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Some properties of the Alday–Maldacena minimum

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Abstract

The Alday–Maldacena solution, relevant to the $n = 4$ gluon amplitude in $N = 4$ SYM at strong coupling, was recently identified as a minimum of the regularized action in the moduli space of solutions of the AdS_5 σ-model equations of motion. Analogous solutions of the Nambu–Goto equations for the $n = 4$ case are presented and shown to form (modulo the reparametrization group) an equally large but different moduli space, with the Alday–Maldacena solution at the intersection of the σ-model and Nambu–Goto moduli spaces. We comment upon the possible form of the regularized action for $n = 5$. A function of moduli parameters z_a is written, whose minimum reproduces the BDDK one-loop five-gluon amplitude. This function may thus be considered as some kind of Legendre transform of the BDDK formula and has its own value independently of the Alday–Maldacena approach.

1. Introduction and conclusions

An ϵ-regularized minimal action in AdS_5 σ-model was defined recently [1], and shown to reproduce the external momentum dependence of the BDS formula [2] for the $n = 4$-gluon amplitude in $N = 4$ Super-Yang–Mills (SYM) theory. In [3–20] one may find generalizations and discussions of this important result. In a previous paper [12] it was demonstrated that the Alday–Maldacena solution is just one member of a large family of solutions; a rather distinct one, though, since it corresponds to a minimum of the classical σ-model action in the moduli space $\mathcal{M}_n^\sigma$ of all solutions in $d = 4$ dimensions (i.e. for $\epsilon = 0$). Throughout this Letter we shall use the notation and results of [12], to which we refer the reader. We shall keep the parameter n explicit in various formulas and symbols, even though, as it will be clear in the text, many of the statements will refer specifically to the cases $n = 4$ or 5.

Let us recall that in $d = 4$ dimensions and for $n = 4$ the moduli space of solutions constructed in [12] was parametrized by $\{z_a, \mathbf{v}_1, \phi\}$ with $a = 1, \ldots, n$ enumerating the sides of the auxiliary polygon Π, Fig. 1, formed by the null 4-momenta $\mathbf{p}_a$ of the external gluons and lying at the boundary of AdS_5 at $z = \infty$.

It is possible, that these are all the solutions with the particular boundary conditions corresponding to the above process. In [12] they were obtained under the assumption (ansatz) that the Lagrangian $L_\sigma = \text{const} = 2$. In any case, in what follows we shall use $\mathcal{M}_n^\sigma$ to denote *this* part of the moduli space.

The $SO(4,2)$ symmetry of AdS_5 relates some of these solutions, but it does not act transitively on $\mathcal{M}_n^\sigma$. Specifically, only the z_a moduli are affected by this group. In addition, $\mathbf{v}_1$ is an inessential modulus, since no physical quantity depends on it. Essential moduli are the ratio z_1z_3/z_2z_4 and the angle ϕ. The latter is not affected by $SO(4,2)$, but only by some larger hidden group, related presumably to the integrability of the σ-model. It is important to point out that by definition the Lagrangian density is constant, namely $L_\sigma = 2$, on the entire $\mathcal{M}_n^\sigma$. Thus, the corresponding action integral diverges and needs regularization. The ϵ-regularization

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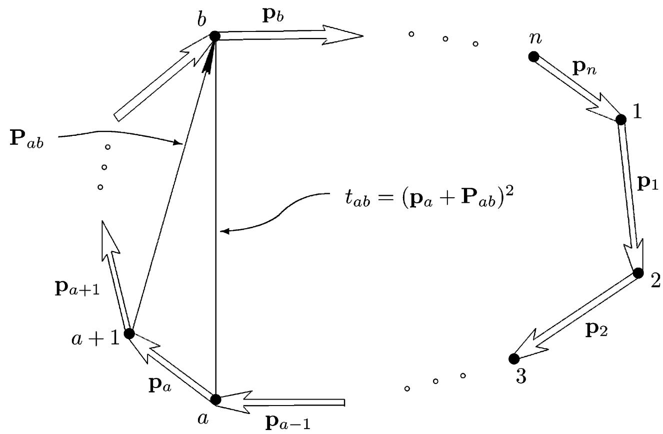

Fig. 1. Auxiliary skew polygon Π, playing a surprisingly important role in the theory of n-point amplitudes: all formulas at the perturbative, as well as the strong-coupling sides of the AdS/CFT correspondence are written in terms of characteristics of Π. Its edges are external gluon 4-momenta $\mathbf{p}_a$, the squares of its diagonals are scattering invariants t_{ab}. Formulas in the text are written in terms of their logarithms, $\tau_{ab} = \log t_{ab}$.

used in [1] breaks not only the integrability, but also the $SO(4,2)$ symmetry, so that the regularized action becomes a non-trivial (z- and ϕ-dependent) function on the moduli space. As shown in [12], the Alday–Maldacena solution is exactly at the minimum of this function. Incidentally, the regularization leaves unbroken the Lorentz subgroup of $SO(4,2)$ (which, however, is partly broken by the boundary conditions) and the two rescalings of z_a that preserve the products $z_1 z_3$ and $z_2 z_4$.

The present Letter is a little further development along the lines of [12]. Our purpose is on the one hand to clarify the difference between the σ-model and Nambu–Goto actions in connection with the above approach, and on the other to attempt a generalization to the five-gluon amplitude.

Specifically, in Section 2, we consider what happens if the σ-model action is replaced by the Nambu–Goto (NG) one—a question raised but left unanswered in Section 4.7 of [12]. We conclude that for $n = 4$ the two moduli spaces are equally large, i.e.

$$\dim\bigl(\mathcal{M}_4^\sigma\bigr) = \dim\bigl(\mathcal{M}_4^{\mathrm{NG}}\bigr) \tag{1}$$

not expected a priori, because of the classical inequivalence of the two actions. We would like to recall here, that the σ-model is being considered without the Virasoro constraints, which would render the two models classically equivalent. As shown, the relevant solutions are parametrized by the same parameters, but they are different in the two models and the corresponding moduli spaces do not coincide. The essential moduli in the σ-model case are the ratio $z_1 z_3 / z_2 z_4$ and ϕ, while in the NG case no essential moduli are made from z_a. Instead, the angle ϕ between the two $\vec{k}$-vectors gets complemented by the ratio of their lengths. This simple description, however, requires careful definition of the manifold $\mathcal{M}_4^{\mathrm{NG}}$. The reason is that, in contrast to the σ-model case, the NG action is invariant under arbitrary reparametrizations of the world sheet and, therefore, the entire space of solutions is infinite dimensional, incomparably larger than that of the σ-model. In such a situation, it is natural to define the moduli space by factoring out the reparametrization group with coordinate transformations vanishing at infinity. Then the moduli space of solutions with a given asymptotic behavior at infinity is finite dimensional and is actually obtained by *linear* transformations of the world-sheet coordinates. Similarly, it is natural to eliminate the $2d$ rotations and displacements, since the $2d$ Poincare invariance is common to the σ-model and NG actions. Next, the ϵ-regularization preserves the $2d$ reparametrization invariance of the NG action, therefore, again in contrast to the σ-model case, the regularized NG action is *constant* on the entire $\mathcal{M}_4^{\mathrm{NG}}$ manifold. The NG valley in the landscape of world-sheet embeddings into AdS_5 is actually flat. It crosses the non-flat σ-model valley exactly at the Alday–Maldacena solution,[1] Fig. 2.

In Section 3, guided by the pictorial representation given in [12] and the results for $n = 4$, we make an attempt to guess the form of the $n > 4$ regularized action $\mathcal{A}_n(z_1, \ldots, z_n; \epsilon)$ on the moduli space $\mathcal{M}_n^\sigma$, parametrized by a conjectured set of parameters z_a with $a = 1, 2, \ldots, n$. In addition, we present an ansatz for the constraint, generalization of its $n = 4$ counterpart, which is argued to be reasonable for $n = 5$. The action is minimized under the constraint and reproduces the BDDK formula [22] for the one-loop 5-gluon amplitude $F_5^{(1)}$, which eventually exponentiates to the BDS formula [2] for the full strong coupling $n = 5$ scattering amplitude. Hopefully, this action will eventually be derived, as in the $n = 4$ case, from exact solutions of the σ-model with subtle growing asymptotics. At this point however, it may just serve as a useful guide through the tedious and not particularly transparent calculations of the regularized integrals in [1] and [12].

It is important to emphasize here, that the finite part $\tilde{\mathcal{A}}_n$ of the n-point action integral will be defined independently of the Alday–Maldacena regularization. Consequently, it may be thought of as a kind of Legendre transform of the BDDK formula

[1] For a small but potentially interesting deviation see [20].

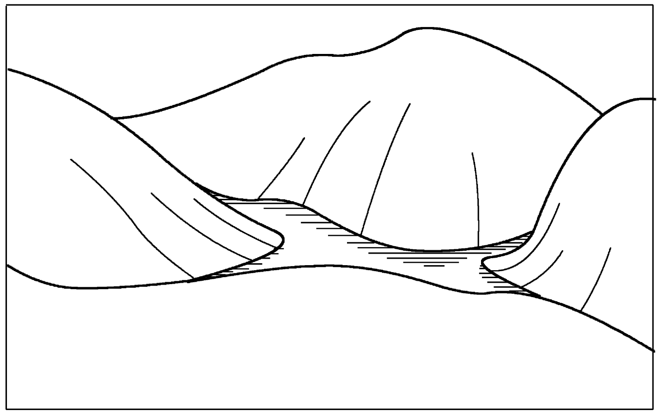

Fig. 2. Symbolic representation of the landscape of world-sheet embeddings into AdS_5 space. The horizontal plane is actually an infinite-dimensional space of mappings $(z(\vec{u}), \mathbf{v}(\vec{u}))$. The "height functionals" on this space are the ϵ-regularized actions. Actually the NG and σ-model "height functions" are different, a fact ignored in this picture. Solutions of the NG and σ-model equations of motion form two valleys in this landscape. The σ-model one is not flat, because the degeneracy is partly broken by the ϵ-regularization. Therefore, there is a minimum in the valley, which coincides with the NG ϵ-regularized action. The Alday–Maldacena solution lies at the crossing of the two valleys.

and in this sense has its own value and significance. Such a function for $n > 5$ would have an advantage, because the $\{z_a\}$ are *independent* variables, while there are many relations between the $n(n-3)/2$ parameters t_{ab}, of which only $3n-10$ are independent. Construction of $\tilde{\mathcal{A}}_n$ for $n > 5$ is a challenging problem beyond the scope of the present Letter.

2. Moduli space of NG solutions for $n = 4$

In this section and in order to make the comparison easier, we shall make a parallel presentation of the solutions of interest in the Nambu–Goto (NG) and the σ-model field equations.

2.1. Solving the NG equations of motion for $n = 4$

As explained in [12] the most relevant variables for the description of the Alday–Maldacena result are $(z, \mathbf{v})$, which are actually five of the six flat coordinates $(Y_-, \mathbf{Y}, Y_+)$, describing the embedding of AdS_5 into $\mathbb{R}^6_{++++--}$. In these variables the equations of motion acquire the simple form

$$\partial_i \bigl(H^{ij} \partial_j z\bigr) = G_{ij} H^{ij} z, \tag{2}$$

$$\partial_i \bigl(H^{ij} \mathbf{V}_j\bigr) = 0 \tag{3}$$

and the difference between the σ-model and NG cases lies in the expression for H, namely, we have

$$H^{ij}_\sigma = \delta^{ij} \tag{4}$$

while

$$H^{ij}_{\mathrm{NG}} = L_{\mathrm{NG}} \bigl(G^{-1}\bigr)^{ij} = \frac{\check{G}^{ij}}{L_{\mathrm{NG}}}. \tag{5}$$

In the above formulas $i, j = 1, 2$ label $2d$ coordinates on the world sheet,

$$G_{ij} = \frac{\partial_i z \partial_j z + \mathbf{V}_i \mathbf{V}_j}{z^2} \tag{6}$$

is the AdS-induced metric on the world sheet, $\check{G}^{ij} = \begin{pmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{pmatrix}$ is made from algebraic complements,

$$\mathbf{V} = z \partial \mathbf{v} - \mathbf{v} \partial z \tag{7}$$

and the two Lagrangian densities are

$$L_\sigma = G^i_i = G_{11} + G_{22} \tag{8}$$

and

$$L_{\mathrm{NG}} = \sqrt{\det G_{ij}} = \sqrt{G_{11}G_{22} - G_{12}^2}, \tag{9}$$

respectively.

In [12] it was suggested to make the ansatz $G_{ij} = \mathrm{const}$ in the differential equations (2) and (3), solve them with appropriate boundary conditions and finally consider the self-consistency of this ansatz as an *algebraic* equation (6) on the parameters of the solution. Many more NG solutions can be produced afterwards by world sheet reparametrizations $u^i \to \tilde{u}^i(\vec{u})$, corresponding to a single point in the moduli space if $\tilde{u}^i = u^i + O(|u|^{-1})$ at large $|\vec{u}|$. For constant G_{ij} both the Lagrangian densities and the coefficient in front of z on the right-hand side of (2) are also constant, in which case the solutions of Eqs. (2), (3) are obviously of the form

$$\begin{aligned} z &= \sum_a z_a e^{\vec{k}_a \vec{u}}, \\ \mathbf{v} &= \sum_a \mathbf{v}_a e^{\vec{k}_a \vec{u}} \end{aligned} \tag{10}$$

where $\vec{u}$ are the world sheet coordinates. The $2d$ vectors $\vec{k}_a$ are constrained in different ways for different actions:

$$\vec{k}_a^2 = \mathrm{Tr}\, G \quad \text{in the } \sigma\text{-model case}, \tag{11}$$

$$\check{G}^{ij} k_i^a k_j^a = 2 \det G \quad \text{in the NG case}. \tag{12}$$

Correspondingly, the parameters $\mathbf{v}_a$ are fixed by the boundary conditions [12],

$$\frac{\mathbf{v}_{a+1}}{z_{a+1}} - \frac{\mathbf{v}_a}{z_a} = \mathbf{p}_a \tag{13}$$

which express them in terms of the external momenta $\mathbf{p}_a$ and $\{z_a\}$. These boundary conditions restrict the number of exponentials in (10) to the number n of sides in the polygon Π: $a = 1, \ldots, n$. One of the $\mathbf{v}$-vectors, say $\mathbf{v}_1$, remains undefined; we called it inessential modulus in the introduction. The essential moduli are $\{z_a\}$ modulo $\vec{u}$ transformations and $\{\vec{k}_a\}$ modulo (11) or (12).

One is left with a set of non-trivial algebraic equations, namely that G_{ij} obtained by substitution of (10) into (6) is constant, i.e. independent of $\vec{u}$:

$$\sum_{a,b=1}^{n} \big(G_{ij} - k_i^a k_j^b\big) z_a z_b E_{a+b} = \sum_{\substack{a<b\\ c<d}} k_i^{ab} k_j^{cd} (\mathcal{P}_{ab}\mathcal{P}_{cd}) z_a z_b z_c z_d E_{a+b+c+d}. \tag{14}$$

Here $E_a = e^{\vec{k}_a \vec{u}}$, $E_{a+b} = E_a E_b$, $\vec{k}^{ab} = \vec{k}^a - \vec{k}^b$, $\mathcal{P}_{ab} = z_a z_b (\mathbf{p}_a + \cdots + \mathbf{p}_{b-1})$, while further details about notation can be found in [12]. In what follows we concentrate on the case of $n = 4$, where this simple ansatz indeed works. Eq. (14) is actually a system of relations for coefficients in front of various exponentials. Many coefficients can be cancelled if we choose $\vec{k}_3 = -\vec{k}_1 = \vec{k}_{-1}$ and $\vec{k}_4 = -\vec{k}_2 = \vec{k}_{-2}$ so that the four $\vec{k}$-vectors form diagonals of a parallelogram. Next, comparison of the coefficients in front of $z_1^2 E_{1+1}$ on both sides of (14) gives:

$$G_{ij} - k_i^1 k_j^1 = \big(k_i^{12} k_j^{14} + k_i^{14} k_j^{12}\big) z_2 z_4 (-\mathbf{p}_1 \mathbf{p}_4) = -\eta^2 \big(k_i^1 k_j^1 - k_i^2 k_j^2\big) \tag{15}$$

where $\eta^2 = z_2 z_4 t_{24}$. Similarly, from the coefficient of $z_2^2 E_{2+2}$ one obtains

$$G_{ij} - k_i^2 k_j^2 = \big(k_i^{12} k_j^{23} + k_i^{23} k_j^{12}\big) z_1 z_3 (\mathbf{p}_1 \mathbf{p}_2) = \xi^2 \big(k_i^1 k_j^1 - k_i^2 k_j^2\big) \tag{16}$$

with $\xi^2 = z_1 z_3 t_{13}$. Together these two equations imply the consistency relation on the parameters z_a,

$$\xi^2 + \eta^2 = z_1 z_3 t_{13} + z_2 z_4 t_{24} = 1 \tag{17}$$

already familiar from [12], and the explicit expression for G_{ij},

$$G_{ij} = \xi^2 k_i^1 k_j^1 + \eta^2 k_i^2 k_j^2. \tag{18}$$

All other relations, following from (14), are then automatically satisfied. For example, the coefficient of $z_1 z_2 E_{1+2}$ on the right-hand side of (14) receives contributions from $a + b + c + d = 1 + 1 + 2 + 3$ and $1 + 2 + 2 + 4$, and using the above relations one has

$$\begin{aligned} &\big(k_i^{12} k_j^{13} + k_i^{13} k_j^{12}\big) z_1 z_3 \big(\mathbf{p}_1(\mathbf{p}_1 + \mathbf{p}_2)\big) + \big(k_i^{12} k_j^{24} + k_i^{24} k_j^{12}\big) z_2 z_4 \big(\mathbf{p}_1(\mathbf{p}_2 + \mathbf{p}_3)\big) \\ &\quad = \big(2k_i^1 k_j^1 - k_i^1 k_j^2 - k_i^2 k_j^1\big) z_1 z_3 t_{13} - \big(2k_i^2 k_j^2 - k_i^1 k_j^2 - k_i^2 k_j^1\big) z_2 z_4 (-t_{24}) \\ &\quad = 2\big(\xi^2 k_i^1 k_j^1 + \eta^2 k_i^2 k_j^2\big) - \big(k_i^1 k_j^2 + k_i^2 k_j^1\big)\big(\xi^2 + \eta^2\big) \\ &\quad = 2G_{ij} - k_i^1 k_j^2 - k_i^2 k_j^1 \end{aligned} \tag{19}$$

the last expression being equal to the coefficient of the same term on the left-hand side. Similarly for the coefficients of $E_0 = 1$.

It remains to substitute G_{ij} from (18) into Eqs. (11) and (12).

In the σ-model case (11) leads to

$$\vec{k}_1^2 = \vec{k}_2^2 = \operatorname{Tr} G = \xi^2 \vec{k}_1^2 + \eta^2 \vec{k}_2^2. \tag{20}$$

As soon as the two vectors $\vec{k}_1$ and $\vec{k}_2$ have equal lengths, the corresponding parallelogram has to be a rectangle. The remaining essential modulus is the angle ϕ between the two vectors, their common length being an inessential modulus (scaling of the Lagrangian). Another essential modulus is ξ^2 or $\eta^2 = 1 - \xi^2$. Rescalings of parameters z_a, which leave ξ^2 and η^2 intact, are induced by constant shifts of the coordinate vectors $\vec{u}$.

Analogously, in the NG case, one obtains from (12)

$$\check{G}^{ij} k_i^1 k_j^1 = \check{G}^{ij} k_i^2 k_j^2 = 2 \det G. \tag{21}$$

If we parametrize the two NG $\vec{k}$-vectors through $\vec{k}_1 = (\alpha, \beta)$ and $\vec{k}_2 = (\gamma, \delta)$, then

$$G_{ij} = \begin{pmatrix} \alpha^2\xi^2 + \gamma^2\eta^2 & \alpha\beta\xi^2 + \gamma\delta\eta^2 \\ \alpha\beta\xi^2 + \gamma\delta\eta^2 & \beta^2\xi^2 + \delta^2\eta^2 \end{pmatrix}, \qquad \check{G}^{ij} = \begin{pmatrix} \beta^2\xi^2 + \delta^2\eta^2 & -\alpha\beta\xi^2 - \gamma\delta\eta^2 \\ -\alpha\beta\xi^2 - \gamma\delta\eta^2 & \alpha^2\xi^2 + \gamma^2\eta^2 \end{pmatrix}, \tag{22}$$

$\det G = (\alpha\delta - \beta\gamma)^2 \xi^2 \eta^2$, and (21) is equivalent to the system of equations

$$\begin{aligned} \check{G}^{ij} k_i^1 k_j^1 &= \left(\alpha^2\delta^2 - 2\alpha\beta\gamma\delta + \beta^2\gamma^2\right)\eta^2 = 2(\alpha\delta - \beta\gamma)^2 \xi^2 \eta^2, \\ \check{G}^{ij} k_i^2 k_j^2 &= \left(\alpha^2\delta^2 - 2\alpha\beta\gamma\delta + \beta^2\gamma^2\right)\xi^2 = 2(\alpha\delta - \beta\gamma)^2 \xi^2 \eta^2. \end{aligned} \tag{23}$$

It follows that

$$\xi_{\mathrm{NG}}^2 = \eta_{\mathrm{NG}}^2 = \frac{1}{2} \tag{24}$$

with no restriction on the vectors $\vec{k}_1$ and $\vec{k}_2$. The corresponding parallelogram in this case can be arbitrary and the two essential moduli are the angle ϕ between $\vec{k}_1$ and $\vec{k}_2$ and the ratio of their lengths, $|\vec{k}_1|/|\vec{k}_2| = \sqrt{(\alpha^2 + \beta^2)/(\gamma^2 + \delta^2)}$. It is clear that *arbitrary* vectors $\vec{k}_1$ and $\vec{k}_2$ give rise to NG solutions, because they can be made arbitrary by linear transformations of u^i, which are part of the $2d$ reparametrization invariance of the NG equations, i.e. part of the symmetry group of $\mathcal{M}_4^{\mathrm{NG}}$.

2.2. *On the relation between NG and σ-model solutions*

In the previous section we determined the moduli spaces of both the σ-model and the NG equations for $n = 4$ in the framework of the ansatz (10). Actually, the σ-model case corresponds to the trace of Eq. (14) with respect to the indices i, j [12], while in the NG case one should contract with $\check{G}^{ij}$ instead of δ^{ij}.

Our result is that the moduli spaces, while both two-dimensional, are essentially different. Moreover, there is no one-to-one correspondence between the solutions. Does this contradict the widespread belief that the NG and the σ-model are equivalent? It does not, as we shall argue next, because of the Virasoro constraints. Notice that the σ-model dealt with in [1] and [12] does not take into account the Virasoro constraints, which are crucial in the proof of the above equivalence. So, the two models are actually different and, not surprisingly, lead to different answers.

More specifically, recall that the idea behind the equivalence of the NG and σ-model formalisms is based on the consideration of the more general Polyakov action [23]:

$$\int \mathcal{L}_P \, d^2u = \int \mathcal{G}_{ab} g^{ij} \partial_i X^a \partial_j X^b \sqrt{g} \, d^2u \tag{25}$$

where $\mathcal{G}_{ab}$ is the target space metric, made from dynamical fields like in (6), $X^a \equiv (r, \mathbf{v})$ and g_{ij} is the auxiliary field of $2d$-metric. The equations of motion for the dynamical fields then read

$$\partial_i \left(g^{ij} \mathcal{G}_{ab} \sqrt{g} \partial_j X^a\right) = \frac{\partial \mathcal{L}_P}{\partial X^b} \tag{26}$$

while variation with respect to the $2d$-metric gives

$$g_{ij} = 2 \frac{\mathcal{G}_{ab} \partial_i X^a \partial_j X^b}{\mathcal{G}_{cd} g^{kl} \partial_k X^c \partial_l X^d} = 2 \frac{G_{ij}}{g^{kl} G_{kl}}. \tag{27}$$

Inserting (27) into (26), one reproduces the NG equations, while, taking advantage of the local symmetries of the Polyakov action to choose $g_{ij} = \delta_{ij}$, one obtains the σ-model equations. This choice is a gauge-fixing and can always be achieved by a proper transformation of the world sheet variables u^i.[2] Based on this, one may argue that any solution of the NG equations can be converted into a solution of the σ-model: once a G_{ij}^{NG} is found, it can always be diagonalized by a coordinate transformation. In our context, with G_{ij} constant, this transformation $\vec{u}^{\mathrm{NG}} \to \vec{u}^{\sigma}$ is linear, and is given simply by

$$\vec{k}_a^{\mathrm{NG}} \vec{u}^{\mathrm{NG}} = \vec{k}_a^{\sigma} \vec{u}^{\sigma}, \tag{28}$$

or equivalently, using the explicit form of k_a^{NG} and k_a^{σ},

$$u_1^{\sigma} = \alpha u_1^{\mathrm{NG}} + \beta u_2^{\mathrm{NG}}, \qquad u_2^{\sigma} = \gamma u_1^{\mathrm{NG}} + \delta u_2^{\mathrm{NG}}. \tag{29}$$

It is always possible to find such a transformation with non-unit Jacobian, in order to convert the two NG $\vec{k}$-vectors with different lengths into two σ-model $\vec{k}$-vectors with equal lengths. Clearly, the above NG $\to \sigma$ mapping has a non-trivial kernel. It has enough parameters to map different NG solutions into the same σ-model solution; it is not an isomorphism of the two moduli spaces.

The converse, however, is not true: one cannot convert an arbitrary σ-model solution into an NG one. For this, one would have in addition to satisfy the gauge condition $G_{ij} \sim \delta_{ij}$. For instance, linear transformations of coordinates $\vec{u}$ cannot change parameters ξ^2 and η^2. The parameters z_a of a particular solution (10) are rescaled by *shifts* of $\vec{u}$, $\vec{u} \to \vec{u} + \vec{a}$, but z_1 and $z_3 = z_{-1}$ or z_2 and $z_4 = z_{-2}$ are rescaled in opposite directions (since $\vec{k}_{-a} = -\vec{k}_a$), so that ξ^2 and $\eta^2 = 1 - \xi^2$ remain intact. This implies that it is not possible to use the gauge freedom of the Polyakov equations to convert σ-model solutions with generic $\xi^2 \neq 1/2$ into NG solutions, which all have $\xi^2 = \eta^2 = 1/2$. Generic coordinate-$\vec{u}$ reparametrizations (linear or otherwise) change the two tensors g_{ij} and G_{ij} simultaneously, and the desired transformation $(g_{ij}^{\sigma}, G_{ij}^{\sigma}) \xrightarrow{?} (g_{ij}^{\mathrm{NG}}, G_{ij}^{\mathrm{NG}})$ is generically in contradiction with the other two properties, namely

$$g_{ij}^{\sigma} = \delta_{ij} \tag{30}$$

and (27)

$$g_{ij}^{\mathrm{NG}} \sim G_{ij}^{\mathrm{NG}}. \tag{31}$$

These relations are all compatible if and only if $G_{ij}^{\sigma} \sim \delta_{ij}$, which is *not* true for a generic σ-model solution, but only for those with $\xi_\sigma^2 = \eta_\sigma^2 = 1/2$. This, as stated in the beginning of this section, is a concrete manifestation of the well-known fact [24] that Polyakov's σ-model, which is classically equivalent to the NG theory, reproduces the ordinary σ-model, but together with the Virasoro constraints.

A consequence of the above discussion is that the regularized σ-model and NG actions of even a common solution do not coincide. Naively, since substitution of the $2d$-metric (27) into the Polyakov action (25) reproduces the NG action, one would expect that the σ-model and NG actions coincide, provided the Virasoro constraint is satisfied. This is true, but ambiguous, since both actions are infinite. The regularization proposed in [1] does not change only the target space metric $\mathcal{G}_{ab}$ in both actions, which would leave them equal. Instead, it spoils the Virasoro constraint and should lead a priori to different actions! Indeed, it was explicitly checked [20] that, even in the $n = 4$ case, the two actions are different. However, they differ by an inessential additive constant. It would be instructive to examine their difference for higher n.

3. Guess of the action integral for $n = 5$

As explained in [12], it is not straightforward to generalize to $n > 4$ our solutions with exponential behavior at infinity. So, it is not obvious how to extend our approach to these cases and have so far been unable to find relevant solutions. Nevertheless, one can still try to *guess* the form of the regularized action integral $\mathcal{A}_n(z_1, \dots, z_n; \epsilon)$ for $n \geqslant 5$, whose minimum will lead to the BDDK formula for the one-loop amplitude $F_n^{(1)}$ of n-gluon scattering. For that, let us *assume* that we have an n-parameter set of solutions of the σ-model with the appropriate asymptotics, parametrized by z_a, with $a = 1, 2, \dots, n$. In addition, we must assume a regularization scheme [1] with parameter ϵ, as well as a constraint analogous to (17).

We split the action integral into the infinite $\mathcal{A}_\epsilon^{(n)}$ and finite $\tilde{\mathcal{A}}_n$ pieces and guided by the pictorial representation of the BDDK formula [12] and by our $n = 4$ results, we write (up to an additive inessential constant $1/\epsilon^2$ term)

$$\mathcal{A}_n(z_1, z_2, \dots, z_n) = \mathcal{A}_\epsilon^{(n)} + \tilde{\mathcal{A}}_n = \frac{1}{\epsilon} \sum_{a=1}^{n} \log z_a + \sum_{a=1}^{n} \log z_a \log z_{a+1} \tag{32}$$

with $a + n \equiv a$ for all values of a. We neglected any additional angular variables like ϕ of the $n = 4$ case, assuming that such parameters enter in an especially simple way, like it happened in the case of $n = 4$ in [12], where $|\sin\phi|^{-1}$ was a common factor in front of the entire $\mathcal{A}_4$. Notice that for $n = 4$ (32) reproduces the expression derived in [12].

[2] It is well known that the freedom of arbitrary reparametrizations of the world sheet is enough to render an arbitrary metric conformally flat; however, the conformal factor is inessential due to the Weyl invariance of the action, $g_{ij} \to \rho g_{ij}$.

To guess a reasonable generalization of the constraint is more difficult. For general n one has to worry about the presence of terms with higher powers of t_{ab} in the expression for the constraint. For instance, the first non-trivial such term would be $\sum_{a<b<c<d} z_a z_b z_c z_d t_{ab} t_{cd}$. However, for $n=4,5$ such a term, as well as all analogous with higher powers of t vanish identically. In what follows, we shall consider the constraint

$$\sum_{a<b}^{n} z_a z_b t_{ab} = 1 \tag{33}$$

but, one should remember, that this particular form may be oversimplified and irrelevant for $n \geqslant 6$.

Our goal is to minimize (32) under the constraint (33). Let us start with the simpler problem of minimizing $\mathcal{A}_\epsilon^{(n)}$ under the above constraint, which is introduced with a Lagrange multiplier λ. The position $z_a^{(0)}$ of the minimum satisfies

$$\frac{1}{z_a^{(0)}} = \lambda \sum_{b=1}^{n} t_{ab} z_b^{(0)}. \tag{34}$$

For $n=4$, the solution is, up to the invariance $z_1^{(0)} \to \zeta z_1^{(0)}$, $z_3^{(0)} \to \frac{1}{\zeta} z_3^{(0)}$ and $z_2^{(0)} \to \zeta' z_2^{(0)}$, $z_4^{(0)} \to \frac{1}{\zeta'} z_4^{(0)}$, given by [12]

$$z_a^{(0)} = \frac{1}{\sqrt{2t_{a,a+2}}}. \tag{35}$$

Similarly, for $n=5$ we obtain instead

$$z_a^{(0)} z_b^{(0)} = \frac{1}{5t_{ab}}, \qquad \lambda = \frac{5}{2}. \tag{36}$$

Multiplying all these pairs together gives

$$z_1^{(0)} z_2^{(0)} z_3^{(0)} z_4^{(0)} z_5^{(0)} = \frac{1}{\sqrt{5^5 t_{13} t_{14} t_{23} t_{24} t_{35}}}. \tag{37}$$

Now, dividing this expression twice by appropriately chosen products $z_a^{(0)} z_b^{(0)}$, one obtains

$$z_a^{(0)} = \sqrt{\frac{t_{a+1,a+3} t_{a+2,a+4}}{5 t_{a,a+2} t_{a,a+3} t_{a+1,a+4}}}. \tag{38}$$

If one denotes $\tau_{ab} = \log t_{ab}$, (38) has a pictorial representation shown in Fig. 3. Incidentally, note that in contrast to the $n=4$ case, there is no rescaling freedom in solutions of Eq. (34) for $n=5$.

Going back to the minimization of $\mathcal{A}$, observe that the presence of the finite correction $\tilde{\mathcal{A}}_n$ in $\mathcal{A}$ will shift the position of the minimum to $z_a = z_a^{(0)} + \epsilon z_a^{(1)}$. The $\mathcal{O}(\epsilon)$ $z_a^{(1)}$-shift of z_a could a priori give a finite correction to $\mathcal{A}$. However, as we will argue, this

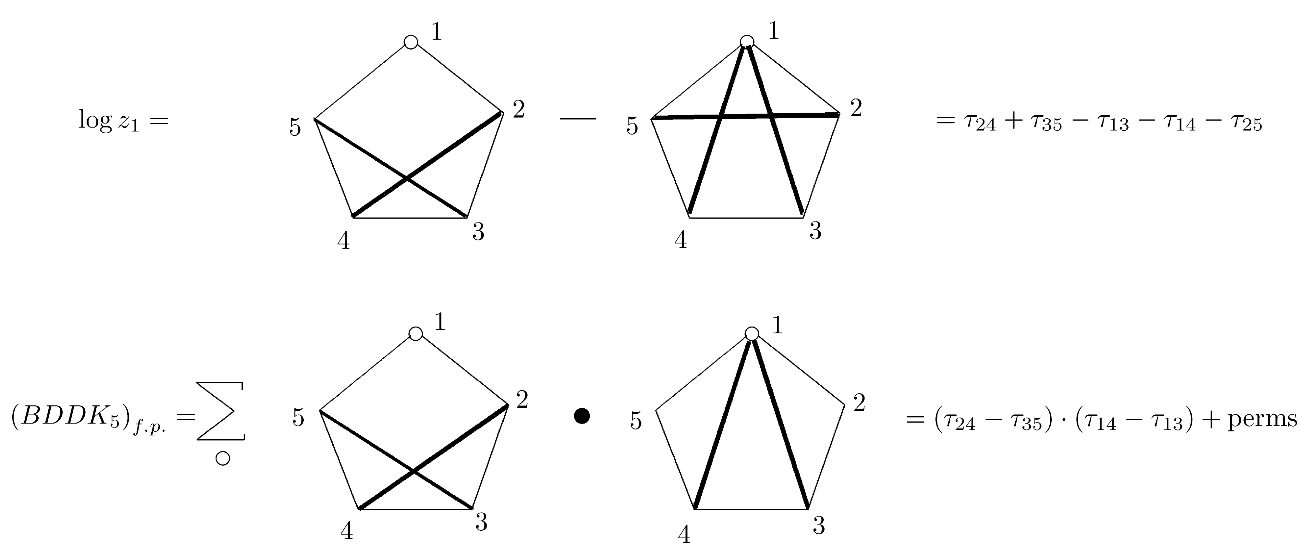

Fig. 3. Pictorial representations of Eqs. (38) and (41). The polygon is nothing but Π from Fig. 1, whose edges are associated with external momenta. τ-parameters $\tau_{ab} = \log t_{ab}$ are associated with diagonals and z_a with the vertices of the polygon. The marked diagonals correspond to differences of τ-parameters in the equations.

$\mathcal{O}(\epsilon^0)$ contribution actually vanishes. Indeed, the finite correction of $\mathcal{A}$ due to $z_a^{(1)}$ is

$$\sum_a^n \frac{\partial \mathcal{A}_\epsilon}{\partial z_a}\bigg|_{z_a=z_a^{(0)}} z_a^{(1)} = \sum_a^n \frac{z_a^{(1)}}{z_a^{(0)}} = \lambda \sum_{a,b}^n t_{ab} z_a^{(0)} z_b^{(1)} = 0 \tag{39}$$

the last two equalities being direct corollaries of (34).

Thus, in order to reproduce the BDDK[3] result, one has to insert the solutions (38) for $z_a^{(0)}$ into the action (32). The result is the BDK formula for $n = 5$ (in this formula one should put $\mu^2 = 1/5$),

$$\mathrm{BDK} = \mathrm{BDDK}_5 = -\frac{1}{\epsilon^2}\prod_a \left(\frac{\mu^2}{t_{a,a+2}}\right)^\epsilon + \sum_a \log\frac{t_{a,a+2}}{t_{a+1,a+3}} \log\frac{t_{a+2,a+4}}{t_{a-2,a}} = -2\mathcal{A}_5|_{z_a=z_a^{(0)}} + O(\epsilon). \tag{40}$$

The finite part of this expression is equal to (see Fig. 3)

$$\sum_{a=1}^n (\tau_{a,a+2} - \tau_{a,a-2})(\tau_{a-1,a+2} - \tau_{a+1,a-2}) = (\tau_{14} - \tau_{13})(\tau_{24} - \tau_{35}) + \text{cyclic permutations}. \tag{41}$$

This generalizes the older result for $n = 4$

$$\text{finite part of } \mathrm{BDDK}_4 = (\tau_{13} - \tau_{24})^2 = \left(\log\frac{s}{t}\right)^2. \tag{42}$$

It is easy to see that the expressions for $n = 5$ are natural generalizations of those for $n = 4$. The main new ingredient for $n > 5$ is that $t^{[r]}$ with $r > 2$ (see [12] for notations) and higher powers of t can enter the constraint. At the same time, dilogarithmic contributions should appear in the action integral. It can happen that they occur after additional integration over some new moduli. We do not go into details of these subtler considerations in the present text.

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[3] In this particular case of $n = 5$ it should rather be named BDK formula [21], this example was actually used as a basis for further calculations at $n \geqslant 6$ in [22] and structures revealed at $n = 5$ are inherited by generic BDDK expressions. The BDK formula has a number of equivalent representations, of which we use just one in this Letter.

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