J. Differential Equations 252 (2012) 988–1002

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Boundary first order derivative estimates for fully nonlinear elliptic equations ☆

Feiyao Ma [a,*], Lihe Wang [b]

[a] Department of Mathematics, Ningbo University, Zhejiang Province, China
[b] Department of Mathematics, University of Iowa, Iowa, IA, USA

ARTICLE INFO

Article history:
Received 26 October 2010
Revised 10 July 2011
Available online 22 October 2011

MSC:
35J25
35J60

Keywords:
Fully nonlinear
Elliptic equations
Viscosity solutions
Dini condition
Boundary regularity

ABSTRACT

In this paper, we prove the estimates of modulus of continuity up to the boundary for the first order derivative of viscosity solutions for fully nonlinear uniformly elliptic equations under Dini boundary data with the domain in the same class. As a corollary we derive $C^{1,\alpha}$ boundary regularity.

1. Introduction

In 1989, Caffarelli [1] developed a general technique using a polynomial approximation to obtain $C^{1,\alpha}$, $C^{2,\alpha}$ and $W^{2,p}$ interior estimates for the viscosity solutions of fully nonlinear uniformly elliptic equations. In 1992, the second author proved similar results for fully nonlinear uniformly parabolic equations as well as the estimates up to the boundary [8,9]. In 1997–1999, Kovats [4,5] obtained interior regularity results for the classical solutions of fully nonlinear uniformly elliptic equations under the Dini condition as well as modulus of continuity estimates, but with a gap in the proof. In 2002 Zou and Chen [10] filled up Kovats's gap and used the approximation lemma to obtain interior regularity results for viscosity solutions of fully nonlinear uniformly parabolic equations under the Dini

☆ Research supported in part by Sequoia Capital China.
* Corresponding author.
E-mail addresses: mafeiyao@nbu.edu.cn (F. Ma), lwang@math.uiowa.edu (L. Wang).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.007

condition. On boundary case, for $C^{2,\alpha}$ regularity is well known [3] and [11]. In 2006, Li and Wang [6] showed the boundary differentiability for solutions of elliptic differential equations in non-divergence form on convex domains under zero Dirichlet boundary conditions. In 2009, Li and Wang [7] generalized their results to the nonhomogeneous Dirichlet boundary conditions. The boundary $C^{1,\alpha}$ theorem of parabolic equations was proved by Wang [8,9]. In this paper, we prove $C^{1,\psi}$ estimates up to the boundary for viscous solutions of fully nonlinear uniformly elliptic equations under Dini conditions. The Dini conditions include that the boundary is $C^{1,dini}$. This is new, and we also emphasize that we have pointwise estimates. As a corollary we derive $C^{1,\alpha}$ boundary regularity. We also like to point out that $C^{1,\alpha}$ regularity holds only on the boundary in our setting but not in the interior.

In the linear case, it is well known that if $u \in C^2(\overline{B_2}(x_0))$ satisfies $\Delta u = f \in C^{0,\omega}(B_2(x_0))$ in $B_2(x_0)$, then for all $0 \leqslant t < 1$, we have

$$\sup_{|x-y|\leqslant t} \left|D^2u(x) - D^2u(y)\right| \leqslant C\left(\int_0^t \frac{\omega(s)}{s}\,ds + t\int_t^1 \frac{\omega(s)}{s^2}\,ds\right), \tag{1.1}$$

where C is independent of t. Of course, when $f \in C^{0,\alpha}(B_2)$, i.e. $\omega(t) = t^\alpha$, $0 < \alpha < 1$, the right hand side of (1.1) is controlled by Ct^α.

Kovats [4] got the similar interior second order derivative estimates for fully nonlinear uniformly elliptic equation $F(D^2u) = f$ in Ω. Under condition

$$\lim_{\mu\to 0+} \sup_{0\leqslant t\leqslant \frac{1}{2}} \frac{\mu^{\tilde{\alpha}}\varphi(t)}{\varphi(t\mu)} = 0,$$

where $\varphi(t) = t^{\tilde{\alpha}} + \omega(t)$, his interior estimate is

$$\|u\|_{C^{2,\psi}(\overline{B_{\frac{r_0}{2}}})} \leqslant C(n, \lambda, \Lambda, \omega, r_0)\left(\|u\|_{0;B_{r_0}} + \|f\|_{0,\omega;B_{r_0}}\right), \tag{1.2}$$

where

$$\psi(r) = r^{\tilde{\alpha}} + \int_0^r \frac{\omega(s)}{s}\,ds,$$

and $\tilde{\alpha} = \tilde{\alpha}(n, \lambda, \Lambda) \in (0, 1)$ is the Hölder exponent given in the Evans–Krylov theorem.

Li and Wang [7] investigated the influence of the convexity of domain on the solutions. Precisely, assuming the domain in R^n is convex, they study the smoothness of the solutions of the following elliptic equation

$$\begin{cases} -a^{ij}D_{ij}u(x) = f(x), & \Omega, \\ u(x) = g(x), & \partial\Omega, \end{cases}$$

where the matrix $\{a_{ij}(x)\}$ is symmetric and satisfies

$$\lambda I_n \leqslant \left(a^{ij}(x)\right)_{n\times n} \leqslant \frac{1}{\lambda} I_n, \quad \forall x \in \Omega.$$

They obtained the following theorem.

Theorem 1.1 *(Li and Wang). Assume $g(x)$ is differentiable at* 0*, i.e.*

$$\left|g(x) - g(0) - Dg(0)x\right| \leqslant r\sigma(r),$$

where σ is a modulus of continuity, then

$$\left|u(x) - u(0) - Dg(0)x\right| \leqslant C\Biggl\{r^\alpha\bigl(\|u\|_{L^\infty(\Omega([1\times 1]))} + \|f\|_{L^n(\Omega([1\times 1]))} + \sigma(\sqrt{2})\bigr) + r^\alpha \int_r^1 \frac{\|f\|_{L^n(\Omega([t\times t]))} + \sigma(\sqrt{2}t)}{t^{1+\alpha}}\,dt + \|f\|_{L^n(\Omega([\Lambda r\times \Lambda r]))} + \sigma(\sqrt{2}\Lambda r)\Biggr\}r.$$

In this paper we give similar estimates. But we don't assume the convexity of domain, and our estimates are with regard to the first order derivative of solutions of fully nonlinear elliptic equation up to the $C^{1,\omega}$ boundary. We consider the following equation

$$\begin{cases} F\bigl(D^2u(x), x\bigr) = f(x), & x \in \Omega, \\ u(x) = \phi(x), & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where F is a uniformly elliptic operation, i.e. F satisfies

$$0 < \lambda\|M\| < F(N+M, x) - F(N, x) \leqslant \Lambda\|M\|, \tag{1.4}$$

where N is any symmetric matrix, M is any definite symmetric matrix, λ, Λ are elliptic constants.

Our primary goal is to answer the following questions: for solutions of (1.3), does Du exist on $\partial\Omega$, and what is the modulus of continuity of Du in terms of the modulus of continuity of f, $\partial\Omega$ and ϕ? Before we claim our main theorems, let us explain in what sense we say the modulus of continuity of them.

Definition 1.2. We say $g(r)$ satisfies doubling condition if there exist θ, C such that $0 < \theta < 1$, $C > 0$ and

$$g\bigl(r'\bigr) \geqslant Cg(\theta r), \quad \text{for each } r' > r > 0. \tag{1.5}$$

Definition 1.3. We say nonnegative function $g(s)$, $s \in [0,1]$, satisfies Dini condition if $g(s)$ satisfies doubling condition and satisfies

$$\int_0^1 \frac{g(s)}{s}\,ds < +\infty.$$

Definition 1.4. We say $\partial\Omega$ is $C^{1,\omega}$ at y, $y \in \partial\Omega$, ω is a modulus of continuity if there exists a vector $\vec{n}$ such that

$$\frac{1}{r}\sup_{x\in\partial\Omega, |y-x|\leqslant r}\left|(x-y)\cdot\vec{n}\right| \leqslant \omega(r), \quad \text{for each } r > 0.$$

We say $\partial\Omega$ is $C^{1,\omega}$ if for any $y \in \partial\Omega$, $\partial\Omega$ is $C^{1,\omega}$ at y. In the same way we define $C^{1,dini}$ if ω above satisfies Dini condition.

Definition 1.5. We say ϕ is $C^{1,\omega}$ on $\partial\Omega$ at y if there exists a linear function $L(x)$ such that

$$\frac{1}{r} \sup_{x \in \partial\Omega, |y-x| \leqslant r} \left|\phi(x) - L(x)\right| \leqslant \omega(r), \quad \text{for each } r > 0.$$

We say ϕ is $C^{1,\omega}$ on $\partial\Omega$ if for any $y \in \partial\Omega$, ϕ is $C^{1,\omega}$ at y. Similarly we can define $C^{1,dini}$ for ϕ.

Definition 1.6. We say $f \in L^p(\Omega)$ is $C_p^{-1,\omega}$ at $y \in \partial\Omega$ if

$$r\left(\fint_{B_r(y)\cap\Omega} f^p \right)^{\frac{1}{p}} = \|f\|_{n,B_r(y)\cap\Omega} \leqslant \omega(r), \quad \text{for each } r > 0.$$

Similarly we can define $C_p^{-1,\omega}$ on $\partial\Omega$ and $C^{-1,dini}$ on a point or on $\partial\Omega$. We will write $C_n^{-1,\omega}$ as $C^{-1,\omega}$ for simplicity.

The following theorems are our main results.

Theorem 1.7. *Let u be a viscosity solution of* (1.4) *in $\Omega \cap B_1(0)$, $0 \in \partial\Omega$, and assume $\partial\Omega$ is $C^{1,\omega}$ at 0, ϕ is $C^{1,\omega}$ at 0, f is $C^{-1,\omega}$ at 0 and ω satisfies Dini condition, then u is differentiable at 0, furthermore, there exist a linear function $L(x)$ and constants $\hat{\alpha} > 0$, $C > 0$ such that*

$$\left|u(x) - L(x)\right| \leqslant Cr\left(r^{\hat{\alpha}} + \int_0^r \frac{\omega(s)}{s}\, ds + r^{\hat{\alpha}} \int_r^1 \frac{\omega(s)}{s^{1+\hat{\alpha}}}\, ds \right), \quad 0 \leqslant r = |x| \leqslant 1,$$

where C only depends on n, λ, Λ, ω, $\|u\|_{C^0(\Omega\cap B_1(0))}$ and $\|f\|_{n,\Omega\cap B_1(0)}$.

Corollary 1.8. *Let u be a viscosity solution of* (1.4) *in $\Omega \cap B_1(0)$, $0 \in \partial\Omega$, and assume $\partial\Omega$ is $C^{1,\alpha}$ at 0, ϕ is $C^{1,\alpha}$ at 0 and f is $C^{-1,\alpha}$ at 0. Then u is $C^{1,\hat{\beta}}$ at 0, $\hat{\beta} = \min(\hat{\alpha}, \alpha)$, i.e. there exists a linear function $L(x)$ such that*

$$\left|u(x) - L(x)\right| \leqslant C|x|^{1+\hat{\beta}}, \quad 0 \leqslant |x| \leqslant 1,$$

where $\hat{\alpha}$, C are constants as in Theorem 1.7.

From Theorem 1.7 we know that solutions of (1.4) are differentiable on boundary under boundary Dini condition. In fact we also have the uniformly estimate of the modulus of continuity of first order derivative under uniformly boundary Dini condition.

Theorem 1.9. *Let u be a viscosity solution of* (1.4) *in $\Omega \cap B_2(0)$, $0 \in \partial\Omega$, and assume $\partial\Omega$ is $C^{1,\omega}$ in $B_2(0)$, ϕ is $C^{1,\omega}$ on $\partial\Omega \cap B_2(0)$, f is $C^{-1,\omega}$ on $\partial\Omega \cap B_2(0)$, ω satisfies Dini condition, $y, z \in \partial\Omega \cap B_1(0)$, $|y - z| = r \leqslant 1$. Then there exist constants $\hat{\alpha} > 0$, $C > 0$ such that*

$$\left|\nabla u(y) - \nabla u(z)\right| \leqslant C\left(r^{\hat{\alpha}} + \int_0^r \frac{\omega(s)}{s}\, ds + r^{\hat{\alpha}} \int_r^1 \frac{\omega(s)}{s^{1+\hat{\alpha}}}\, ds \right),$$

where C only depends on $n, \lambda, \Lambda, \omega$, $\|u\|_{C^0(\Omega\cap B_2(0))}$ *and* $\|f\|_{n,\Omega\cap B_2(0)}$, $\hat{\alpha}$ *is the same constant as in Theorem* 1.7.

Corollary 1.10. *Let u be a viscosity solution of* (1.4) *in* $\Omega \cap B_2(0)$, $0 \in \partial\Omega$, *and assume* $\partial\Omega$ *is* $C^{1,\alpha}$ *in* $B_2(0)$, ϕ *is* $C^{1,\alpha}$ *on* $\partial\Omega \cap B_2(0)$, f *is* $C^{-1,\alpha}$ *on* $\partial\Omega \cap B_2(0)$ *and* $y, z \in \partial\Omega \cap B_1(0)$, $|y-z| \leqslant 1$. *Then there exist constants* $\hat{\beta} = \min(\alpha, \hat{\alpha}) > 0$, $C > 0$ *such that*

$$\left|\nabla u(y) - \nabla u(z)\right| \leqslant C|y-z|^{\hat{\beta}},$$

where $\hat{\alpha}, C$ *are constants as in Theorem* 1.9.

This paper is organized as follows: In Section 2, we present some notations and tools for fully nonlinear elliptic equations. In Section 3, we prove Theorem 1.7 and Theorem 1.9.

2. Some auxiliary materials

We review the viscosity solutions and some important tools for fully nonlinear elliptic equations in this section. In order to avoid talking about a specific operator, we introduce Pucci's extremal operators as introduced in [2].

Definition 2.1 *(Pucci's extremal operator).*

$$\mathcal{M}^-\left(D^2u\right) = \lambda\left(\sum_{e_j>0} e_j\right) + \Lambda\left(\sum_{e_j<0} e_j\right),$$

$$\mathcal{M}^+\left(D^2u\right) = \Lambda\left(\sum_{e_j>0} e_j\right) + \lambda\left(\sum_{e_j<0} e_j\right),$$

where λ and Λ are the elliptic constants as in (1.4), and $\{e_j, 1 \leqslant j \leqslant n\}$ are the eigenvalues of D^2u.

Definition 2.2. We say that u belongs to the class $\overline{S}(f) = \overline{S}(\lambda, \Lambda, f)$ if for any C^2 function φ which satisfies $\mathcal{M}^+(D^2\varphi) \leqslant f$, $u - \varphi$ cannot have a local maximum. Similarly we say that u belongs to the class $\underline{S}(f) = \underline{S}(\lambda, \Lambda, f)$ if for any C^2 function φ which satisfies $\mathcal{M}^+(D^2\varphi) \geqslant f$, $u - \varphi$ cannot have a local minimum. The set $\overline{S}(f) \cap \underline{S}(f)$ is denoted by $S(f)$.

We introduce barriers for uniformly elliptic operators based on $P(x) = \frac{1}{|x|^{2N}}$. First we observe that $P(x)$ is a smooth subsolution except $x = 0$ when $N > \frac{(n-1)\Lambda}{\lambda} - 1$. Indeed, since $P(x)$ is rotationally symmetric, we only check at point $(r, 0, \ldots, 0)$.

$$\mathcal{M}^-\left(D^2P\right)(x) = \left(\lambda(N+1) - \Lambda(n-1)\right)N|x|^{-N-2} > 0, \quad x \neq 0.$$

Let

$$P_{x_0,R}(x) = \frac{\frac{1}{|x-x_0|^{2N}} - \frac{1}{(R)^{2N}}}{(R/2)^{-2N} - (R)^{-2N}},$$

then $P_{x_0,R}$ is rotational symmetric and satisfies the following properties:

$$\begin{cases} \mathcal{M}^- P_{x_0,R} > 0, & x \neq x_0, \\ P_{x_0,R} = 1, & \partial B_{\frac{R}{2}}(x_0), \\ P_{x_0,R} = 0, & \partial B_R(x_0), \\ P_{x_0,R} \leqslant 0, & B_R^c(x_0), \\ \frac{\partial P_{x_0,R}}{\partial r} < 0, \quad \frac{\partial^2 P_{x_0,R}}{\partial r^2} > 0, & x \neq x_0, \end{cases} \tag{2.1}$$

where $r = |x - x_0|$.

We also need the following two classic tools: the A-B-P estimate and Harnack inequality [2].

Theorem 2.3 *(A-B-P estimate). Assume $u \in \overline{S}(f)$ in Ω, then*

$$\sup_{\Omega}\Big(u - \inf_{\partial\Omega} u\Big)^- \leqslant Cd(\Omega)\big\|f^+\big\|_{L^n(u-\inf_{\partial\Omega} u=\Gamma(u-\inf_{\partial\Omega} u))},$$

where $d(\Omega)$ is the diameter of Ω, C is a universal constant which only depends on λ, Λ and n.

Theorem 2.4 *(Harnack inequality). Let u be a nonnegative function and $u \in S(f)$ in B_2. Then*

$$\sup_{B_1} u \leqslant C\Big(\inf_{B_1} u + \|f\|_{L^n(B_2)}\Big),$$

where C is a universal constant which only depends on λ, Λ and n.

3. Estimates

In this section, we prove the theorems discussed in the Introduction. For convenience, we always assume that $0 \in \partial\Omega$ and the direction of the vector $\vec{n}$ in Definition 1.4 is the direction of x_n axis. Since the equation doesn't include Du, we may assume $\phi(0) = 0$ and $D\phi(0) = 0$ by subtracting a linear function from the solution and the boundary value condition. Throughout this paper, we use the following notations:

$$x = \big(x', x_n\big), \qquad \|f\|_{n,\Omega} = \|f\|_{L^n(\Omega)}, \qquad B_r = B_r(0),$$

$$T_r = \big\{x' \in \mathbb{R}^{n-1}\colon \big|x'\big| < r\big\}, \qquad T_r(x) = T_r + x,$$

$$\partial\Omega_r = \partial\Omega \cap \big(T_r \times (-r, r)\big), \qquad \Omega_r = \Omega \cap \big(T_r \times (-r, r)\big),$$

$$\operatorname{osc} \partial\Omega_r = \sup_{(x',x_n)\in\partial\Omega_r} x_n - \inf_{(x',x_n)\in\partial\Omega_r} x_n, \qquad \operatorname{osc}_{\partial\Omega_r} \phi = \sup_{(x',x_n)\in\partial\Omega_r} \phi(x) - \inf_{(x',x_n)\in\partial\Omega_r} \phi(x).$$

Constant C may take different values independent of solution in different places.

Consider the following normalization of solution

$$\tilde{u}(x) = \frac{\epsilon_0 u(\delta_0 x)\delta_0^{-2}}{\delta_0^{-2}|u(\delta_0 x)|_{C^0(\Omega_1)} + \|f(\delta_0 x)\|_{n,\Omega_1}}, \quad x \in \widetilde{\Omega}_1, \tag{3.1}$$

and the normalization of domain

$$\widetilde{\Omega} = \frac{\Omega}{\delta_0} = \{x, \delta_0 x \in \Omega\}, \tag{3.2}$$

specially $\widetilde{\Omega}_1 = \frac{\Omega_1}{\delta_0} = \{x, \delta_0 x \in \Omega_1\}$.

Obviously $\tilde{u}(x)$ satisfies

$$G\big(D^2\tilde{u}(x), x\big) = \frac{1}{K} F\big(K D^2\tilde{u}(x), x\big) = \frac{1}{K} f(\delta_0 x),$$

where $K = (\delta_0^{-2}|u(\delta_0 x)|_{C^0(\Omega_1)} + \|f(\delta_0 x)\|_{n,\Omega_1})/\epsilon_0$. We know that $G(\cdot)$ and $F(\cdot)$ have the same elliptic constants. When ϵ_0 and δ_0 are small enough, (3.1) and (3.2) may satisfy the assumptions in Lemma 3.1 as well as conditions (3.5) and (3.6) in Lemma 3.2. The value of ϵ_0 and δ_0 will be decided as in Remark 3.3.

Lemma 3.1. *Let $u \in S(f)$ in Ω_2 and $u|_{\partial\Omega_1} = \phi$. Assume*

$$B + \beta x_n \leqslant u \leqslant A + \alpha x_n \quad \text{in } \Omega_1,$$

and $|A - B|, |\alpha - \beta|, \|f\|_{n,\Omega_1} \leqslant 1$, $B \leqslant 0 \leqslant A$, $\partial\Omega_1 \subset T_1 \times (-l, k)$ for $l, k \leqslant \frac{1}{8}$.

Then there exist constants $C, A_1, B_1, \alpha_1, \beta_1$ and ϵ, $C > 0$, $\epsilon \in (0, 1)$, such that

$$\begin{cases} B_1 + \beta_1 x_n \leqslant u \leqslant A_1 + \alpha_1 x_n \quad \text{in } \Omega_{\frac{1}{4}}, \\ B_1 \leqslant 0 \leqslant A_1, \\ \alpha_1, \beta_1 \leqslant \alpha - \epsilon|\alpha - \beta| + C|A - B| + C\|f\|_{n,\Omega_1}, \\ \alpha_1, \beta_1 \geqslant \beta + \epsilon|\alpha - \beta| - C|A - B| - C\|f\|_{n,\Omega_1}, \\ |B_1 - A_1| \leqslant \operatorname{osc}_{\partial\Omega_1} \phi + C\|f\|_{n,\Omega_1} + C(l + k), \\ |\alpha_1 - \beta_1| \leqslant \epsilon|\alpha - \beta| + C|A - B| + C\|f\|_{n,\Omega_1}, \end{cases} \tag{3.3}$$

where C, ϵ depend only on λ, Λ and n.

Proof. Without loss of generality, we may suppose $B \leqslant \inf \phi \leqslant 0 \leqslant \sup \phi \leqslant A$ and $\beta = 0$. We also suppose that

$$(u - B)\left(0, \frac{1}{2}\right) \geqslant \frac{1}{2}(A - B) + \frac{\alpha}{4}, \tag{3.4}$$

otherwise we can replace u by $A + \alpha x_n - u$, then the argument would be similar to the case above with small variation.

From (3.4), and applying Harnack inequality to the function $u - B$, we obtain

$$u - B \geqslant C_1\left[\frac{1}{2}(A - B) + \frac{\alpha}{4}\right] - C_2\|f\|_{n,\Omega_1},$$

in $T_{\frac{3}{4}} \times (\frac{1}{4} + k, \frac{3}{4} + l)$.

For any $x_0' \in \mathbb{R}^{n-1}$ satisfying $|x_0'| < \frac{1}{4}$, let

$$v = u - B - \Big(\inf_{\partial\Omega_1} \phi - B\Big) P_{(x_0', -\frac{1}{4} - l), \frac{1}{2}} - \left\{C_1\left[\frac{1}{2}(A - B) + \frac{\alpha}{4}\right] - C_2\|f\|_{n,\Omega_1}\right\} P_{(x_0', \frac{1}{2} + k), \frac{1}{2}}.$$

We may assume that

$$C_1\left[\frac{1}{2}(A-B)+\frac{\alpha}{2}\right]-C_2\|f\|_{n,\Omega_1}\geqslant 0,$$

otherwise (3.3) is obvious by choosing $A_1=A$, $B_1=B$, $\alpha_1=\alpha$, $\beta_1=\beta=0$ and $\epsilon=\frac{1}{2}$.

Now, let us restrict ourself to the domain

$$\widetilde{\Omega}=\left(B_{\frac{1}{2}}\left(x_0',\frac{1}{2}+k\right)\cup B_{\frac{1}{2}}\left(x_0',-\frac{1}{4}-l\right)-B_{\frac{1}{4}}\left(x_0',\frac{1}{2}+k\right)\right)\cap\Omega_1.$$

We have that $v\in\overline{S}(f)$ in $\widetilde{\Omega}$ and $v|_{\partial\widetilde{\Omega}}\geqslant 0$.

By the A-B-P estimate,

$$v\geqslant -C_3\|f\|_{L^n(\Omega_1)}.$$

Using the properties of P on the line $\{x_0'\}\times(-l,\frac{1}{4})$, we obtain

$$\begin{aligned}
u&\geqslant B-C_3\|f\|_{L^n(\Omega_1)}+\Big(\inf_{\partial\Omega_1}\phi-B\Big)P_{(x_0',-\frac{1}{4}-l),\frac{1}{2}}\\
&\quad+\left(C_1\left[\frac{1}{2}(A-B)+\frac{\alpha}{4}\right]-C_2\|f\|_{n,\Omega_1}\right)P_{(x_0',\frac{1}{2}+k),\frac{1}{2}}\\
&\geqslant B-C_3\|f\|_{L^n(\Omega_1)}+\Big(\inf_{\partial\Omega_1}\phi-B\Big)\big(1-C_4(x_n+l)\big)\\
&\quad+C_5\left(C_1\left[\frac{1}{2}(A-B)+\frac{\alpha}{4}\right]-C_2\|f\|_{n,\Omega_1}\right)(x_n-k)\\
&=\left(\frac{C_5C_1}{4}\alpha+\frac{C_5C_1}{2}(A-B)-C_4\Big(\inf_{\partial\Omega_1}\phi-B\Big)-C_5C_2\|f\|_{n,\Omega_1}\right)x_n\\
&\quad+\inf_{\partial\Omega_1}\phi-C_3\|f\|_{n,\Omega_1}+C_4\Big(\inf_{\partial\Omega_1}\phi-B\Big)l-C_5\left(C_1\left[\frac{1}{2}(A-B)+\frac{\alpha}{4}\right]-C_2\|f\|_{n,\Omega_1}\right)k\\
&\geqslant\left(\frac{C_5C_1}{4}\alpha-C_4|A-B|-C_5C_2\|f\|_{n,\Omega_1}\right)x_n\\
&\quad+\inf_{\partial\Omega_1}\phi-C_3\|f\|_{n,\Omega_1}-C_4|A-B|(k+l)-C_5\left(C_1\left[\frac{1}{2}(A-B)+\frac{\alpha}{4}\right]-C_2\|f\|_{n,\Omega_1}\right)(k+l).
\end{aligned}$$

We may assume C_5 is a small positive number such that $0<C_6:=\frac{1}{4}C_5C_1<1$. Noticing x_0' is arbitrary in $T_{\frac{1}{4}}$, hence

$$B_1+\beta_1x_n\leqslant u\leqslant A_1+\alpha_1x_n,$$

where

$$\begin{cases}B_1=\inf\limits_{\partial\Omega_1}\phi-C_3\|f\|_{n,\Omega_1}-C_7(k+l), & A_1=A,\\ \beta_1=C_6\alpha-C_4|A-B|-C_8\|f\|_{n,\Omega_1}, & \alpha_1=\alpha,\end{cases}$$

with $C_7=C_4(A-B)+C_5(C_1[\frac{1}{2}(A-B)+\frac{\alpha}{4}]-C_2\|f\|_{n,\Omega_1})$ and $C_8=C_5C_2$. Now let $\epsilon=1-C_6$, $C=\max\{C_3,C_4,C_6,C_7,C_8\}$, then (3.3) is satisfied.

As we mentioned at the beginning of this proof, if (3.4) is not satisfied, then similarly we have

$$B_1 + \beta_1 x_n \leqslant u \leqslant A_1 + \alpha_1 x_n,$$

where

$$\begin{cases} A_1 = \sup\limits_{\partial\Omega_1} \phi + C_9\|f\|_{n,\Omega_1} + C_{10}(k+l), & B_1 = B, \\ \alpha_1 = \alpha - C_{11}\alpha + C_{12}|A-B| + C_{13}\|f\|_{n,\Omega_1}, & \beta_1 = \beta = 0. \end{cases}$$

Then (3.3) is also satisfied. $\square$

Lemma 3.2. *Assume $u \in S(f)$. For any $k = 0, 1, 2, 3, \ldots$, there are constants A_k, B_k, α_k and β_k such that if*

$$\frac{\operatorname{osc}_{\partial\Omega_{\theta^l}} \phi}{\theta^l} + C\|f\|_{n,\Omega_{\theta^l}} + C\frac{\operatorname{osc} \partial\Omega_{\theta^l}}{\theta^l} \leqslant 1, \quad 0 \leqslant l < k, \tag{3.5}$$

and

$$|\alpha_l - \beta_l| \leqslant 1, \quad 0 \leqslant l < k, \tag{3.6}$$

are satisfied, then we have

$$\theta^k B_k + \beta_k x_n \leqslant u \leqslant \theta^k A_k + \alpha_k x_n, \quad \text{in } \Omega_{\theta^k},$$

and

$$\begin{cases} B_k \leqslant 0 \leqslant A_k, \\ \alpha_k, \beta_k \leqslant \alpha_{k-1} - \epsilon|\alpha_{k-1} - \beta_{k-1}| + C|A_{k-1} - B_{k-1}| + C\|f\|_{n,\Omega_{\theta^{k-1}}}, \\ \alpha_k, \beta_k \geqslant \beta_{k-1} + \epsilon|\alpha_{k-1} - \beta_{k-1}| - C|A_{k-1} - B_{k-1}| - C\|f\|_{n,\Omega_{\theta^{k-1}}}, \\ |A_k - B_k| \leqslant \dfrac{\operatorname{osc}_{\partial\Omega_{\theta^{k-1}}} \phi}{\theta^{k-1}} + C\|f\|_{n,\Omega_{\theta^{k-1}}} + C\dfrac{\operatorname{osc} \partial\Omega_{\theta^{k-1}}}{\theta^{k-1}}, \\ |\alpha_k - \beta_k| \leqslant \epsilon|\alpha_{k-1} - \beta_{k-1}| + C|A_{k-1} - B_{k-1}| + C\|f\|_{n,\Omega_{\theta^{k-1}}}. \end{cases} \tag{3.7}$$

Actually θ can be $\frac{1}{4}$. C and ϵ are the same constants as in Lemma 3.1.

Proof. We prove this lemma by induction.

Case $k = 0$, by taking $B_0 = \inf_{\Omega_1} u$, $A_0 = \sup_{\Omega_1} u$, $\alpha_0 = \beta_0 = 0$, is obviously true.

Case $k = 1$ is derived by Lemma 3.1 directly.

Suppose it is right for case k. Now we show that case $k+1$ is true. Let $v(x) = \frac{u(\theta^k x)}{\theta^k}$, then $v \in S(\tilde{f})$ in $\widetilde{\Omega}_1$ for

$$\tilde{f} = \theta^k f(\theta^k x), \qquad \tilde{\phi}(x) = \frac{\phi(\theta^k x)}{\theta^k}, \qquad \widetilde{\Omega}_1 = \frac{\Omega_{\theta^k}}{\theta^k},$$

where $\frac{\mathfrak{D}}{\theta^k} = \{x, \theta^k x \in \mathfrak{D}\}$, $\mathfrak{D}$ is a domain.

Since we have (3.5), (3.6) and $\|\tilde{f}\|_{n,\widetilde{\Omega}_1} \leqslant 1$, and use Lemma 3.1 again for v, then we derive

$$\begin{aligned}\beta_{k+1}x_n + B_{n+1} \leqslant v \leqslant \alpha_{k+1}x_n + A_{k+1} \quad \text{in } \widetilde{\Omega}_\theta,\\ |B_{k+1} - A_{k+1}| \leqslant \operatorname{osc}_{\partial\widetilde{\Omega}_1} \tilde{\phi} + C\|\tilde{f}\|_{n,\widetilde{\Omega}_1} + C \operatorname{osc} \partial\widetilde{\Omega}_1\\ \leqslant \frac{\operatorname{osc}_{\partial\Omega_{\theta^k}} \phi}{\theta^k} + C\|f\|_{n,\Omega_{\theta^k}} + C\frac{\operatorname{osc} \partial\Omega_{\theta^k}}{\theta^k},\end{aligned}$$

and

$$|\beta_{k+1} - \alpha_{k+1}| \leqslant \epsilon|\alpha_k - \beta_k| + C|A_k - B_k| + C\|f\|_{n,\Omega_{\theta^k}}.$$

Of course we also have the other three inequalities of (3.7). Scaling back, we obtain that case $k+1$ is true. □

Remark 3.3. The conditions (3.5) and (3.6) can be satisfied by normalization. For (3.5), we have $\frac{\operatorname{osc}_{\partial\Omega_{\theta^l}} \phi}{\theta^l} + C\|f\|_{n,\Omega_{\theta^l}} + C\frac{\operatorname{osc} \partial\Omega_{\theta^l}}{\theta^l} \leqslant C\omega(\theta^k)$. For (3.6), observing (3.8), we know

$$|\alpha_l - \beta_l| \leqslant \epsilon^l|\alpha_0 - \beta_0| + C\sum_{i=0}^{l-1}\epsilon^i\big(|A_{l-1-i} - B_{l-1-i}| + \|f\|_{n,\Omega_{\theta^{l-1-i}}}\big) \leqslant \epsilon^l|\alpha_0 - \beta_0| + \frac{C}{1-\epsilon}\omega(1).$$

Consequently $|\alpha_0 - \beta_0| \leqslant \frac{1}{2}$ and $\omega(1) \leqslant \frac{1-\epsilon}{2C}$ derive (3.5) and (3.6). Since $\partial\Omega$ is $C^{1,\omega}$ at 0, let δ_0 be small enough to make $\operatorname{osc} \partial\Omega_{\delta_0} \leqslant \frac{1-\epsilon}{2C}$ in the normalization (3.1) and (3.2), then $\omega(1) \leqslant \frac{1-\epsilon}{2C}$ is satisfied if we choose $\epsilon_0 \leqslant \frac{1-\epsilon}{2C}$.

If we calculate the last two inequalities of (3.7), then we have the following lemma which will be used in the proof of Theorem 1.7.

Lemma 3.4. *If positive functions $h(x)$ and $g(x)$ satisfy $h(\theta^{k+1}) \leqslant \epsilon h(\theta^k) + g(\theta^k)$, $k = 0, 1, 2, 3, \ldots,$ where $0 \leqslant \theta, \epsilon < 1$, then*

$$\sum_{i=k}^{\infty} h(\theta^i) \leqslant \frac{\epsilon^k}{1-\epsilon}h(1) + \sum_{i=0}^{\infty}\frac{\epsilon^{(k-1-i)^+}}{1-\epsilon}g(\theta^i),$$

where $(s)^+ = \begin{cases} s, & s \geqslant 0,\\ 0, & s < 0.\end{cases}$

Proof. We compute

$$\begin{aligned} h(\theta^k) &\leqslant \epsilon h(\theta^{k-1}) + g(\theta^{k-1})\\ &\leqslant \epsilon^2 h(\theta^{k-2}) + \epsilon g(\theta^{k-2}) + g(\theta^{k-1})\\ &\cdots\\ &\leqslant \epsilon^k h(1) + \sum_{i=0}^{k-i}\epsilon^i g(\theta^{k-1-i}), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\sum_{i=k}^{\infty} h(\theta^i) &\leqslant \sum_{i=k}^{\infty}\big(\epsilon h(\theta^{j-1}) + g(\theta^{i-1})\big)\\ &\leqslant \sum_{i=k}^{\infty}\left(\epsilon^j h(1) + \sum_{i=k}^{\infty}\sum_{j=0}^{i-1}\epsilon^j g(\theta^{k-1-j})\right)\\ &= \frac{\epsilon^k}{1-\epsilon}h(1) + \sum_{i=0}^{\infty}\frac{\epsilon^{(k-1-i)^+}}{1-\epsilon}g(\theta^i). \qquad \square\end{aligned}$$

We also need an elementary lemma for linear functions.

Lemma 3.5. *If $L(\cdot)$ is linear function, then*

$$\begin{cases} \big|L(\cdot)\big|_{L^\infty(B_{\theta^k})} \leqslant \dfrac{1}{\theta}\big|L(\cdot)\big|_{L^\infty(B_{\theta^{k+1}})}, & 0<\theta<1,\ k=0,1,2,\dots,\\ \big|\nabla L(\cdot)\big| \leqslant \dfrac{1}{r}\big|L(\cdot)\big|_{L^\infty(B_r)}, & r>0, \end{cases}$$

where $|\cdot|$ is standard norm. We skip the proof of this lemma.

Now we prove the main theorem of this paper.

Proof of Theorem 1.7. From Lemma 3.2, we know that there exists a linear function $L_{\theta^k}(x)$ s.t.

$$|u - L_{\theta^k}|_{L^\infty(\Omega_{\theta^k})} \leqslant |A_k - B_k|\theta^k + |\alpha_k - \beta_k|\theta^k. \tag{3.9}$$

We need the convergence of L_{θ^k}. From (3.9) and Lemma 3.5, we have

$$\begin{aligned}|L_{\theta^k} - L_{\theta^{k+1}}|_{L^\infty(\Omega_{\theta^k})} &\leqslant \frac{1}{\theta}|L_{\theta^k} - L_{\theta^{k+1}}|_{L^\infty(\Omega_{\theta^{k+1}})}\\ &\leqslant \frac{1}{\theta}\big(|u - L_{\theta^k}|_{L^\infty(\Omega_{\theta^{k+1}})} + |u - L_{\theta^{k+1}}|_{L^\infty(\Omega_{\theta^{k+1}})}\big)\\ &\leqslant \frac{1}{\theta}\big(|u - L_{\theta^k}|_{L^\infty(\Omega_{\theta^k})} + |u - L_{\theta^{k+1}}|_{L^\infty(\Omega_{\theta^{k+1}})}\big)\\ &\leqslant \frac{1}{\theta}\big(\big(|A_k - B_k| + |\alpha_k - \beta_k|\big)\theta^k + \big(|A_{k+1} - B_{k+1}| + |\alpha_{k+1} - \beta_{k+1}|\big)\theta^{k+1}\big)\\ &\leqslant \theta^{k-1}\big(|A_k - B_k| + |\alpha_k - \beta_k| + |A_{k+1} - B_{k+1}| + |\alpha_{k+1} - \beta_{k+1}|\big).\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}|L_{\theta^{k+i}} - L_{\theta^{k+i+1}}|_{L^\infty(\Omega_{\theta^k})} &\leqslant \frac{1}{\theta^{i+1}}|L_{\theta^{k+i}} - L_{\theta^{k+i+1}}|_{L^\infty(\Omega_{\theta^{k+i+1}})}\\ &\leqslant \theta^{k-1}\big(|A_{k+i} - B_{k+i}| + |\alpha_{k+i} - \beta_{k+i}|\\ &\quad + |A_{k+i+1} - B_{k+i+1}| + |\alpha_{k+i+1} - \beta_{k+i+1}|\big).\end{aligned}$$

Thus, using (3.7) and assumption, we know that the limit of L_{θ^k}, $k \to \infty$, exists. Let L_0 denote the limit of L_{θ^k}, $k \to \infty$.

From Lemma 3.4, we have

$$\frac{1}{\theta^k}|L_{\theta^k}-L_0|_{L^\infty(\Omega_{\theta^k})} \leqslant \sum_{i=k}^{\infty} C\big(|A_k-B_k|+|\alpha_k-\beta_k|\big)$$
$$=\sum_{i=k}^{\infty} C|A_i-B_i| + \frac{C\epsilon^k}{1-\epsilon}|\alpha_0-\beta_0| + \sum_{i=0}^{\infty}\frac{C\epsilon^{(k-1-i)^+}}{1-\epsilon}\big(C|A_i-B_i|+C\|f\|_{n,\Omega_{\theta^i}}\big).$$

Among these terms,

$$\frac{\epsilon^k}{1-\epsilon}|\alpha_0-\beta_0| \leqslant \frac{(\theta^k)^{\log_\theta \epsilon}}{1-\epsilon}.$$

Noticing

$$\begin{cases} \dfrac{\operatorname{osc}_{\partial\Omega_{\theta^i}}\phi}{\theta^i} \leqslant \dfrac{1}{\theta}\dfrac{\operatorname{osc}_{\partial\Omega_{\theta^{i-s}}}\phi}{\theta^{i-s}}, \\ \dfrac{\operatorname{osc}\partial\Omega_{\theta^i}}{\theta^i} \leqslant \dfrac{1}{\theta}\dfrac{\operatorname{osc}\partial\Omega_{\theta^{i-s}}}{\theta^{i-s}}, \\ \|f\|_{n,\Omega_{\theta^i}} \leqslant \dfrac{1}{\theta}\|f\|_{n,\Omega_{\theta^{i-s}}}, \quad 0\leqslant s\leqslant i, \end{cases}$$

we obtain

$$\sum_{i=k}^{\infty}|A_i-B_i| \leqslant C\sum_{i=k-1}^{\infty}\left(\frac{\operatorname{osc}_{\partial\Omega_{\theta^i}}\phi}{\theta^i}+\frac{\operatorname{osc}\partial\Omega_{\theta^i}}{\theta^i}+\|f\|_{n,\Omega_{\theta^i}}\right)$$
$$\leqslant \frac{C}{\theta}\sum_{i=k-2}^{\infty}\int_i^{i+1}\left(\frac{\operatorname{osc}_{\partial\Omega_{\theta^t}}\phi}{\theta^t}+\frac{\operatorname{osc}\partial\Omega_{\theta^t}}{\theta^t}+\|f\|_{n,\Omega_{\theta^t}}\right)dt$$
$$\leqslant \frac{C}{\theta}\int_0^{\theta^{k-2}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)\frac{ds}{s},$$

and

$$\sum_{i=0}^{k-1}\epsilon^{k-1-i}|A_i-B_i| \leqslant C\sum_{i=2}^{k-1}\epsilon^{k-1-i}\left(\frac{\operatorname{osc}_{\Omega_{\theta^{i-1}}}\phi}{\theta^{i-1}}+\frac{\operatorname{osc}\partial\Omega_{\theta^{i-1}}}{\theta^{i-1}}+\|f\|_{n,\Omega_{\theta^{i-1}}}\right)$$
$$=C\sum_{i=1}^{k-2}\epsilon^{k-2-i}\left(\frac{\operatorname{osc}_{\partial\Omega_{\theta^i}}\phi}{\theta^i}+\frac{\operatorname{osc}\partial\Omega_{\theta^i}}{\theta^i}+\|f\|_{n,\Omega_{\theta^i}}\right)$$
$$+\epsilon^{k-1}|A_0-B_0|+\epsilon^{k-2}\big(\operatorname{osc}_{\partial\Omega_1}\phi+\operatorname{osc}\partial\Omega_1+\|f\|_{n,\Omega_1}\big)$$
$$\leqslant C\sum_{i=0}^{k-3}\epsilon^{k-3-i}\int_i^{i+1}\frac{1}{\theta}\left(\frac{\operatorname{osc}_{\partial\Omega_{\theta^t}}\phi}{\theta^t}+\frac{\operatorname{osc}\partial\Omega_{\theta^t}}{\theta^t}+\|f\|_{n,\Omega_{\theta^t}}\right)dt$$
$$+\epsilon^{k-1}|A_0-B_0|+\epsilon^{k-2}\big(\operatorname{osc}_{\partial\Omega_1}\phi+\operatorname{osc}\partial\Omega_1+\|f\|_{n,\Omega_1}\big)$$

$$
\begin{aligned}
&\leqslant C\sum_{i=0}^{k-3}\epsilon^{k-3}\int_i^{i+1}\frac{1}{\theta\epsilon^t}\left(\frac{\operatorname{osc}_{\partial\Omega_{\theta^t}}\phi}{\theta^t}+\frac{\operatorname{osc}\partial\Omega_{\theta^t}}{\theta^t}+\|f\|_{n,\Omega_{\theta^t}}\right)dt\\
&\quad+\epsilon^{k-1}|A_0-B_0|+\epsilon^{k-2}\left(\operatorname{osc}_{\partial\Omega_1}\phi+\operatorname{osc}\partial\Omega_1+\|f\|_{n,\Omega_1}\right)\\
&=\frac{C}{\theta\epsilon}\left(\theta^{k-2}\right)^{\log_\theta\epsilon}\int_0^{k-2}\frac{1}{(\theta^t)^{\log_\theta\epsilon}}\left(\frac{\operatorname{osc}_{\partial\Omega_{\theta^t}}\phi}{\theta^t}+\frac{\operatorname{osc}\partial\Omega_{\theta^t}}{\theta^t}+\|f\|_{n,\Omega_{\theta^t}}\right)dt\\
&\quad+\epsilon^{k-1}|A_0-B_0|+\epsilon^{k-2}\left(\operatorname{osc}_{\partial\Omega_1}\phi+\operatorname{osc}\partial\Omega_1+\|f\|_{n,\Omega_1}\right)\\
&=\frac{C}{\theta\epsilon}\left(\theta^{k-2}\right)^{\log_\theta\epsilon}\int_{\theta^{k-2}}^{1}\frac{1}{s^{1+\log_\theta\epsilon}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)ds\\
&\quad+\epsilon^{k-1}|A_0-B_0|+\epsilon^{k-2}\left(\operatorname{osc}_{\partial\Omega_1}\phi+\operatorname{osc}\partial\Omega_1+\|f\|_{n,\Omega_1}\right).
\end{aligned}
$$

Then it's easy to conclude

$$
\begin{aligned}
|L_{\theta^k}-L_0|_{L^\infty(\Omega_{\theta^k})}&\leqslant C\theta^k\Bigg(\left(\theta^k\right)^{\log_\theta\epsilon}+\int_0^{\theta^{k-2}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)\frac{ds}{s}\\
&\quad+\left(\theta^k\right)^{\log_\theta\epsilon}\int_{\theta^{k-2}}^{1}\frac{1}{s^{1+\log_\theta\epsilon}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)ds\Bigg)\\
&\leqslant C\theta^k\Bigg(\left(\theta^k\right)^{\log_\theta\epsilon}+\int_0^{\theta^{k}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)\frac{ds}{s}\\
&\quad+\left(\theta^k\right)^{\log_\theta\epsilon}\int_{\theta^{k}}^{1}\frac{1}{s^{1+\log_\theta\epsilon}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)ds\Bigg).
\end{aligned}
$$

By the triangle inequality

$$
|u-L_0|_{L^\infty(\Omega_{\theta^k})}\leqslant|u-L_{\theta^k}|_{L^\infty(\Omega_{\theta^k})}+|L_{\theta^k}-L_0|_{L^\infty(\Omega_{\theta^k})},
$$

and noticing $|A_k-B_k|+|\alpha_k-\beta_k|\leqslant\sum_{i=k}^{\infty}(|A_i-B_i|+|\alpha_k-\beta_k|)$, we have

$$
\begin{aligned}
|u-L_0|_{L^\infty(\Omega_{\theta^k})}&\leqslant C\theta^k\Bigg(\left(\theta^k\right)^{\log_\theta\epsilon}+\int_0^{\theta^{k}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)\frac{ds}{s}\\
&\quad+\left(\theta^k\right)^{\log_\theta\epsilon}\int_{\theta^{k}}^{1}\frac{1}{s^{1+\log_\theta\epsilon}}\left(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s}+\frac{\operatorname{osc}\partial\Omega_s}{s}+\|f\|_{n,\Omega_s}\right)ds\Bigg).
\end{aligned}
$$

Let $\hat{\alpha}:=\log_\theta\epsilon$. Since $0<\theta,\ \epsilon<1$, we have $\hat{\alpha}>0$. By a standard process, for $0<r<1$, we have

$$|u - L_0|_{L^\infty(\Omega_r)} \leqslant Cr\Biggl(r^{\hat\alpha} + \int_0^r \biggl(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s} + \frac{\operatorname{osc}\partial\Omega_s}{s} + \|f\|_{n,\Omega_s}\biggr)\frac{ds}{s}$$
$$+ r^{\hat\alpha}\int_r^1 \frac{1}{s^{1+\hat\alpha}}\biggl(\frac{\operatorname{osc}_{\partial\Omega_s}\phi}{s} + \frac{\operatorname{osc}\partial\Omega_s}{s} + \|f\|_{n,\Omega_s}\biggr)ds\Biggr),$$

where C only depends on $n, \lambda, \Lambda, \omega$ and $\|u\|_{C^0(\Omega_1)}$. The proof is finished. □

Remark 3.6. It's easy to verify that $r^{\hat\alpha}\int_r^1 \frac{\omega(s)}{s^{1+\hat\alpha}}ds$ is a modulus of continuity if ω is a modulus of continuity. The corollary is just the special case of the theorem when $\omega = C_0 r^\alpha$. And in this case, the constant C in Corollary 1.8 and Corollary 1.10 depends on C_0 instead of ω.

Proof of Theorem 1.9. We just consider $|\nabla u(y) - \nabla u(z)|$, where $y, z \in \partial\Omega$ and $|y - z| = \theta^k$. From Lemma 3.1 we have

$$\begin{aligned} \bigl|u - L^y_{\theta^k}\bigr|_{L^\infty(\Omega_{\theta^k}(y))} &\leqslant \bigl|A^y_k - B^y_k\bigr|\theta^k + \bigl|\alpha^y_k - \beta^y_k\bigr|\theta^k, \\ \bigl|u - L^z_{\theta^k}\bigr|_{L^\infty(\Omega_{\theta^k}(z))} &\leqslant \bigl|A^z_k - B^z_k\bigr|\theta^k + \bigl|\alpha^z_k - \beta^z_k\bigr|\theta^k, \end{aligned} \tag{3.10}$$

where $L^y_{\theta^k}(x), L^z_{\theta^k}(x)$ are linear functions of $x \in \overline{\Omega}$. Noticing $\partial\Omega$ is $C^{1,\omega}$ and our normalization makes ω small, then there exists a point $q \in \Omega$ such that $B_{\frac{\theta^k}{4}}(q) \subset B_{\theta^k}(y) \cap B_{\theta^k}(z) \cap \Omega$. Using the argument above, we obtain

$$\bigl|L^y_{\theta^k}(x) - L^z_{\theta^k}(x)\bigr|_{L^\infty(B_{\frac{\theta^k}{4}}(z))} \leqslant C\bigl(\bigl|A^y_k - B^y_k\bigr|\theta^k + \bigl|\alpha^y_k - \beta^y_k\bigr|\theta^k + \bigl|A^z_k - B^z_k\bigr|\theta^k + \bigl|\alpha^z_k - \beta^z_k\bigr|\theta^k\bigr).$$

Consequently by Lemma 3.5, we have

$$\begin{aligned} \bigl|\nabla L^y_{\theta^k}(x) - \nabla L^z_{\theta^k}(x)\bigr| &\leqslant \frac{1}{\theta^k}\bigl|L^y_{\theta^k}(x) - L^y_{\theta^k}(x)\bigr|_{L^\infty(B_{\frac{\theta^k}{4}}(q))} \\ &\leqslant C\bigl(\bigl|A^y_k - B^y_k\bigr| + \bigl|\alpha^y_k - \beta^y_k\bigr| + \bigl|A^z_k - B^z_k\bigr| + \bigl|\alpha^z_k - \beta^z_k\bigr|\bigr). \end{aligned}$$

Hence

$$\begin{aligned} \bigl|\nabla u(y) - \nabla u(z)\bigr| &= \bigl|\nabla L^y_0 - \nabla L^z_0\bigr| \\ &\leqslant \bigl|\nabla L^y_0 - \nabla L^y_{\theta^k}\bigr| + \bigl|\nabla L^z_0 - \nabla L^z_{\theta^k}\bigr| + \bigl|\nabla L^y_{\theta^k} - \nabla L^z_{\theta^k}\bigr| \\ &\leqslant \frac{1}{\theta^k}\bigl(\bigl|L^y_0 - L^y_{\theta^k}\bigr|_{L^\infty(\Omega\cap B_{\theta^k}(y))} + \bigl|L^z_0 - L^z_{\theta^k}\bigr|_{L^\infty(\Omega\cap B_{\theta^k}(z))}\bigr) + \bigl|\nabla L^y_{\theta^k} - \nabla L^z_{\theta^k}\bigr|. \end{aligned}$$

By the proof of Theorem 1.7 and boundary condition, we have

$$\bigl|\nabla u(y) - \nabla u(z)\bigr| \leqslant C\Biggl(r^{\hat\alpha} + \int_0^r \frac{\omega(s)}{s}ds + r^{\hat\alpha}\int_r^1 \frac{\omega(s)}{s^{1+\hat\alpha}}ds\Biggr), \quad |y - z| \leqslant r \leqslant 1,$$

where C only depends on $n, \lambda, \Lambda, \omega$, $\|u\|_{C^0(\Omega_1)}$ and $\|f\|_{n,\Omega_1}$. This completes our proof. □

Acknowledgment

We would like to thank the referee for helpful suggestions.

References

[1] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. 130 (1989) 189–213.
[2] L.A. Caffarelli, X. Cabre, Fully Nonlinear Elliptic Equations, Amer. Math. Soc., Providence, RI, 1995.
[3] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1983.
[4] J. Kovats, Fully nonlinear elliptic equations and the Dini condition, Comm. Partial Differential Equations 22 (1997) 1911–1927.
[5] J. Kovats, Dini–Campanato spaces and applications to nonlinear elliptic equations, Electron. J. Differential Equations 37 (1999) 1–20.
[6] D. Li, L. Wang, Boundary differentiability of solutions of elliptic equations on convex domains, Manuscripta Math. 121 (2006) 137–156.
[7] D. Li, L. Wang, Elliptic equations on convex domains with nonhomogeneous Dirichlet boundary conditions, J. Differential Equations 246 (2009) 1723–1743.
[8] L. Wang, On the regularity theory of fully nonlinear parabolic equations. I, Comm. Pure Appl. Math. 45 (1992) 27–76.
[9] L. Wang, On the regularity theory of fully nonlinear parabolic equations. II, Comm. Pure Appl. Math. 45 (1992) 141–178.
[10] X. Zou, Y.Z. Chen, Fully nonlinear parabolic equations and the Dini condition, Acta Math. Sin. 18 (2002) 473–480.
[11] X. Zou, Y.Z. Chen, Boundary regularity for elliptic equations and the Dini condition, Acta Math. Sinica 45 (2002) 701–710 (in Chinese).