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Distinguished orbits and the L–S category of simply connected compact Lie groups

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ABSTRACT

We show that the Lusternik–Schnirelmann category of a simple, simply connected, compact Lie group G is bounded above by the sum of the relative categories of certain distinguished conjugacy classes in G corresponding to the vertices of the fundamental alcove for the action of the affine Weyl group on the Lie algebra of a maximal torus of G.

1. Introduction

1.1. The (normalized) *Lusternik–Schnirelmann category* [1], of a topological space X, denoted $\operatorname{cat}(X)$, is the least integer m such that X can be covered by $m+1$ open sets that are contractible in X. One of the problems on Ganea's list [3] from 1971 asks to find the L–S category of (compact) Lie groups. In 1975, Singhof [9] proved that $\operatorname{cat}(\mathrm{SU}(n+1)) = n$. For the other families of simply connected compact Lie groups, the answer is only known when the rank is small (cf. [7] for a nice summary of what is known for simply connected and non-simply connected compact Lie groups of small rank).

1.2. The purpose of this short note is to show that the L–S category of a simple, simply connected, compact Lie group G is bounded above by the sum of the relative categories of certain distinguished conjugacy classes in G. More precisely, suppose $\{v_0, \ldots, v_n\}$ is the set of vertices of the fundamental alcove for the action of the affine Weyl group on the Lie algebra of a maximal torus of G. For $0 \leqslant k \leqslant n$, let $\mathcal{O}_k$ be the conjugacy class of $\exp v_k$ in G. Then we will show in Section 4 that

$$\operatorname{cat}(G) + 1 \leqslant \sum_{k=0}^{n} \bigl(\operatorname{cat}_G(\mathcal{O}_k) + 1\bigr),$$

where $\operatorname{cat}_G(\mathcal{O}_k)$ is the *relative L–S category* of $\mathcal{O}_k$ in G. (If $Y \subseteq X$ is a topological subspace, $\operatorname{cat}_X(Y)$ is the least integer m such that there is a covering of Y by $m+1$ open subsets of X, each contractible in X.)

1.3. For $G = \mathrm{SU}(n+1)$, the conjugacy classes $\mathcal{O}_k$ turn out to be the points of the center of G and we recover Singhof's result that $\operatorname{cat}(\mathrm{SU}(n+1)) \leqslant n$. For $G = \mathrm{Sp}(n)$, we conjecture that $\operatorname{cat}_G(\mathcal{O}_k) \leqslant \min\{k, n-k\}$ (with respect to an appropriate numbering) which would imply that

$$\operatorname{cat}\bigl(\mathrm{Sp}(n)\bigr) \leqslant \left\lfloor \frac{(n+2)^2}{4} \right\rfloor - 1.$$

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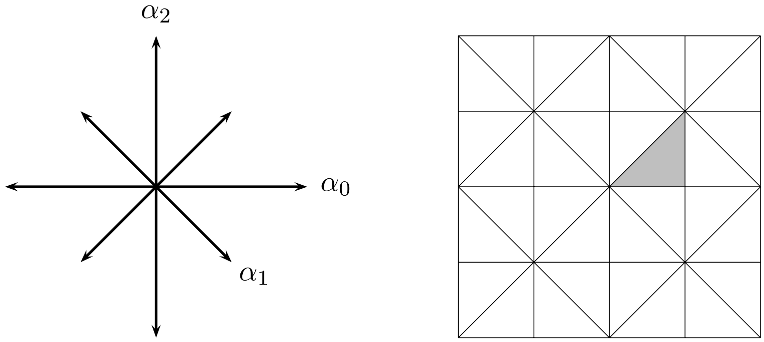

Fig. 1. Roots and alcoves for Sp(2).

Thus for $n = 1, 2, 3, 4, 5, 6, \ldots$ our conjectured upper bound is $1, 3, 5, 8, 11, 15, \ldots$. For $n = 1, 2, 3$ it is known [2] that $\mathrm{cat}(\mathrm{Sp}(n)) = 1, 3, 5$. Also, for $n = 1, 2, 3, 4$ it is known [5] that $\mathrm{cat}(\mathrm{Spin}(2n+1)) = 1, 3, 5, 8$. Based on this small set of data, we conjecture that $\mathrm{cat}(\mathrm{Sp}(n)) = \mathrm{cat}(\mathrm{Spin}(2n+1))$ and that the inequality above is in fact an equality. We remark that the best known lower bound is $\mathrm{cat}(\mathrm{Sp}(n)) \geqslant n + 2$ for $n \geqslant 3$ [2,6].

1.4. Acknowledgments

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2. Notation

2.1. Let G be a simple, simply connected, compact Lie group with Lie algebra $\mathfrak{g}$. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Then $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{R}} = \mathbf{i}\mathfrak{t}$. Write $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ for the set of roots and choose a positive system Δ^+ with corresponding set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. With respect to this system, write α_0 for the highest root. For the classical Lie groups and with respect to standard notation, Π and α_0 can be taken as in the following table:

G	Π	α_0
$\mathrm{SU}(n+1)$	$\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant n\}$	$\varepsilon_1 - \varepsilon_{n+1}$
$\mathrm{Sp}(n)$	$\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{\alpha_n = 2\varepsilon_n\}$	$2\varepsilon_1$
$\mathrm{Spin}(2n+1)$	$\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{\alpha_n = \varepsilon_n\}$	$\varepsilon_1 + \varepsilon_2$
$\mathrm{Spin}(2n)$	$\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$	$\varepsilon_1 + \varepsilon_2$

2.2. Write $R^\vee$ for the coroot lattice in $\mathfrak{h}$ (which is the same as the dual to the weight lattice in $\mathfrak{h}^*$) so that

$$R^\vee = \mathrm{span}_{\mathbb{Z}}\{h_\alpha \mid \alpha \in \Delta\}.$$

Here $h_\alpha = 2u_\alpha / B(u_\alpha, u_\alpha) \in \mathfrak{h}_{\mathbb{R}}$ where $B(\cdot,\cdot)$ is the Killing form and $u_\alpha \in \mathfrak{h}_{\mathbb{R}}$ is uniquely determined by the equation $\alpha(H) = B(H, u_\alpha)$ for all $H \in \mathfrak{h}_{\mathbb{R}}$. Since G is simply connected, it follows that

$$\ker(\exp|_{\mathfrak{t}}) = 2\pi \mathbf{i} R^\vee.$$

2.3. The connected components of

$$\{t \in \mathfrak{t} \mid \alpha(t) \notin 2\pi\mathbf{i}\mathbb{Z} \text{ for } \alpha \in \Delta\}$$

are called *alcoves*. Write $W = W(G, \mathfrak{t})$ for the Weyl group of G with respect to $\mathfrak{t}$ viewed as acting on $\mathfrak{t}$ (and extended to $\mathfrak{h}$ as needed). The *affine Weyl group*, $\widehat{W}$, is the group generated by the transformations of $\mathfrak{t}$ of the form $t \mapsto wt + z$ for $w \in W$ and $z \in \ker(\exp|_{\mathfrak{t}})$. It acts simply transitively on the set of alcoves. The *fundamental alcove*, A_0, is the alcove given by

$$\begin{aligned} A_0 &= \{t = \mathbf{i}H \in \mathfrak{t} \mid 0 < \alpha(H) < 2\pi \text{ for } \alpha \in \Delta^+\} \\ &= \{t = \mathbf{i}H \in \mathfrak{t} \mid \alpha_0(H) < 2\pi \text{ and } 0 < \alpha_j(H) \text{ for } 1 \leqslant j \leqslant n\}. \end{aligned}$$

The closure of the fundamental alcove, $\overline{A}_0$, is a fundamental domain for the $\widehat{W}$-action (cf. [4, Theorem 4.8]). For $G = \mathrm{Sp}(2)$, the roots and the fundamental alcove are shown in Fig. 1.

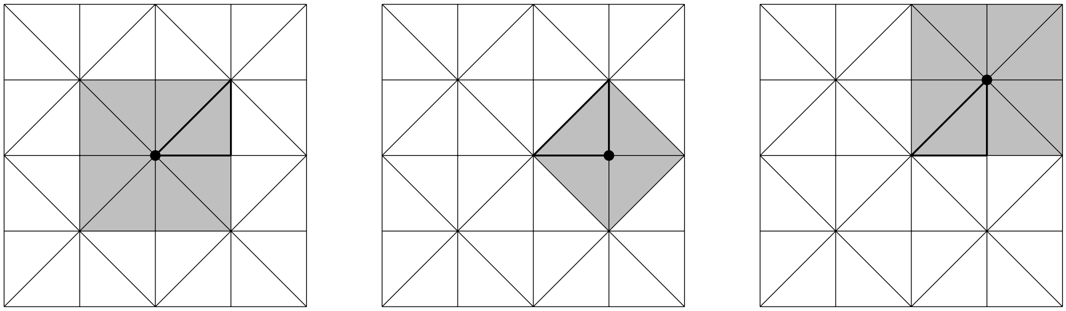

Fig. 2. The cells C_0, C_1, and C_2 for Sp(2).

3. Cells

3.1. Define $v_0 = 0 \in \mathfrak{t}$ and for $1 \leqslant k \leqslant n$, define $v_k \in \mathfrak{t}$ by the equations

$$\alpha_j(v_k) = \begin{cases} 2\pi\mathbf{i} & \text{if } j = 0, \\ 0 & \text{if } 1 \leqslant j \leqslant n \text{ and } j \neq k. \end{cases}$$

Then $\{v_0, \ldots, v_n\}$ is the set of vertices of the n-simplex $\overline{A}_0$. Notice that if we write $\alpha_0 = \sum_{j=1}^n m_j \alpha_j$ with $m_j \in \mathbb{N}$, we get $2\pi\mathbf{i} = \alpha_0(v_k) = \sum_{j=1}^n m_j \alpha_j(v_k) = m_k \alpha_k(v_k)$. Therefore,

$$\alpha_k(v_k) = \frac{2\pi\mathbf{i}}{m_k} \quad \text{for } 1 \leqslant k \leqslant n.$$

(For classical G, the $m_k \in \{1, 2\}$; however, for exceptional G, the m_k can be as large as 6.)

3.2. Define

$$F_0 = \left\{ t = \mathbf{i}H \in \mathfrak{t} \;\middle|\; \alpha_0(t) = 2\pi\mathbf{i} \text{ and } 0 \leqslant \alpha_j(H) \text{ for } 1 \leqslant j \leqslant n \right\}$$

and for $1 \leqslant k \leqslant n$,

$$F_k = \left\{ t = \mathbf{i}H \in \mathfrak{t} \;\middle|\; \alpha_0(H) \leqslant 2\pi,\ 0 \leqslant \alpha_j(H) \text{ for } 1 \leqslant j \leqslant n \text{ with } j \neq k, \text{ and } 0 = \alpha_k(t) \right\}.$$

Then $\{F_0, \ldots, F_n\}$ is the set of faces of $\overline{A}_0$. For $0 \leqslant k \leqslant n$, we will call F_k the *face opposite to* v_k. In the following, we will write $r_k \in \widehat{W}$ for the reflection across F_k. Explicitly, $r_0(t) = t - (\alpha_0(t) - 2\pi\mathbf{i})h_{\alpha_0}$ and $r_k(t) = t - \alpha_k(t)h_{\alpha_k}$ for $1 \leqslant k \leqslant n$.

3.3. For $0 \leqslant k \leqslant n$, let $\widehat{W}_k$ be the stabilizer of v_k,

$$\widehat{W}_k = \left\{ w \in \widehat{W} \;\middle|\; w(v_k) = v_k \right\}.$$

Lemma 1. *For $0 \leqslant k \leqslant n$, the group $\widehat{W}_k$ is generated by $\{r_j \mid 0 \leqslant j \leqslant n$ and $j \neq k\}$ and*

$$\{\textit{alcoves } A \textit{ such that } v_k \in \overline{A}\} = \left\{ w(A_0) \;\middle|\; w \in \widehat{W}_k \right\}.$$

Proof. For the first statement, recall that it is well known (cf. [4, Chapter 4]) that the stabilizer of any point in $\overline{A}_0$ is generated by the set of reflections across the alcove faces that contain the point. In particular, v_k lies on every face except F_k and the result follows. For the second statement, observe that any alcove A can be uniquely written as $A = w(A_0)$ for some $w \in \widehat{W}$. Since the vertices of $w(A_0)$ are $\{w(v_j) \mid 0 \leqslant j \leqslant n\}$, it follows that $v_k \in \overline{A}$ if and only if $v_k = w(v_j)$ for some j, $0 \leqslant j \leqslant n$. However, it is well known that $\overline{A}_0$ is a fundamental domain for the action of $\widehat{W}$. Therefore $v_k = w(v_j)$ if and only if $k = j$ if and only if $w \in \widehat{W}_k$ as desired. □

3.4. For $0 \leqslant k \leqslant n$, define

$$C_k = \bigcup_{w \in \widehat{W}_k} w(\overline{A}_0 \setminus F_k).$$

For $G = \mathrm{Sp}(2)$, the cells are shown in Fig. 2.

By Lemma 1 and construction, the following result is immediate.

Proposition 2.

(a) *C_k is an open neighborhood of v_k that is contractible to v_k via a straight line contraction.*
(b) *Each alcove wall having nonempty intersection with C_k contains v_k.*
(c) *Suppose $u_1, u_2 \in C_k$ satisfy $u_2 = w(u_1)$ for some $w \in \widehat{W}$. Then $v_k = w(v_k)$.*
(d) *$\overline{A}_0 \subseteq \bigcup_{k=0}^{n} C_k$.*

4. A cover of G

4.1. For $0 \leqslant k \leqslant n$, define

$$U_k = \{c_g(\exp t) \mid g \in G,\ t \in C_k\} \quad \text{and} \quad \mathcal{O}_k = \{c_g(\exp v_k) \mid g \in G\},$$

where $c_g(x) = gxg^{-1}$ for $g, x \in G$.

Theorem 3.

(a) *$\{U_k \mid 0 \leqslant k \leqslant n\}$ is an open cover of G.*
(b) *$\mathcal{O}_k$ is a deformation retract of U_k.*

Proof. Since $\exp(C_k)$ is open in T and since conjugation takes the exponential of the closure of an alcove onto G, part (a) is automatic. For part (b), we claim the deformation retract is given by $R_k : U_k \times I \to U_k$ where $I = [0, 1]$ and

$$R_k(c_g(\exp t), s) = c_g(\exp((1-s)t + sv_k)).$$

It remains to see that R_k is actually well defined.

Suppose $c_{g_1}(\exp t_1) = c_{g_2}(\exp t_2)$ for $g_j \in G$ and $t_j \in C_k$. Writing $c_{g_2^{-1}g_1}(\exp t_1) = \exp t_2$, there exists $h \in Z_G(\exp t_2)^0$ so that $\widetilde{w} = hg_2^{-1}g_1 \in N_G(T)$ (cf. [8, Section 6.4]). Let $\Sigma_{t_2} = \{\alpha \in \Delta \mid \alpha(t_2) \in 2\pi \mathbf{i}\mathbb{Z}\}$, i.e., the set of α for which t_2 lies on an α-alcove wall. Then $Z_G(\exp t_2)^0$ is the exponential of the direct sum of $\mathfrak{t}$ and all $\mathfrak{su}(2)$-triples corresponding to roots in Σ_{t_2}. Since v_k also lies on all such α-alcove walls, it follows that $h \in Z_G(\exp((1-s)t + sv_k))^0$.

Setting $w = \mathrm{Ad}_{\widetilde{w}} \in W$, we have $c_{\widetilde{w}}(\exp t_1) = \exp t_2$. Thus $\exp(wt_1) = \exp(t_2)$ so that $t_2 = wt_1 + z$ for some $z \in \ker(\exp|_{\mathfrak{t}})$. By Proposition 2, it follows that $v_k = wv_k + z$. Then

$$\begin{aligned} c_{g_1}(\exp((1-s)t_1 + sv_k)) &= c_{g_2h^{-1}\widetilde{w}}(\exp((1-s)t_1 + sv_k)) \\ &= c_{g_2h^{-1}}(\exp((1-s)wt_1 + swv_k)) \\ &= c_{g_2h^{-1}}(\exp((1-s)(t_2 - z) + s(v_k - z))) \\ &= c_{g_2h^{-1}}(\exp((1-s)t_2 + sv_k - z)) \\ &= c_{g_2}(\exp((1-s)t_2 + sv_k)) \end{aligned}$$

and we are finished. □

4.2. The results of the previous subsection give immediately the following main result.

Theorem 4.

$$\mathrm{cat}(G) + 1 \leqslant \sum_{k=0}^{n} (\mathrm{cat}_G(\mathcal{O}_k) + 1).$$

5. The orbits $\mathcal{O}_k$

We present some remarks and explicit realizations for the $\mathcal{O}_k$ in the classical cases.

5.1. $G = \mathrm{SU}(n+1)$

Trivial calculations show that

$$v_k = \frac{2\pi \mathbf{i}}{n+1}(\overbrace{n+1-k, \ldots, n+1-k}^{k}, -k, \ldots, -k)$$

for $0 \leqslant k \leqslant n$. Therefore $\exp v_k = e^{\frac{-2\pi \mathbf{i} k}{n+1}}\,\mathrm{Id}$. In particular, $\mathcal{O}_k = \{e^{\frac{-2\pi \mathbf{i} k}{n+1}}\,\mathrm{Id}\}$ and so $\mathrm{cat}(\mathcal{O}_k) = 0$ for all $0 \leqslant k \leqslant n$. Thus, Theorem 4 implies $\mathrm{cat}(\mathrm{SU}(n+1)) \leqslant n$, i.e., we recover Singhof's result [9].

5.2. $G = \mathrm{Sp}(n)$

Let $\mathbb{H}$ denote the division algebra of quaternions $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, $a, b, c, d \in \mathbb{R}$. View $\mathbb{H}^n$ as a right vector space and identify the set of quaternionic matrices, $M_n(\mathbb{H})$, with the set of $\mathbb{H}$-linear endomorphisms of $\mathbb{H}^n$ via standard matrix multiplication on the left. Write $\nu : M_n(\mathbb{H}) \to \mathbb{R}$ for the reduced norm. In particular, if $\varphi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ is the $\mathbb{C}$-linear injective homomorphism given by

$$\varphi(A + \mathbf{j}B) = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

for $A, B \in M_n(\mathbb{C})$, then $\nu = \det \circ \varphi$. We then realize $\mathrm{GL}(n, \mathbb{H}) = \{g \in M_n(\mathbb{H}) \mid \nu(g) \neq 0\}$, $\mathrm{SL}(n, \mathbb{H}) = \{g \in M_n(\mathbb{H}) \mid \nu(g) = 1\}$, and

$$G = \mathrm{Sp}(n) = \{g \in \mathrm{SL}(n, \mathbb{H}) \mid gg^* = I_n\},$$

where g^* denotes the quaternionic conjugate transpose of g. We also fix the maximal torus

$$T = \{\mathrm{diag}(e^{\mathbf{i}\theta_1}, \ldots, e^{\mathbf{i}\theta_n}) \mid \theta_j \in \mathbb{R}\}.$$

With this set-up, it is straightforward to check that

$$v_k = \mathbf{i}\pi\ \mathrm{diag}(\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0)$$

for $0 \leqslant k \leqslant n$. Therefore

$$\exp v_k = \begin{pmatrix} -I_k & \\ & I_{n-k} \end{pmatrix}.$$

In particular, $\mathcal{O}_0 = \{\mathrm{Id}\}$ and $\mathcal{O}_n = \{-\mathrm{Id}\}$ so that $\mathrm{cat}(\mathcal{O}_0) = \mathrm{cat}(\mathcal{O}_n) = 0$.

The other $\mathcal{O}_k$ require more work, though they are easy to identify. For this we realize the quaternionic Grassmannian of k-planes in $\mathbb{H}^n$, $Gr_k(\mathbb{H}^n)$, by $\{x \in M_{n\times k}(\mathbb{H}) \mid \mathrm{rk}(x) = k\}$ equipped with the equivalence relation $x \sim xh$ where $x \in M_{n\times k}(\mathbb{H}^n)$ and $h \in \mathrm{GL}(k, \mathbb{H})$. The following result is immediate.

Lemma 5. *Let $1 \leqslant k \leqslant n-1$ and set $d_k = \min\{k, n-k\}$. Then there is a diffeomorphism $\tau_k : \mathcal{O}_k \to Gr_{d_k}(\mathbb{H}^n)$,*

$$\mathcal{O}_k \cong \mathrm{Sp}(n)/(\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)) \cong Gr_{d_k}(\mathbb{H}^n),$$

given by

$$\tau_k(c_g(\exp v_k)) = g \begin{pmatrix} I_k \\ 0_{(n-k)\times k} \end{pmatrix}$$

when $d_k = k$ and by

$$\tau_k(c_g(\exp v_k)) = g \begin{pmatrix} 0_{k\times(n-k)} \\ I_{n-k} \end{pmatrix}$$

when $d_k = n-k$.

Conjecture 1. $\mathrm{cat}_{\mathrm{Sp}(n)}(\mathcal{O}_k) = d_k$.

As we observed already in the introduction, if the conjecture is true, then Theorem 4 quickly shows that

$$\mathrm{cat}(\mathrm{Sp}(n)) \leqslant \left\lfloor \frac{(n+2)^2}{4} \right\rfloor - 1.$$

In terms of trying to show that $\mathrm{cat}_{\mathrm{Sp}(n)}(\mathcal{O})_k \leqslant d_k$, there is an obvious choice of a cover of $\mathcal{O}_k$. For this, we introduce the following notation. For the sake of clarity, we assume we are in the case of $d_k = k$, i.e., $1 \leqslant k \leqslant n/2$.

For $1 \leqslant j \leqslant k+1$, write $x \in Gr_{k-1}(\mathbb{H}^{n-1})$ as

$$x = \begin{pmatrix} x_{j,1} \\ x_{j,2} \end{pmatrix}$$

with $x_{j,1} \in M_{(j-1)\times(k-1)}(\mathbb{H})$ and $x_{j,2} \in M_{(n-j)\times(k-1)}(\mathbb{H})$. Let $X_{j,k} \cong Gr_{k-1}(\mathbb{H}^{n-1}) \subseteq Gr_k(\mathbb{H}^n)$ be given by

$$\left\{ \begin{pmatrix} 0_{(j-1)\times 1} & x_{j,1} \\ 1 & 0_{1\times(k-1)} \\ 0_{(n-j)\times 1} & x_{j,2} \end{pmatrix} \,\middle|\, x \in Gr_{k-1}(\mathbb{H}^{n-1}) \right\}.$$

Write $y \in Gr_k(\mathbb{H}^{n-1})$ as

$$y = \begin{pmatrix} y_{j,1} \\ y_{j,2} \end{pmatrix}$$

with $y_{j,1} \in M_{(j-1)\times k}(\mathbb{H})$ and $y_{j,2} \in M_{(n-j)\times k}(\mathbb{H})$. Let $Y_{j,k} \cong Gr_k(\mathbb{H}^{n-1}) \subseteq Gr_k(\mathbb{H}^n)$ be given by

$$\left\{ \begin{pmatrix} y_{j,1} \\ 0_{1\times k} \\ y_{j,2} \end{pmatrix} \;\middle|\; y \in Gr_k\big(\mathbb{H}^{n-1}\big) \right\}.$$

Proposition 6.

(a) $\{Gr_k(\mathbb{H}^n) \setminus X_{j,k} \mid 1 \leqslant j \leqslant k+1\}$ *is an open cover of* $Gr_k(\mathbb{H}^n)$.
(b) $Y_{j,k}$ *is a deformation retract of* $Gr_k(\mathbb{H}^n) \setminus X_{j,k}$.
(c) *Written in* $(j-1) \times 1 \times (n-j)$ *block form,* $\tau_k^{-1}(Y_{j,k})$ *is*

$$\left\{ \begin{pmatrix} A & & B \\ & 1 & \\ C & & D \end{pmatrix} \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n-1) \textit{ and conjugate to } \exp v_{k-1,n-1} \right\}$$

where $v_{k,n} = \mathbf{i}\,\mathrm{diag}(\overbrace{\pi,\ldots,\pi}^{k}, \overbrace{0,\ldots,0}^{n-k})$.

Proof. For part (a), simply observe that a k-plane in $X_{1,k} \cap \cdots \cap X_{k+1,k}$ would have to contain $k+1$ independent vectors which is impossible. For part (b), observe that $Gr_k(\mathbb{H}^n) \setminus X_{j,k}$ is the set of

$$\begin{pmatrix} x_{(j-1)\times k} \\ y_{1\times k} \\ z_{(n-j)\times k} \end{pmatrix} \in Gr_k\big(\mathbb{H}^n\big) \quad \text{so that} \quad \begin{pmatrix} x_{(j-1)\times k} \\ z_{(n-j)\times k} \end{pmatrix} \in Gr_k\big(\mathbb{H}^{n-1}\big).$$

Therefore, the retraction $R : Gr_k(\mathbb{H}^n) \setminus X_{j,k} \times I \to X_{j,k}$ given by

$$R\left(\begin{pmatrix} x_{(j-1)\times k} \\ y_{1\times k} \\ z_{(n-j)\times k} \end{pmatrix}, s \right) = \begin{pmatrix} x_{(j-1)\times k} \\ (1-s)y_{1\times k} \\ z_{(n-j)\times k} \end{pmatrix}$$

does the trick. For part (c), observe that $\tau_k^{-1}(Y_{j,k})$ can be written in $(j-1) \times 1 \times (n-j)$ block form as

$$\left\{ g = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \zeta \\ \eta & \iota & \kappa \end{pmatrix} \in G \right\}.$$

Making note that $gg^* = I$, part (c) follows immediately by explicit matrix multiplication using $(j-1) \times 1 \times (k-j) \times (n-k)$ block form when $j \leqslant k$ and by using $k \times 1 \times (n-k-1)$ block form when $j = k+1$. □

Proposition 7. *If the sets* $\tau_k^{-1}(Y_{j,k})$ *are contractible in* $\mathrm{SL}(n,\mathbb{H})$*, then* $\mathrm{cat}_{\mathrm{Sp}(n)}(\mathcal{O}_k) \leqslant k$.

Proof. Let $F_1 : \tau_k^{-1}(Y_{j,k}) \times I \to \mathrm{SL}(n,\mathbb{H})$ be a contraction that takes $\tau_k^{-1}(Y_{j,k})$ to a point. Using the Cartan decomposition, there is a diffeomorphism $\mathrm{SL}(n,\mathbb{H}) \cong G \times \mathfrak{p}$ where $\mathfrak{p}$ is the -1 eigenspace of the Cartan involution corresponding to $\mathfrak{sp}(n)$, i.e., the involution given by $\theta(x) = -x^*$. For $g \in \mathrm{SL}(n,\mathbb{H})$, uniquely write $g = \kappa(g)\exp(\rho(g))$ with $\kappa(g) \in G$ and $\rho(g) \in \mathfrak{p}$. Finally, define $F_2 : \tau_k^{-1}(Y_{j,k}) \times I \to G$ by $F_2(g,s) = \kappa(F_1(g,s))$. By construction, F_2 contracts $\tau_k^{-1}(Y_{j,k})$ to a point. Thus, if the sets $\tau_k^{-1}(Y_{j,k})$ are contractible in $\mathrm{SL}(n,\mathbb{H})$ then they are also contractible in $G = \mathrm{Sp}(n)$. The proposition then follows from Proposition 6. □

At the present time, we do not know whether $\tau_k^{-1}(Y_{j,k})$ is contractible in $\mathrm{SL}(n,\mathbb{H})$. It is worth noting that a similar result can be obtained by showing that $\tau_k^{-1}(Y_{j,k})$ is contractible in $\mathrm{Sp}(2n,\mathbb{C})$. This too is unknown.

5.3. $G = \mathrm{Spin}(2n+1)$

Write the tensor algebra over $\mathbb{R}^m$ as $\mathcal{T}_m(\mathbb{R})$. Then the Clifford algebra is $\mathcal{C}_m(\mathbb{R}) = \mathcal{T}_m(\mathbb{R})/\mathcal{I}$ where $\mathcal{I}$ is the ideal of $\mathcal{T}_m(\mathbb{R})$ generated by $\{(x \otimes x + \|x\|^2) \mid x \in \mathbb{R}^m\}$. By way of notation for Clifford multiplication, write $x_1x_2\cdots x_k$ for the element $x_1 \otimes x_2 \otimes \cdots \otimes x_k + \mathcal{I} \in \mathcal{C}_m(\mathbb{R})$ where $x_1, x_2, \ldots, x_m \in \mathbb{R}^m$. Write $\mathcal{C}_m^+(\mathbb{R})$ for the subalgebra of $\mathcal{C}_m(\mathbb{R})$ spanned by all products of an even number of elements of $\mathbb{R}^m$. Conjugation, an anti-involution on $\mathcal{C}_m(\mathbb{R})$, is defined by $(x_1x_2\cdots x_k)^* = (-1)^k x_k\cdots x_2x_1$ for $x_i \in \mathbb{R}^m$.

Then

$$\mathrm{Spin}(m) = \{g \in \mathcal{C}_m^+(\mathbb{R}) \mid gg^* = 1 \text{ and } gxg^* \in \mathbb{R}^m \text{ for all } x \in \mathbb{R}^m\}.$$

In fact, it is the case that $\mathrm{Spin}(m) = \{x_1x_2\cdots x_{2k} \mid x_i \in S^{m-1}$ for $2 \leqslant 2k \leqslant 2m\}$. If we write $(\mathcal{A}g)x = gxg^*$ when $g \in \mathrm{Spin}(m)$ and $x \in \mathbb{R}^m$, then $\mathcal{A}$ gives the double cover of $\mathrm{SO}(m)$:

$$\{1\} \to \{\pm 1\} \to \mathrm{Spin}(m) \xrightarrow{\mathcal{A}} \mathrm{SO}(m) \to \{I_m\}.$$

A maximal torus T_0 for $\mathrm{SO}(2n+1)$ is given by

$$T_0 = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_n & \sin\theta_n & \\ & & & -\sin\theta_n & \cos\theta_n & \\ & & & & & 1 \end{pmatrix} \middle| \; \theta_i \in \mathbb{R} \right\}$$

with Lie algebra

$$\mathfrak{t}_0 = \left\{ \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_n & \\ & & & -\theta_n & 0 & \\ & & & & & 0 \end{pmatrix} \middle| \; \theta_i \in \mathbb{R} \right\}.$$

We write $\exp_{\mathrm{SO}(2n+1)}$ for the exponential map from $\mathfrak{t}_0$ onto T_0 and condense notation by writing E_k for the element of $\mathfrak{t}$ given by

$$E_k = \mathrm{blockdiag}\Bigg(\overbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}^{k}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\Bigg).$$

Writing e_k for the kth standard basis vector in $\mathbb{R}^n$, observe that $\mathcal{A}(\cos\theta - \sin\theta e_{2k-1}e_k)$ acts by the rotation $\left(\begin{smallmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{smallmatrix}\right)$ in the $e_{2k-1}e_k$ plane. It follows that

$$T = \{(\cos\theta_1 - \sin\theta_1 e_1e_2)\cdots(\cos\theta_n - \sin\theta_n e_{2n-1}e_{2n}) \mid \theta_k \in \mathbb{R}\}$$

is a maximal torus of $\mathrm{Spin}(2n+1)$. If we identify $\mathfrak{t}$ with the Lie algebra of T and write exp for the exponential map of $\mathrm{Spin}(2n+1)$ taking $\mathfrak{t}$ onto T, then $\exp_{\mathrm{SO}(n)} = \mathcal{A} \circ \exp$. It follows that

$$\exp(\theta E_k) = \big(\cos(\theta/2) - \sin(\theta/2)e_{2k-1}e_{2k}\big).$$

Using the definitions, it is straightforward to check that

$$v_0 = 0,$$
$$v_1 = 2\pi E_1,$$
$$v_k = \pi \sum_{j=1}^{k} E_j,$$

for $2 \leqslant k \leqslant n$. Therefore $\exp v_0 = 1$, $\exp v_1 = -1$, and $\exp v_k = (-1)^k \prod_{j=1}^{k} e_{2j-1}e_j$. Of course, $\mathcal{O}_0 = \{1\}$ and $\mathcal{O}_1 = \{-1\}$ so $\mathrm{cat}(\mathcal{O}_0) = \mathrm{cat}(\mathcal{O}_1) = 0$.

The other orbits are easy to describe, though calculating $\mathrm{cat}_G(\mathcal{O}_k)$ is not easy.

Proposition 8. *For $2 \leqslant k \leqslant n$,*

$$\mathcal{O}_k \cong \mathrm{Spin}(2k)/\mathrm{Spin}(2k)\,\mathrm{Spin}(2n+1-2k)$$
$$\cong \mathrm{SO}(2n+1)/\big(\mathrm{SO}(2k) \times \mathrm{SO}(2n+1-2k)\big) \cong \widetilde{Gr_{2k}}\big(\mathbb{R}^{2n+1}\big),$$

the Grassmannian of oriented $2k$-planes in $\mathbb{R}^{2n+1}$.

Proof. Since $\mathcal{A}(\exp v_k) = \begin{pmatrix} -I_{2k} & \\ & I_{n-2k} \end{pmatrix}$,

$$\mathcal{A}(\mathcal{O}_k) \cong \mathrm{SO}(2n+1)/S\big(\mathrm{O}(2k) \times \mathrm{O}(2n+1-2k)\big) \cong Gr_{2k}\big(\mathbb{R}^{2n+1}\big),$$

the Grassmannian of $2k$-planes in $\mathbb{R}^{2n+1}$. Moreover, $\mathcal{A} : \mathcal{O}_k \to \mathcal{A}(\mathcal{O}_k)$ is a double cover. To see this, observe that there is a Weyl group (isomorphic to $S_n \ltimes \mathbb{Z}_2^n$) element taking v_k to $-\pi E_1 + \pi \sum_{j=2}^{k} E_j$ which exponentiates to $-\exp v_k$.

To prove the proposition, first observe that the stabilizer of $\exp v_k$ under conjugation must be contained in $S = \mathcal{A}^{-1}(S(\mathrm{O}(2k) \times \mathrm{O}(2n+1-2k))) = S(\mathrm{Pin}(2k)\,\mathrm{Pin}(2n+1-2k))$. Since $\mathrm{Pin}(2k) \cap \mathrm{Pin}(2n+1-2k) \subseteq \mathbb{R}$, it follows that the connected component of the identity of S is $S_0 = \mathrm{Spin}(2k)\,\mathrm{Spin}(n-2k) \cong \mathrm{Spin}(2k) \times \mathrm{Spin}(n-2k)/\{\pm(1,1)\}$ and the other component is diffeomorphic to $\mathrm{Pin}(2k)_1 \times \mathrm{Pin}(2n+1-2k)_1$ where $\mathrm{Pin}(j)_1$ is the non-identity component of $\mathrm{Pin}(j)$. Recalling that the center of $\mathrm{Spin}(2k)$ is $\{\pm 1, \pm \exp v_k\}$, it follows that S_0 is contained in the stabilizer of $\exp v_k$. However, $\mathrm{Pin}(2k)_1$ anticommutes with $\exp v_k$ while $\mathrm{Pin}(2n+1-2k)_1$ commutes. Therefore, the stabilizer of $\exp v_k$ is S_0. Finally, since $S_0 = \mathcal{A}^{-1}(\mathrm{SO}(2k) \times \mathrm{SO}(n-2k))$, the proof is complete. □

The relative cat calculation of $\mathcal{O}_k$ in $\mathrm{Spin}(2n+1)$ is not known.

5.4. $G = \mathrm{Spin}(2n)$

A maximal torus T_0 for $\mathrm{SO}(2n)$ is given by

$$T_0 = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_n & \sin\theta_n \\ & & & -\sin\theta_n & \cos\theta_n \end{pmatrix} \,\middle|\, \theta_i \in \mathbb{R} \right\}$$

with Lie algebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix} \,\middle|\, \theta_i \in \mathbb{R} \right\}.$$

As before, write

$$E_k = \mathrm{blockdiag}\Bigg(\overbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}^{k}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\Bigg).$$

From the definitions, it is straightforward to check that

$$\begin{aligned} v_0 &= 0, \\ v_1 &= 2\pi E_1, \\ v_k &= \pi \sum_{j=1}^{k} E_j, \\ v_{n-1} &= \pi \sum_{j=1}^{n-1} E_j - \pi E_n, \end{aligned}$$

for $2 \leqslant k \leqslant n$, $k \neq n-1$. Therefore $\exp v_0 = 1$, $\exp v_1 = -1$, $\exp v_k = (-1)^k \prod_{j=1}^{k} e_{2j-1} e_j$, and $\exp v_{n-1} = (-1)^{n-1} \times \prod_{j=1}^{n} e_{2j-1} e_j$. Of course, $\mathcal{O}_0 = \{1\}$ and $\mathcal{O}_1 = \{-1\}$ so $\mathrm{cat}(\mathcal{O}_0) = \mathrm{cat}(\mathcal{O}_1) = 0$. As in Proposition 8, the remaining conjugacy classes are

$$\begin{aligned} \mathcal{O}_k &\cong \mathrm{Spin}(2k)/\,\mathrm{Spin}_{2k}(\mathbb{R})\,\mathrm{Spin}(2n-2k) \\ &\cong \mathrm{SO}(2n)/\,\mathrm{SO}(2k) \times \mathrm{SO}(2n-2k) \cong \widetilde{Gr_{2k}}\big(\mathbb{R}^{2n}\big), \end{aligned}$$

the Grassmannian of oriented $2k$-planes in $\mathbb{R}^{2n}$. Again, the relative category in $\mathrm{Spin}(2n)$ is not known.

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