^G| = n$. It follows that $x \in K \subseteq C_G(z_\pi)$, whence $z_\pi \in C_G(x) = H \times K_x$. This implies that $z_\pi \in K_x$ and $H \subseteq C_G(z_\pi)$, contradicting the fact that $|z_\pi^G| = n$.

The consequence in the statement can be easily obtained by considering the $\{\pi, \pi'\}$-decomposition of any element of index m. □

Step 2. There are no π'-elements of index n. As a consequence, there exist π-elements of index n and if x is an element of index n, then $x_{\pi'}$ is central.

Proof. Suppose that y is a π'-element of index n. We can assume without loss that y is a p-element for some prime p. If we take any p'-element, say w, of $C_G(y)$, we have $C_G(yw) = C_G(y) \cap C_G(w)$ and then the maximality of n implies $C_G(y) \subseteq C_G(w)$, so $w \in Z(C_G(y))$. Accordingly, we can write $C_G(y) = P_y \times T$ with P_y a p-subgroup and T an abelian p'-subgroup. Now, by Step 1, we can take some π'-element of index m, say z, and up to conjugacy we can assume that $P_y \subseteq C_G(z)$. So in particular, $z \in C_G(y)$ and $C_G(zy) = C_G(y) \cap C_G(z)$. But this forces zy to have index nm, which is a contradiction.

The consequence of the statement follows as in the above step by considering de decomposition of an element. □

Step 3. If x is a π-element of index mk, then $C_G(x) = K_x \times H_x$, with K_x a π-subgroup and H_x a non-central abelian π'-subgroup of G.

Proof. Let y be a π'-element of $C_G(x)$. Then $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$. The maximality of mk implies that $C_G(x) \subseteq C_G(y)$, so $y \in Z(C_G(x))$. Then we can write $C_G(x) = K_x \times H_x$, with K_x a π-subgroup and H_x an abelian π'-subgroup of G.

We show that H_x cannot be central in G. Suppose that $H_x \subseteq Z(G)$. It follows that $H_x = Z(G)_{\pi'}$ and $k = |G : Z(G)|_{\pi'}$, but this certainly leads to a contradiction with the existence of elements of index n. □

Step 4. The Hall π-subgroups of G are abelian. Consequently, any π-element has index 1 or n and if x is an element of index m or mk, then x_π is central.

Proof. Suppose that G has non-abelian Hall π-subgroups. By applying Step 1 and Lemma 1(a), we can take some π-element $x \in G$ of index mk. By Step 3, we write $C_G(x) = K_x \times H_x$, with K_x a π-subgroup and H_x an abelian non-central π'-subgroup. We shall prove now that K_x is abelian too. Let us choose some non-central $y \in H_x$ and notice that $C_G(x) \subseteq C_G(y)$, so y must have index m or mk. If $|y^G| = mk$, then certainly $C_G(x) = C_G(y)$. In this case, for any π-element of $C_G(y)$, say $t \in K_x$, we have $C_G(ty) = C_G(y) \cap C_G(t) = C_G(y) \subseteq C_G(t)$, so $t \in Z(C_G(y))$. Therefore, K_x is abelian as wanted. Assume now that $|y^G| = m$. If w is a π-element of $C_G(y)$ then $|C_G(y) : C_G(y) \cap C_G(w)| = |C_G(y) : C_G(wy)|$ is equal to 1 or k. Since k is a π'-number, by Lemma 1(a), the Hall π-subgroups of $C_G(y)$ are abelian, so in particular, K_x is abelian too. Consequently, $C_G(x)$ is abelian.

By Step 2, there exists a π-element w of index n and it easily follows that $w \in Z(K)$, for some Hall π-subgroup K of G. Moreover, up to conjugacy we can assume that $x \in K$, so $w \in C_G(x)$ and as $C_G(x)$ is abelian, then $C_G(x) \subseteq C_G(w)$. This leads to a contradiction.

Now the consequence in the statement trivially follows by applying Lemma 1(a). □

Step 5. $|G : Z(G)| = mn$.

Proof. We show first that $|G : Z(G)|_\pi = m$. Let z be an element of index mk and consider the $\{\pi, \pi'\}$-decomposition of $z = z_\pi z_{\pi'}$. As $C_G(z) \subseteq C_G(z_\pi)$, by applying Step 4, we deduce that $z_\pi \in Z(G)$. Consequently, $z_{\pi'}$ has index mk and thus, we may suppose that z is a π'-element. Now, if t is a non-central π-element of $C_G(z)$, then $C_G(zt) = C_G(z) \cap C_G(t) = C_G(z) \subseteq C_G(t)$ and again by applying Step 4, $t \in Z(G)$. Then $|G : Z(G)|_\pi = m$.

Now by Step 2, we can take a π-element z of index n. If w is a π'-element of $C_G(z)$, then $C_G(zw) = C_G(z) \cap C_G(w) = C_G(z) \subseteq C_G(w)$ and, by Step 2, $w \in Z(G)$. Hence $Z(G)_{\pi'}$ is a Hall π'-subgroup of $C_G(z)$ and thus, $|G : Z(G)|_{\pi'} = n$. Now the step is clearly proved. □

Step 6. If x is a non-central π-element, then $C_G(x) = K \times Z(G)_{\pi'}$, for some Hall π-subgroup K of G. If y is a non-central π'-element, then y has index m or mk and $C_G(y) = H_y \times Z(G)_\pi$, with H_y a π'-subgroup of G.

Proof. If x is a non-central π-element, then we know by Step 4 that it has index n, and by applying Step 5, it follows that $C_G(x) = K \times Z(G)_{\pi'}$, for some Hall π-subgroup K of G. On the other hand, by Step 2 there are no π'-elements of index n and, by applying Step 5 again, we obtain that any non-central π'-element y satisfies $C_G(y) = H_y \times Z(G)_\pi$, with H_y a π'-subgroup of G. □

Step 7. Let H be a Hall π'-subgroup of G. Then one of the following two properties holds:

(a) $k = q^a$ for some prime q and $H = QA$, with $Q \in \mathrm{Syl}_q(G)$ and A is an abelian q'-group.

(b) $k = q^t r^l$ for some distinct primes q and r and $l, t > 0$, and $H = QR \times V$, where Q and R are q- and r-Sylow subgroups of G and $V \subseteq Z(G)$.

Proof. We can choose a π'-element y of index m by Step 1, and up to conjugacy, such that $H \subseteq C_G(y)$. Also, by considering the primary decomposition, y can be assumed to be a q-element for some prime $q \in \pi'$. Let z be a q'-element of $C_G(y)$ and observe that $C_G(zy) = C_G(z) \cap C_G(y) \subseteq C_G(y)$. This forces z to have index 1 or k in $C_G(y)$. Suppose first that all these indexes are 1 and consequently, we can write $C_G(y) = C_G(y)_q \times C_G(y)_{q'}$ with $C_G(y)_{q'}$ abelian. By Step 4, we can take some π'-element x of index mk, which up to conjugacy, can be assumed to belong to $C_G(y)$. Then $C_G(y)_{q'} \subseteq C_G(x)$, and this yields to $k = q^a$. From now on, we will assume that both indexes, 1 and k, appear, so we can apply Theorem 3 to obtain two possibilities for k: either $k = q^a$ and $C_G(y) = QA$, with A abelian and $Q \in \mathrm{Syl}_q(G)$ (notice that this case is exactly the one obtained above), or $k = q^t r^l$, for some prime r, and $t \geqslant 0$, $l > 0$ and also $C_G(y) = QR \times S$ with Q and R Sylow q- and a r-subgroups of $C_G(y)$ (and consequently, of G) and S abelian. In any case, by Step 6, we also know that $C_G(y) = H \times Z(G)_\pi$ for some π-complement H of G.

Assume the first possibility, that is, $k = q^a$ and $C_G(y) = QA$. Then $H = H \cap QA = Q(H \cap A)$ with $A_1 = H \cap A$ abelian, so we get case (a).

If we assume the second possibility for $C_G(y)$, we have $QR \subseteq H$, whence $H = QR \times (H \cap S)$ with $S_1 := H \cap S$ abelian. We distinguish two cases: $t = 0$ and $t > 0$. If $t = 0$, then Theorem 3 asserts that $H = Q \times R \times S_1$. Notice that then the q-elements of G have index 1, m or mr^l, so by Lemma 1(a), we have that Q is abelian. So we can write $H = RA$ where $A := Q \times S_1$ is abelian and then H has again the structure described in (a). Now assume that $t > 0$ and it only remains to see that S_1 is central. Let us take some non-central $w \in S_1$ of order a power of a prime $s \in \pi'$. As $H \subseteq C_G(w)$, then w has index m. If z is any s'-element of $C_G(w)$, by arguing as above, it follows that z has index 1 or k in $C_G(w)$. We analyze first the case in which every such s'-element z has index 1 in $C_G(w)$. This implies that $C_G(w)$ can be written as $C_G(w) = C_G(w)_s \times C_G(w)_{s'}$ with $C_G(w)_{s'}$ abelian. Note that $C_G(w)_s \subseteq H$, so we can write $H = C_G(w)_s \times (C_G(w)_{s'} \cap H)$. Now, if x is a π'-element of index mk lying in H and we factorize $x = x_s x_{s'}$, then each one of these factors, and accordingly x, must be centralized by $C_G(w)_{s'} \cap H$. Hence k is necessarily a power of s, contradicting the fact that k is divisible by q and r. Thus, both integers 1 and k really appear as indexes of q-elements of $C_G(w)$, so we can apply Theorem 3 to obtain that either s divides k or k is a power of a single prime. Both cases yield to a contradiction. Therefore, S_1 is central in G and in this case the structure of H is as described in (b). □

Step 8. $\mathbf{O}_\pi(G) \subseteq Z(G)$ or $\mathbf{O}_{\pi'}(G) \subseteq Z(G)$.

Proof. Suppose that $\mathbf{O}_\pi(G) \nsubseteq Z(G)$ and take $w \in \mathbf{O}_\pi(G) - Z(G)$. We have

$$Z(G) \subseteq C_G\big(\mathbf{O}_\pi(G)\big) \subseteq C_G(w) \subseteq G.$$

Since $|G : Z(G)|_{\pi'} = n$ by Step 5 and $|G : C_G(w)| = n$, this implies that $Z(G)_{\pi'} = C_G(\mathbf{O}_\pi(G))_{\pi'}$. As $\mathbf{O}_{\pi'}(G) \subseteq C_G(\mathbf{O}_\pi(G))$, then $\mathbf{O}_{\pi'}(G) \subseteq Z(G)$. □

Step 9. Suppose that $\mathbf{O}_{\pi'}(G) \subseteq Z(G)$. Then G is a semidirect product $G = K(Q \times A)$, where K is an abelian normal Hall π-subgroup of G, A is abelian and $Q \in \mathrm{Syl}_q(G)$ for some prime q. Moreover, G is quasi-Frobenius, $k = q^a$ and $n < m$.

Proof. Any non-central π-element has index n, so by applying Lemma 1(c), it belongs to $\mathbf{O}_{\pi',\pi}(G) = \mathbf{O}_\pi(G) \times Z(G)_{\pi'}$, and hence, it belongs to $\mathbf{O}_\pi(G)$. As a result, $K = \mathbf{O}_\pi(G)$ is a normal Hall π-subgroup of G. It is also abelian by Step 4.

Write $G^* = G/K$. We assert that every conjugacy class of G^* has size 1 or k. For any $g^* \in G^*$ we can clearly assume that g is a π'-element, and thus, g has index 1, m or mk in G. If g has index 1 or m, then g^* has index 1 in G^* because the index of g^* in G^* divides $(m, |G^*|) = 1$. Therefore, in order to prove the assertion we shall assume that g has index mk in G and show that g^* has index k in G^*. We notice that

$$\big|G : C_G(g)K\big|\big|C_G(g)K : C_G(g)\big| = mk.$$

But, on the other hand, by applying Steps 6 and 5, we have

$$\big|K : C_K(g)\big| = \big|K : Z(G)_\pi\big| = \big|G : Z(G)\big|_\pi = m,$$

and consequently, $k = |G : C_G(g)K|$. Now, let $y^* \in C_{G^*}(g^*)$ and choose H to be a π-complement of G with $g \in H$. As $G = KH$, we can clearly write $y^* = h^*$ for some $h \in H$.

As we are assuming that $[h^*, g^*] = 1$, then $[h, g] \in K \cap H = 1$. Therefore, $h \in C_G(g)$ and so, $y \in KC_G(g)$. It follows that $C_{G^*}(g^*) = C_G(g)^*$ and that g^* has index k in G^*, as we wanted to prove.

Now, we are able to apply Theorem 2 so as to obtain that $k = q^a$ for some prime q and $G^* = Q^* \times A^*$, where $Q \in \mathrm{Syl}_q(G)$ and A is abelian. Since $G^* \cong H$, then H has the structure described in the statement of this step.

By Steps 2 and 4, every element of $\overline{G} := G/Z(G)$ is a π-element or a π'-element. Since $\overline{K} \neq 1$ is a normal Hall π-subgroup and $\overline{H} \neq 1$, it follows that $\overline{G}$ is a Frobenius group with kernel $\overline{K}$ of order m. In particular, $m > n$. □

We remark that when $\mathbf{O}_{\pi'}(G) \subseteq Z(G)$ then, in view of Step 9, the structure of the π-complements of G is as described in case (a) of Step 7.

From now on to the end of the proof we shall assume that $\mathbf{O}_{\pi'}(G) \nsubseteq Z(G)$ and then, by Step 8, $\mathbf{O}_{\pi}(G) \subseteq Z(G)$.

Step 10. $\mathbf{O}_{\pi',\pi}(G)$ is a quasi-Frobenius group and $n > m$.

Proof. Since the Hall π-subgroups of G are abelian, it is known then that the π-length of G is less or equal to 1. Therefore, $N := \mathbf{O}_{\pi',\pi}(G) = \mathbf{O}_{\pi'}(G)K$, with K a Hall π-subgroup of G. Also notice that $\mathbf{O}_{\pi'}(G)$ and K are non-central in G.

It is trivial that $Z(G) \subseteq Z(N)$. By Steps 2 and 4, every element of $\overline{N} := N/Z(G)$ is a π-element or a π'-element. Since $\overline{\mathbf{O}_{\pi'}(G)} \neq 1$ is a normal Hall π'-subgroup of $\overline{N}$ and $\overline{K} \neq 1$, it follows that $\overline{N}$ is a Frobenius group with kernel $\overline{\mathbf{O}_{\pi'}(G)}$. In particular, we have $|\overline{\mathbf{O}_{\pi'}(G)}| > |\overline{K}|$, so by Step 5, $n = |G : Z(G)|_{\pi'} \geqslant |\overline{\mathbf{O}_{\pi'}(G)}| > |\overline{K}| = m$. Moreover, since $\overline{N}$ is a Frobenius group, $Z(\overline{N}) = 1$, so $Z(N) = Z(G)$ and N is quasi-Frobenius. □

In view of Steps 9 and 10, when $n < m$ then $\mathbf{O}_{\pi'}(G)$ is central and Step 9 provides case (1) of the theorem. If $n > m$, then $\mathbf{O}_{\pi}(G)$ must be central and thus, Step 10 proves case (2). Now we are going to distinguish the two cases given in Step 7. First, from Steps 11 to 13, we analyze case (a) and assume the conditions given there, and later, from Steps 14 to 19, we shall prove that case (b) cannot happen. We notice that if $n = q^r$ (so the class sizes of G are $(\pi \cup \{q\})$-numbers) then $A \subseteq Z(G)$, so we obtain case (2.1). Thus, in the following three steps we shall assume that n is not a q-power (so A is not central in G) and show that one of the situations (2.2)(a) or (2.2)(b) occurs.

Step 11. G has a normal π-complement or a normal Hall $(\pi \cup \{q\})$-subgroup.

Proof. We fix K and $H = QA$ a Hall π-subgroup and a π-complement of G, respectively. Let $g \in G$ and consider the factorization $g = g_\pi g_{\pi'}$. If g_π is non-central, by Step 6, we have $g_{\pi'} \in Z(G)$, so $g \in K^t\, Z(G)$, for some $t \in G$. Since K is abelian, then $g \in C_G(K^t)$. On the other hand, if $g_{\pi'}$ is non-central, then again by Step 6, g_π is central and the index of g is $1, m$ or mq^a. This implies that there is some $t \in G$ such that $A^t \subseteq C_G(g)$, whence $g \in C_G(A^t)$. These properties yield to the following equality

$$G = \bigcup_{t\in G} C_G\big(K^t\big) \cup \bigcup_{t\in G} C_G\big(A^t\big)$$

and by counting elements we get

$$|G| \leqslant \left|G : N_G\left(C_G(K)\right)\right|\left(\left|C_G(K)\right| - 1\right) + \left|G : N_G\left(C_G(A)\right)\right|\left(\left|C_G(A)\right| - 1\right) + 1$$

or equivalently,

$$1 \leqslant \frac{|C_G(K)|}{|N_G(C_G(K))|} - \frac{1}{|N_G(C_G(K))|} + \frac{|C_G(A)|}{|N_G(C_G(A))|} - \frac{1}{|N_G(C_G(A))|} + \frac{1}{|G|}.$$

Now we denote by $n_1 = |N_G(C_G(K))|$ and $n_2 = |N_G(C_G(A))|$. If $|C_G(K)| < n_1$ and $|C_G(A)| < n_2$, then

$$1 \leqslant \frac{1}{2} - \frac{1}{n_1} + \frac{1}{2} - \frac{1}{n_2} + \frac{1}{|G|},$$

so we obtain the following contradiction

$$\frac{|G|}{n_1} + \frac{|G|}{n_2} \leqslant 1.$$

Hence, $N_G(C_G(K)) = C_G(K)$ or $N_G(C_G(A)) = C_G(A)$, and the step follows as a consequence of Theorem 5. □

Step 12. If G has a normal π-complement H, then it factorizes as $H = Q \times A$. Also, G is a quasi-Frobenius group.

Proof. Suppose that G has a normal π-complement $H = QA$ with A abelian. By Step 6, any $x \in Q$ has index $1, m$ or mq^a. On the other hand, we have

$$\left|x^G\right|\left|C_G(x) : C_H(x)\right| = |G : H|\left|x^H\right|.$$

As $|G : H|$ is a π-number and $|x^H|$ divides $|x^G|$, the above equality implies that $|x^H|$ is equal to 1 or q^a. By Lemma 1(b), we conclude that $H = Q \times A$. The fact that G is quasi-Frobenius follows from Step 10. □

The above step provides the properties given in (2.2)(a) of the theorem. The next step will provide case (2.2)(b).

Step 13. If G has a normal Hall $(\pi \cup \{q\})$-subgroup, then G has an abelian normal Sylow q-subgroup.

Proof. Let $L := KQ$ be a normal Hall $(\pi \cup \{q\})$-subgroup of G, where K is a Hall π-subgroup and Q is a Sylow q-subgroup of G such that $H = QA$. Since $\mathbf{O}_{\pi'}(G) \subseteq H$, we can certainly write $\mathbf{O}_{\pi'}(G) = A_1 Q_1$, where $A_1 \subseteq A$ and $Q_1 \subseteq Q$. By Step 10, $\mathbf{O}_{\pi'}(G)/Z(G)_{\pi'}$ is a Frobenius kernel, so $\mathbf{O}_{\pi'}(G)$ is nilpotent. Consequently, $\mathbf{O}_{\pi'}(G) = A_1 \times Q_1$ and $Q_1 = \mathbf{O}_q(G)$.

Suppose first that $\mathbf{O}_q(G) \subseteq Z(G)$. Since A is abelian, this implies that $A \subseteq C_G(\mathbf{O}_{\pi'}(G))$. On the other hand, as $\mathbf{O}_\pi(G) \subseteq Z(G)$ it is easy to see that $C_G(\mathbf{O}_{\pi'}(G)) \subseteq \mathbf{O}_{\pi'}(G)Z(G)$. So

$A \subseteq A_1 Z(G)$ and $A = A_1(A \cap Z(G))$. Hence A_1 is normal in G, so we get $G = L \times A$ and consequently, $A \subseteq Z(G)$, which contradicts our assumption that $\mathbf{O}_{\pi'}(G)$ is not central.

Thus, we can assume that there is some non-central $x \in \mathbf{O}_q(G)$. If y is a π-element of $C_G(\mathbf{O}_q(G))$, then $y \in C_G(x)$ and, by Step 6, $y \in Z(G)$. Hence,

$$C_G(\mathbf{O}_q(G)) = C_G(\mathbf{O}_q(G))_{\pi'} \times C_G(\mathbf{O}_q(G))_{\pi} \subseteq \mathbf{O}_{\pi'}(G)Z(G).$$

Now suppose that Q has an element z of index mk. Replacing z by some conjugated we can assume that $A \subseteq C_G(z)$. If $t \in A$, then the maximality of mk implies that $C_G(zt) = C_G(z) \subseteq C_G(t)$, so $C_{\mathbf{O}_q(G)}(z) \subseteq C_{\mathbf{O}_q(G)}(t)$. By applying Theorem 4, we obtain $t \in C_G(\mathbf{O}_q(G))$, so

$$A \subseteq C_G(\mathbf{O}_q(G)) \subseteq \mathbf{O}_{\pi'}(G)Z(G)$$

and $A \subseteq \mathbf{O}_{\pi'}(G)$. Accordingly, $A = A_1$ and as above this yields to a contradiction. Therefore, there are no q-elements of index mk, so any q-element has index 1 or m, and by Lemma 1(a), we conclude that Q is abelian.

Now, as Q is abelian, then $Q \subseteq C_G(\mathbf{O}_q(G)) \subseteq \mathbf{O}_{\pi'}(G)Z(G)$. Then $Q \subseteq \mathbf{O}_{\pi'}(G)$ and $Q = \mathbf{O}_q(G)$, so the step is proved. □

The rest of the proof consists of proving that case (b) of Step 7 is not possible and then the theorem will be proved. Thus, from now on we shall assume the conditions given in that case in order to get a contradiction in Step 19.

Step 14. Let $L_{\pi'} = \langle x \in G\colon\ x \text{ is } \pi'\text{-element of index 1 or } m\rangle$. Then $L_{\pi'}$ is a (normal) abelian π'-subgroup and $L_{\pi'} \subseteq Z(\mathbf{O}_{\pi'}(G))$.

Proof. Notice that by Lemma 1(c) any π'-element of index m must lie in $\mathbf{O}_{\pi,\pi'}(G) = \mathbf{O}_{\pi'}(G) \times Z(G)_{\pi}$, so it belongs to $\mathbf{O}_{\pi'}(G)$. We deduce that $L_{\pi'} \subseteq \mathbf{O}_{\pi'}(G)$, whence $L_{\pi'}$ is a (normal) π'-group. In order to see that $L_{\pi'}$ lies in the center of $\mathbf{O}_{\pi'}(G)$, it is enough to note that the index in $\mathbf{O}_{\pi'}(G)$ of any generator of $L_{\pi'}$ must divide $(|\mathbf{O}_{\pi'}(G)|, m) = 1$. □

From the conditions given in 7(b) we note that the only primes dividing n are exactly q and r. As $L_{\pi'}$ is abelian, then we can factor $L_{\pi'} = L_q \times L_r \times Z$, where $Z \subseteq Z(G)$ and the subgroups L_q and L_r are defined as follows:

$$L_q = \langle x \in G\colon\ x \text{ is a } q\text{-element of index 1 or } m\rangle,$$

$$L_r = \langle x \in G\colon\ x \text{ is an } r\text{-element of index 1 or } m\rangle.$$

Step 15. $\mathbf{O}_{\pi'}(G) = L_{\pi'}$.

Proof. We claim that if z is a q-element of index mk, then $C_G(z) = Q_z \times R_z \times Z$, with Q_z a q-group, R_z an abelian r-group centralizing L_q and $Z = Z(G)_{\{q,r\}'}$. By applying Step 6, we can write $C_G(z) = Q_z R_z \times Z$ with Q_z, R_z and Z as described above, so it remains to show that R_z is an abelian direct factor of $C_G(z)$ which centralizes L_q. Let $y \in R_z$ and notice that $C_G(zy) = C_G(y) \cap C_G(z) \subseteq C_G(z)$. The maximality of mk implies that $C_G(z) \subseteq C_G(y)$, so R_z is an abelian direct factor of $Z(C_G(z))$ as wanted. Also, in particular we have $C_{L_q}(z) \subseteq C_{L_q}(y)$, and by applying Theorem 4, we get $y \in C_G(L_q)$. The same assertion can be done for r-elements of index mk. In this case, if z is such an element, then $C_G(z) = Q_z \times R_z \times Z$, with Q_z an abelian q-group centralizing L_r, and R_z an r-group and $Z = Z(G)_{\{q,r\}'}$.

We shall assume that $w \in \mathbf{O}_{\pi'}(G) - L_{\pi'}$ and work to get a contradiction. Observe that, up to some central element, we may factor $w = w_q w_r$, with w_q and w_r the q- and r-part of w, respectively. Certainly one of them, say w_q, does not lie in $L_{\pi'}$, although both factors must belong to $\mathbf{O}_{\pi'}(G)$. Then

$$L_{\pi'} \subseteq C_G\big(\mathbf{O}_{\pi'}(G)\big) \subseteq C_G(w_q) = Q_{w_q} \times R_{w_q} \times Z,$$

where the last equality is written by using the above paragraph with the notation given there for $C_G(w_q)$. The first inclusion follows because any generator of $L_{\pi'}$ has π-index. In particular, $L_q \subseteq Q_{w_q}$ and $L_r \subseteq R_{w_q}$, which provides the following inequalities:

$$|G|_q/|L_q| > k_q = q^t \quad \text{and} \quad |G|_r/|L_r| \geqslant k_r = r^l. \tag{I}$$

Observe that the first one cannot be an equality since $w_q \in Q_{w_q} - L_q$.

We claim that L_q is centralized by any Sylow q-subgroup of G. Suppose not and choose some q-element $z \notin C_G(L_q)$. Certainly, z has index mk and again by the first paragraph, we write $C_G(z) = Q_z \times R_z \times Z$, with $R_z \subseteq C_G(L_q)$. We claim now that $R_z = L_r$. If $y \in R_z - L_r$, then y has index mk and by maximality, $C_G(z) = C_G(y)$. But then $L_q \subseteq C_G(y) = C_G(z)$ and so, $z \in C_G(L_q)$, which is a contradiction. Thus, $R_z \subseteq L_r$ and by (I), we get $R_z = L_r$, as claimed. Suppose now that there exists some r-element x of index mk and write $C_G(x) = Q_x \times R_x \times Z$, with $Q_x \subseteq C_G(L_r)$. Then we may choose some q-element $s \in Q_x$ of index mk, otherwise every element of Q_x would have index m, so $Q_x \subseteq L_q$ and this yields to a contradiction with the first inequality of (I). By maximality of mk, it follows that $C_G(x) = C_G(s)$, so in particular, $L_r \subseteq C_G(s) = C_G(x)$, that is, $L_r \subseteq R_x$. As $|R_x| = |R_z| = |L_r|$, we conclude that $R_x = L_r$, so $x \in L_r$. We have proved that $L_r \in \mathrm{Syl}_r(G)$, but this is not possible since $L_r \subseteq C_G(w_q)$ and w_q has index mk. Therefore, the claim is proved.

We prove now that L_r is also centralized by any Sylow r-subgroup of G. Choose any r-element x of index mk and write, taking into account the first paragraph, $C_G(x) = Q_x \times R_x \times Z$, with $Q_x \subseteq C_G(L_r)$. Notice that (I) implies that $Q_x \nsubseteq L_q$, so there exists some $z \in Q_x$ of index mk. By maximality, $L_r \subseteq C_G(z) = C_G(x)$, whence $x \in C_G(L_r)$. Since any r-element of index 1 or m also lies in L_r, we conclude that any Sylow r-subgroup of G centralizes L_r.

We prove that any Sylow q-subgroup of G centralizes $L_{\pi'}$. Let z be a q-element of index mk and write again by using the first paragraph, $C_G(z) = R_z \times Q_z \times Z$ with $R_z \subseteq C_G(L_q)$. If $R_z \subseteq L_r$ then (I) implies that $R_z = L_r$, so in particular, $z \in C_G(L_r)$. Suppose then that $R_z \neq L_r$ and choose $x \in R_z$ of index mk. By maximality $C_G(x) = C_G(z)$ and by the above paragraph $L_r \subseteq C_G(z) = C_G(x)$. Thus, in both cases $z \in C_G(L_r) \cap C_G(L_q) = C_G(L_{\pi'})$. On the other hand, if z is a q-element of index 1 or m then it trivially centralizes $L_{\pi'}$.

Finally, we assert that any Sylow r-subgroup of G also centralizes $L_{\pi'}$. The proof is similar to the one of the above paragraph. It is enough to consider an r-element x of index mk, write $C_G(x) = Q_x \times R_x \times Z$ and take into account that $Q_x \nsubseteq L_q$ by (I).

All of the above results show that $L_{\pi'}$ is centralized by any Hall π'-subgroup of G. On the other hand, if we choose some $w \in L_{\pi'}$ of index m, we have $H \subseteq C_G(L_{\pi'}) \subseteq C_G(w)$. By Step 6,

$C_G(L_{\pi'}) = H \times Z(G)_\pi$, whence $H \trianglelefteq G$. We see that this provides the final contradiction. For any $x \in H$, which has index 1, m or mk, we have

$$|G:H|\left|x^H\right| = \left|x^G\right|\left|C_G(x):C_H(x)\right|,$$

and as $|x^H|$ divides $|x^G|$, it follows that $|x^H|$ is necessarily equal to 1 or k. By Theorem 2, then k would be a power of a single prime, contradicting the fact that $k = q^t r^l$. □

Step 16. For a q-element $x \notin L_q$, we can write $C_G(x) = Q_x \times R_x \times Z$, with Q_x a q-group, $R_x \subseteq L_r$ and $Z = Z(G)_{\{q,r\}'}$. Analogously, this property is satisfied for any r-element $x \notin L_r$, but with $Q_x \subseteq L_q$.

Proof. Notice that x has index mk and then by the first paragraph of the proof of Step 15, $C_G(x)$ has the structure described in the statement, although it remains to prove that $R_x \subseteq L_r$. Suppose that there is some $y \in R_x - L_r$. Clearly y has index mk and by maximality, it follows that $C_G(x) = C_G(y)$. In particular, $C_{L_q}(x) = C_{L_q}(y)$ and $C_{L_r}(x) = C_{L_r}(y)$, so we can apply Theorem 4 to obtain that both L_q and L_r centralize x, that is, $x \in C_G(L_{\pi'})$. But $C_G(L_{\pi'}) = C_G(\mathbf{O}_{\pi'}(G)) \subseteq \mathbf{O}_{\pi'}(G) \times Z(G)_\pi$ and then $x \in \mathbf{O}_{\pi'}(G) = L_{\pi'}$, which is a contradiction.

The same property for r-elements can be demonstrated in a similar way. □

Step 17. $L_{\pi'} = \{x \in G\colon x \text{ is a } \pi'\text{-element of index 1 or } m\}$.

Proof. By definition of $L_{\pi'}$ and Step 2 it is enough to assume that there exists some $w \in L_{\pi'}$ of index mk and get a contradiction. Let us factor $w = w_q w_r$ with $w_q \in L_q$ and $w_r \in L_r$ (omitting without loss the central factor). Now, as $L_{\pi'}$ is abelian, we have $L_{\pi'} \subseteq C_G(w)$, so we get $|G|_r/|L_r| \geqslant k_r = r^l$.

Let z be any q-element of G. If $z \in L_q$, then it trivially centralizes w_r. If $z \notin L_q$, then it has index mk, and by using Step 16 (with the notation given for $C_G(z)$) together with the above inequality, we obtain $R_z = L_r$. Then $z \in C_G(L_r)$, so in particular $z \in C_G(w_r)$. Therefore, w_r is centralized by any Sylow q-subgroup of G. But we can argue similarly with any r-element of G to obtain that w_q is centralized by any Sylow r-subgroup of G. This implies that w_q has index 1 or m, so w_q is centralized by some Sylow q-subgroup of G. It follows that $w = w_q w_r$ must be centralized by some Sylow q-subgroup of G, which contradicts the fact that w has index mk. □

Step 18. If $L_q \nsubseteq Z(G)$, then $L_q \in \mathrm{Syl}_q(G)$.

Proof. We shall assume that $L_q \nsubseteq Z(G)$ and that L_q is not a Sylow q-subgroup of G in order to get a contradiction. Suppose first that $L_r \subseteq Z(G)$, that is, every non-central r-element has index mk. We can take some q-element, x, of index mk. By Step 16, we write $C_G(x) = Q_x \times R_x \times Z$ with $R_x \subseteq L_r$. Consequently, $R_x = Z(G)_r$ and $|G|_r/|Z(G)|_r = k_r = r^l$, and this contradicts the existence of r-elements of index mk. Notice that there are r-elements of index mk as the Sylow r-subgroups are not central since r divides k.

Therefore, we can assume that $L_r \nsubseteq Z(G)$ and then we may choose some non-central $y \in L_r$. By Steps 6 and 17, we have $C_G(y) = H \times Z(G)_\pi$, for some Hall π'-subgroup H of G. In the rest of this step, we shall prove that any element of $C_G(y)$ has index 1 or k in $C_G(y)$. This yields, via Theorem 2, to either H is abelian or k is a power of a single prime. In both cases, there is a contradiction. Note that H is not abelian, since there are π'-elements of index mk by Step 4.

To see the above assertion, we choose first any q-element $z \in C_G(y)$. As $C_G(zy) = C_G(z) \cap C_G(y) \subseteq C_G(y)$, then z has index 1 or k in $C_G(y)$. Now suppose that z is an r-element of $C_G(y)$ and take some non-central $x \in L_q$ (of index m by Step 17). As in the above paragraph, we have $C_G(x) = H^g \times Z(G)_\pi$, for some $g \in G$. Replace x by some G-conjugated to assume that $C_G(y) = C_G(x)$. Then

$$C_G(z) \cap C_G(y) = C_G(z) \cap C_G(x) = C_G(zx) \subseteq C_G(x) = C_G(y),$$

so z has again index 1 or k in $C_G(y)$, as wanted. Finally, we check the same property for an arbitrary element w of $C_G(y)$. We can assume without loss that w is a $\{q, r\}$-element and write $w = w_q w_r$, since other any factor in the primary decomposition would be central. Moreover, if one of both is central, then the result follows by the above argument. Suppose now that w_r has index m and notice that one can take $x \in L_q$ with $C_G(x) = C_G(y)$ by arguing as in the above paragraph. Then

$$C_G(w_r) \cap C_G(y) = C_G(w_r) \cap C_G(x) = C_G(w_r x).$$

But $w_r x \in L_{\pi'}$, so by Step 17, it has index 1 or m. Since y has also index m, it follows that $C_G(w_r) \cap C_G(y) = C_G(y)$ and as a result, $C_G(w_r) = C_G(y)$. Then $C_G(w) \cap C_G(y) = C_G(w_q) \cap C_G(y)$ and this forces w to have index 1 or k in $C_G(y)$. Analogously, if we assume that w_q has index m, we can argue as above to obtain that w has also index 1 or k in $C_G(y)$. Therefore, suppose finally that both w_q and w_r have index mk. By applying Step 16, and taking the notation given in that step, we write $C_G(w_q) = Q_{w_q} \times R_{w_q} \times Z$, with $R_{w_q} \subseteq L_r$. Since $w_r \in R_{w_q}$, this provides a contradiction with the fact that w_r has index mk. □

Step 19. Final contradiction.

Proof. We can assume that one of the subgroups L_q or L_r lies in $Z(G)$, otherwise Step 18 implies that G possesses an abelian π-complement, which provides a contradiction using Lemma 1(a). We shall suppose for instance that $L_r \subseteq Z(G)$, or equivalently, that every r-element of G has index 1 or mk. Consequently, L_q is non-central and it is a Sylow q-subgroup of G by Step 18. On the other hand, as the Hall π-subgroups of G are abelian, we know then that the π-length of G is less or equal to 1. Hence, $\mathbf{O}_{\pi'}(G)K \trianglelefteq G$ and we deduce that $L_q K \trianglelefteq G$. Let $R \in \mathrm{Syl}_r(G)$ and let us take any prime $s \in \pi$. Since R acts coprimely on $L_q K$, there exists some $S \in \mathrm{Syl}_s(G)$ fixed by R. Also, since S is abelian, certain coprime action properties allow us to write $S = T \times C_S(R)$, where $T := [S, R]$. Notice that if $a \in C_S(R)$, and hence $R \subseteq C_G(a)$, then $a \in Z(G)$ by Step 6. Therefore, $C_S(R) = Z(G)_s$, and consequently, $T \cap Z(G) = 1$.

We consider the group $\overline{R} = R/Z(G)_r$ which acts fixed-point-freely on T by $x^{\bar{y}} = x^y$ for all $x \in T$ and all $y \in R$. Certainly this action has no fixed points because if $\bar{y}$ fixes some $1 \neq x \in T$, then $y \in C_G(x)$ and by Step 6 it follows that $y \in Z(G)$ and $\bar{y} = 1$. It is also known that in this case $\overline{R}$ must be cyclic or generalized quaternion (see for instance Theorem 16.12 of [9]). Note that $T \neq 1$ since there are elements of index n.

On the other hand, T also acts coprimely on L_q, so we write $L_q = [L_q, T] \times C_{L_q}(T)$. Moreover, if $y \in C_{L_q}(T)$, then $S \subseteq C_G(y)$, and since y has index 1 or m and s divides m, we obtain $y \in Z(G)$. Hence, $C_{L_q}(T) = Z(G)_q$. Moreover, T acts fixed-point-freely on $[L_q, T]$. To see this, it is enough to notice that by Step 6 any $a \in [L_q, T] - \{1\}$ cannot be centralized by any non-central element of K, so in particular, by any element of $T - \{1\}$. Now, again by Theorem 16.12 of [9], T must be cyclic if $s \neq 2$. Finally, as the automorphism group of a cyclic group is abelian, we conclude that $\overline{R}$ is abelian. Therefore, $\overline{R}$ must be cyclic, whence R is abelian. By using Lemma 1(a), we see that this contradicts the existence of r-elements of index mk in G. $\square$

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