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On uniqueness of Cartesian products of surfaces with boundary

J. Malešič [a], D. Repovš [a], W. Rosicki [b,*], A. Zastrow [b]

[a] *Institute for Mathematics, Physics and Mechanics, University of Ljubljana, PO Box 2964, Ljubljana 1001, Slovenia*
[b] *Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, Gdańsk PL-80-952, Poland*

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Abstract

It is known that if one of the factors of a decomposition of a manifold into Cartesian product is an interval then the decomposition is not unique. We prove that the decomposition of a 4-manifold (possibly with boundary) into 2-dimensional factors is unique, provided that the factors are not products of 1-manifolds.

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1. Introduction

In 1945 Borsuk [2] showed that any connected compact n-dimensional manifold without boundary has at most one decomposition into a Cartesian product of factors of dimension $\leqslant 2$. If we consider Cartesian products of higher-dimensional manifolds then such

* Corresponding author.
E-mail addresses: joze.malesic@uni-lj.si (J. Malešič), dusan.repovs@uni-lj.si (D. Repovš), wrosicki@math.univ.gda.pl (W. Rosicki), mataz@univ.gda.pl (A. Zastrow).

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uniqueness property does not hold (see Theorem 11.5 in [4] and [11]). Even if we consider the classical Ulam problem [17] of uniqueness of Cartesian squares, one can find counterexamples for 3-manifolds (cf. [12]).

The uniqueness of the decomposition into Cartesian products fails if the factors are 2-manifolds with boundary. A torus with a hole and a disk with two holes are not homeomorphic, however, their Cartesian products with the interval $I = [0, 1]$ are homeomorphic.

Similarly, the product of a Möbius band with a hole and the interval I is homeomorphic to the product of a Klein bottle with a hole and the interval I. All 2-manifolds in the examples above can be constructed by identifying two pairs of disjoint arcs in the boundary of a disk. After multiplication by the interval I, the order of identified arcs on the boundaries of disks becomes inessential. If 3-manifold or more general 3-polyhedron has two different decompositions into Cartesian product then one of the factors in these decompositions must be an interval (see [14]).

The uniqueness property holds for Cartesian squares (cf. [5]) and Cartesian powers (cf. [15]) of 2-manifolds with boundary. The uniqueness (up to permutation of factors) of a Cartesian product of circles and intervals is obvious. We have the uniqueness of decomposition into a finite Cartesian product of 1-polyhedra (cf. [1]) and 1-dimensional locally connected continua (cf. [3]). A Cartesian product of 1-polyhedra does not have another decomposition into a Cartesian product of polyhedra of dimension $\leqslant 2$ (cf. [16]). Before we begin to consider uniqueness of Cartesian products of connected 2-manifolds with boundary we need some preliminaries.

Definition 1.1. Let X be a compact connected 2-manifold with nonempty boundary. We associate to X the following number:

$$s(X) = \operatorname{rank} H_1(X) - \operatorname{rank} H_1(\partial X) + 1.$$

Lemma 1.1. *Let X, Y, X', and Y' be any compact connected 2-manifolds with nonempty boundary and suppose that the Cartesian products $X \times Y$ and $X' \times Y'$ are homeomorphic. Then*

$$s(X)s(Y) = s(X')s(Y').$$

Proof. We use an argument similar to the one in [15, Theorem 2.1]. We consider the map

$$i_*: H_2(X \times Y) \to H_2\big(X \times Y, \partial(X \times Y)\big),$$

which is induced by the inclusion of the pair $(X \times Y, \emptyset)$. The image of this map is generated by all products $\zeta_1 \otimes \zeta_2$ such that $\zeta_1 \in H_1(X)$ and $\zeta_2 \in H_1(Y)$, such that $j_{k*}(\zeta_k) \neq 0$, for $k = 1, 2$, where

$$j_{1*}: H_1(X) \to H_1(X, \partial X) \quad \text{and} \quad j_{2*}: H_1(Y) \to H_1(Y, \partial Y)$$

are given by inclusions. The number $s(X)$ is equal to $\operatorname{rank}\operatorname{im} j_{1*}$ and the number $s(Y)$ is equal to $\operatorname{rank}\operatorname{im} j_{2*}$. So $s(X)s(Y)$ is equal to $\operatorname{rank}\operatorname{im} i_*$.

Hence if $X \times Y$ and $X' \times Y'$ are homeomorphic it follows that $s(X)s(Y) = s(X')s(Y')$. □

Lemma 1.2. *Let X, Y, X', and Y' be any compact connected 2-manifolds with nonempty boundary and suppose that the Cartesian products $X \times Y$ and $X' \times Y'$ are homeomorphic. Then with respect to the order of the factors we have*:

(i) $H_1(X) = H_1(X')$ *and* $H_1(Y) = H_1(Y')$;
(ii) $H_1(X, \partial X) = H_1(X', \partial X')$ *and* $H_1(Y, \partial Y) = H_1(Y', \partial Y')$.

Proof. Let $H_1(X) = Z^x$, $H_1(Y) = Z^y$, $H_1(X') = Z^{x'}$ and $H_1(Y') = Z^{y'}$. By the Künneth formula we conclude that:

$$Z^{xy} \cong H_2(X \times Y) \cong H_2\big(X' \times Y'\big) \cong Z^{x'y'} \quad \text{and}$$
$$Z^{x+y} \cong H_1(X \times Y) \cong H_1\big(X' \times Y'\big) \cong Z^{x'+y'}.$$

Hence, $x = x'$ and $y = y'$ or $x = y'$ and $y = x'$. We can assume that the first case holds. This completes the proof of (i).

If X is orientable then $H_1(X, \partial X) = Z^x$. If it is not then $H_1(X, \partial X) = Z^{x-1} \oplus Z_2$. Similarly, for Y, X', and Y'. By the relative Künneth formula,

$$H_2\big(X \times Y, \partial(X \times Y)\big) = Z^{xy - o_2 x - o_1 y + o_1 o_2} \oplus Z_2^{o_2 x + o_1 y - o_1 o_2},$$

where $o_1 = 1$ if X is nonorientable and $o_1 = 0$ if X is orientable, and $o_2 = 1$ if Y is nonorientable and $o_2 = 0$ if Y is orientable. Similarly for X' and Y'. Hence $xy - o_2 x - o_1 y + o_1 o_2 = xy - o'_2 x - o'_1 y + o'_1 o'_2$. So, if $x > 1$ and $y > 1$ then $H_1(X, \partial X) = H_1(X', \partial X')$ and $H_1(Y, \partial Y) = H_1(Y', \partial Y')$.

If $x = 0$ then X and X' are homeomorphic to the disk. Therefore Y and Y' are both orientable or both nonorientable, and their relative first homology groups are the same.

If $x = 1$ then X can be the annulus $A = S^1 \times I$ or the Möbius band M. Similarly for X'.

If X is an annulus then $H_2(X \times Y, \partial(X \times Y)) = Z \otimes H_1(Y, \partial Y) = H_1(Y, \partial Y)$. If X' is a Möbius band then $H_2(X' \times Y', \partial(X' \times Y')) = Z_2 \otimes H_1(Y', \partial Y')$. These groups can be isomorphic only if $H_1(Y, \partial Y) = Z_2$ and if $H_1(Y', \partial Y')$ is equal to Z or Z_2. The spaces $A \times M$ and $M \times M$ are not homeomorphic by Lemma 1.1; by definition $s(A) = 0$, and $s(M) = 1$, so $s(A)s(M) \neq s(M)s(M)$. □

We start the consideration of the Cartesian products of connected 2-manifolds with boundary by presenting the case where one of the factors is not prime. In this paper a *prime* manifold is a manifold which is not a nontrivial Cartesian product. There exist three nonprime surfaces: $I \times I$, $I \times S^1$, and $S^1 \times S^1$. We have the following:

Proposition 1.1. *Let X and Y be any compact 2-manifolds, possibly with boundary, and suppose that the Cartesian products $X \times Y$ and $X' \times Y'$ are homeomorphic. If X is prime and Y is a product of two 1-manifolds, then X' is also a prime 2-manifold and Y' is a product of two 1-manifolds* (*up to a permutation of X' and Y'*). *In both cases, Y and Y' are homeomorphic. Furthermore, if X and X' are not homeomorphic, then Y and Y' are homeomorphic either to I^2 or to $S^1 \times I$.*

Proof. By Kosiński's theorem [10], all 2-dimensional Cartesian factors of a polyhedron are polyhedra, so X' and Y' are 2-manifolds, possibly with boundary. If $\partial X = \emptyset$ and $Y = S^1 \times S^1$ then we have the uniqueness by a classical result of Borsuk [2].

If $\partial X = \emptyset$ and $Y = I \times S^1$, then one of the factors X', Y', say X' has an empty boundary, because $H_3(X \times Y; Z_2) = H_3(X' \times Y'; Z_2) \neq 0$. Since $\partial(X \times Y) = X \times \partial Y = X' \times \partial Y' = \partial(X' \times Y')$, the surfaces X and X' are homeomorphic. Hence, comparing the homology groups we obtain that Y' is an annulus, also.

Now, let $\partial X = \emptyset$ and $Y = I^2$. If X is nonorientable then $0 = H_2(X) = H_2(X \times Y) = H_2(X' \times Y')$, so one of the factors X', Y' is a disk. The second factor is homeomorphic to X. If X is orientable, $\partial X' \neq \emptyset$ and $\partial Y' \neq \emptyset$ then $Z = H_2(X) = H_1(X') \otimes H_1(Y')$. Therefore X' and Y' are homeomorphic to $S^1 \times I$ and X is a torus. If $\partial X' = \emptyset$ then the boundaries $\partial(X \times Y)$ and $\partial(X' \times Y')$ are homeomorphic, so X and X' are homeomorphic and Y' is a disk.

If $\partial X \neq \emptyset$ and $Y = S^1 \times S^1$, then $Y' = S^1 \times S^1$ because $\partial(X \times Y)$ is a disjoint union of the sets homeomorphic to $S^1 \times S^1 \times S^1$. Hence X and X' are homeomorphic by a special case of Theorem 2 [16]. If Y is homeomorphic to a disk or to an annulus and $\partial X \neq \emptyset$, then by Lemma 1.2, Y' is also homeomorphic to a disk or to an annulus. □

2. The main result

The following is the main result of our paper:

Theorem 2.1. *Any connected* 4*-dimensional manifold, possibly with boundary, has at most one decomposition into Cartesian products of prime* 2*-manifolds, possibly with boundary.*

The techniques which were used in a similar lemma in [13] are not strong enough for our purpose. We shall use the Splitting theorem in the proof of our theorem above (see [8,9])—for investigation of the boundaries of the manifolds $X \times Y$ and $X' \times Y'$. So we use this theorem in the case when ∂M is empty.

In [8,9] manifolds are orientable, so we must also assume that the manifold M is orientable. We denote by $\sigma_W(M)$ the 3-manifold obtained by splitting M along W. Similarly we define the 2-manifold $\sigma_{\partial W}(\partial M)$, which can be naturally identified with a submanifold of the boundary of $\sigma_W(M)$.

Theorem 2.2 (Splitting theorem [8, p. 157]). *Let M be any compact, orientable, sufficiently-large, irreducible and boundary-irreducible* 3*-manifold. Then there exists a two-sided, incompressible* 2*-manifold, W properly embedded in M, unique up to ambient isotopy, having the following three properties*:

(a) *The components of W are annuli and tori, and none of them is boundary-parallel in M*;
(b) *Each component of $(\sigma_W(M), \sigma_{\partial W}(\partial M))$ is either a Seifert pair or a simple pair*; *and*
(c) *W is minimal with respect to inclusion among all two-sided* 2*-manifolds in M having properties* (a) *and* (b).

Proof of Theorem 2.1. If both surfaces X and Y are without boundary, the uniqueness holds by Borsuk's theorem [2].

If $\partial X = \emptyset$ and $\partial Y \neq \emptyset$ then $\partial(X \times Y) = X \times \partial Y$. Since $Y \neq I^2$, like in the proof of Proposition 1.1, one of the factors X', Y', say X' has an empty boundary, because $H_3(X \times Y; Z_2) = H_3(X' \times Y'; Z_2) \neq 0$ and $\partial Y' \neq \emptyset$. So, $\partial(X' \times Y') = X' \times \partial Y'$. Therefore X and X' are homeomorphic and the numbers of the components of the boundaries ∂Y and $\partial Y'$ are the same. Looking at the homology and relative homology groups we obtain that the surfaces Y and Y' are also homeomorphic.

Now we consider the case when ∂X and ∂Y are nonempty. Again by Lemma 1.2, the first Betti numbers of X and X' are the same and the first Betti numbers of Y and Y', are also the same. The coincidence of the first relative homology groups implies that the orientability of X and Y agree with the orientability of X' and Y', respectively. We consider three cases.

In the *first case*, X and Y are orientable, $M = \partial(X \times Y)$, $W = \partial X \times \partial Y$. Since by assumption, X and Y are not homeomorphic to I^2 or $S^1 \times I$, the manifolds M and W satisfy the hypotheses of the Splitting theorem. Since the boundary of M is empty, the manifold W is a disjoint union of tori.

For somebody who is familiar with 3-manifolds the irreducibility of M is a simple exercise, but for the reader's covenience we outline a proof. If S is a 2-sphere contained in M we can assume that it is in a general position with W, so the intersection $S \cap W$ is a disjoint union of closed curves. Some of them bound innermost disks in S. Such a disk lies in one of components of $\sigma_W(M)$. The boundaries of the components are incompressible [8, II.2.4], so the boundary of the disk bound a disk in W. The components of $\sigma_W(M)$ are irreducible [8, II.2.3], so the union of our two disk bounds a ball. Via this ball we isotope parts of S into the adjacent component of $\sigma_W(M)$ eliminating one closed curve of $S \cap W$. We repeat this operation as many times as S lies in one component and it bounds a ball.

We will show that W is minimal. Assume that $V = W \setminus (S_1 \times S_2)$ where $S_1 \times S_2$ is a component of W also gives a splitting in the sense of Theorem 2.2. According to V, we have $U = (X \times S_2) \cup (S_1 \times Y)$ as a component of $\sigma_V(M)$. It must be either a Seifert pair or a simple pair. The set U is not a simple pair because the incompressible torus $S_1 \times S_2$ is not boundary-parallel in U (see [8, p. 154]).

The fundamental group of U is infinite, so by Corollary 8.3 in [6] or VI.11.a in [7], the manifold U is a Seifert manifold if and only if its fundamental group has a normal cyclic infinite subgroup. Let an element α of $\pi_1(U)$ be a generator of this subgroup. By Seifert–van Kampen theorem $\pi_1(U)$ is a sum with amalgamation of the groups $\pi_1(X \times S_2)$ and $\pi_1(S_1 \times Y)$. The natural projections map the element α onto elements of the centers of $\pi_1(X \times S_2)$ and $\pi_1(S_1 \times Y)$. So, if $\pi_1(X)$ and $\pi_1(Y)$ have more than one generator, it is impossible.

The same holds for X' and Y', where $M' = \partial(X' \times Y')$, $W' = \partial X' \times \partial Y'$. The components of $\sigma_W(M)$ are homeomorphic to spaces $X \times S^1$ and $S^1 \times Y$. Because the manifolds M and M' are homeomorphic and W is unique up to ambient isotopy, the components of $\sigma_W(M)$ and the components of $\sigma_{W'}(M')$ are homeomorphic. The components of $\sigma_{W'}(M')$ are homeomorphic to spaces $X' \times S^1$ and $S^1 \times Y'$, so the manifolds X and Y are homeomorphic to X' and Y'.

In the *second case* only one manifold is orientable. Let X be nonorientable and Y be orientable. We consider the oriented double covers $\widetilde{X}$ and $\widetilde{X}'$ of X and X'. The manifolds $\widetilde{X} \times Y$, and $\widetilde{X}' \times Y'$ are orientable double covers of the homeomorphic manifolds $X \times Y$ and $X' \times Y'$, so our manifolds are homeomorphic.

If X is the Möbius band, then X' is also nonorientable and $H_1(X) = H_1(X') = Z$, by Lemma 1.2, so X' is the Möbius band, too.

If X is not the Möbius band, then as before, we have homeomorphy either according to $\widetilde{X} \approx \widetilde{X}'$ and $Y \approx Y'$ or according to $\widetilde{X} \approx Y'$ and $Y \approx \widetilde{X}'$ by the Splitting theorem. In the first case X and X' are also homeomorphic. In the second case if $H_1(X) = Z^x$ then $H_1(Y) = Z^{2x-1}$. Putting $s(X') = s(X) + a$, $s(Y') = s(Y) + b$, $s(\widetilde{X}) = 2(s(X) - 1)$ and $s(\widetilde{X}') = 2(s(X') - 1)$ to the equations

$$s(X)s(Y) = s(X')s(Y'), \qquad s(\widetilde{X})s(Y) = s(\widetilde{X}')s(Y')$$

we obtain $s(Y) = s(Y')$, so Y and Y' are homeomorphic. Then

$$\widetilde{X} \approx Y' \approx Y \approx \widetilde{X}',$$

so $X \approx X'$ also.

If X and X' are Möbius bands then we use Lemma 1.1. We have that $s(X)s(Y) = s(X')s(Y')$. Hence $s(Y) = s(Y')$, because $s(X) = s(X') = 1$. Since $H_1(Y) = H_1(Y')$ and $s(Y) = s(Y')$, they have the same number of components of their boundaries, so they are homeomorphic.

In the *third case* both surfaces X and Y are nonorientable. We cannot use exactly the same argument, but we make a similar consideration. First, we know by Lemma 1.2 that both surfaces X' and Y' are also nonorientable. We consider the manifolds $X \times S_i$ where S_i are components of ∂Y, and $S_j \times Y$ where S_j are components of ∂X.

Next, we take the oriented double covers $\widetilde{X}$ and $\widetilde{Y}$ of X and Y. The manifolds $\widetilde{X} \times S_i$ and $S_j \times \widetilde{Y}$ are the oriented double covers of $X \times S_i$ and $S_j \times Y$. Each of the tori $S_j \times S_i$ is covered by tori $S_j' \times S_i$ and $S_j'' \times S_i$ in $\widetilde{X} \times S_i$ and is covered by tori $S_j \times S_i'$ and $S_j \times S_i''$ in $S_j \times \widetilde{Y}$.

By identifying $S_j' \times S_i$ with $S_j \times S_i'$ and $S_j'' \times S_i$ with $S_j \times S_i''$, we obtain the oriented double cover M of $\partial(X \times Y)$. It is not essential which circles we denoted by S_i', S_j' and S_i'', S_j'' because in every case we obtain the unique the oriented double cover of $\partial(X \times Y)$.

Analogously, we construct the oriented double cover M' of $\partial(X' \times Y')$. Of course M and M' are homeomorphic. If the manifolds X and Y are not the Möbius bands then we solve the problem by the Splitting theorem.

If X is a Möbius band then we solve the problem using Lemma 1.1, like in the second case. □

We also include the following new related result:

Theorem 2.3. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be any surfaces with nonempty boundary and suppose that their Cartesian products $X_1 \times \cdots \times X_n$ and $Y_1 \times \cdots \times Y_n$ are homeomorphic. Then there exists a one-to-one correspondence between them* (*assume X_i corresponds to Y_i*) *such that* $\operatorname{rank} H_1(X_i) = \operatorname{rank} H_1(Y_i)$ *and if*

$$s(X_i) = \operatorname{rank} H_1(X_i) - \operatorname{rank} H_1(\partial X_i) + 1$$

for $i = 1, 2, \dots, n$ *then*

$$s(X_1)s(X_2)\cdots s(X_n) = s(Y_1)s(Y_2)\cdots s(Y_n).$$

Proof. Let $H_1(X_i) = Z^{n_i}$ and $H_1(Y_1) = Z^{m_i}$. We can conclude from the Künneth formula that

$$H_1(X_1 \times \cdots \times X_n) = Z^{\sum_{i=1}^{n} n_i},$$
$$H_2(X_1 \times \cdots \times X_n) = Z^{\sum_{i_1 \neq i_2} n_{i_1} n_{i_2}}, \quad \text{and}$$
$$\vdots$$
$$H_n(X_1 \times \cdots \times X_n) = Z^{n_1 \cdots n_n}.$$

We obtain similar formulae for the product $Y_1 \times \cdots \times Y_n$. Because rank $H_i(X_1 \times \cdots \times X_n) =$ rank $H_i(Y_1 \times \cdots \times Y_n)$ we can conclude that $n_i = m_i$ for $i = 1, 2, \dots, n$. This follows from the fact that the ranks of the homology groups above are the coefficients of the polynomials $\prod_{i=1}^{n}(x - n_i)$ and $\prod_{i=1}^{n}(x - m_i)$. The polynomials are equal, so the numbers n_i and m_i are the same.

We obtain the equality $s(X_1)s(X_2)\cdots s(X_n) = s(Y_1)s(Y_2)\cdots s(Y_n)$ like in the previous proof. □

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