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Continuity of the Fenchel Correspondence and Continuity of Polarities

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The polar cone of the limit inferior of a sequence of cones of a Banach space is shown to be the Cesari's limit superior of the sequence of polar cones in the bounded weak* topology. In a similar way the lower semicontinuity of a not necessarily countable family of closed convex cones is characterized in terms of Castaing's notion of pseudo-upper semicontinuity. More general results are given within the framework of convergence spaces and the pseudo-upper semicontinuity of a closed-valued multifunction is characterized as closedness with respect to a new convergence. Applications to the Young–Fenchel correspondence are pointed out.
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It is a well-known fact that if a vector subspace of a Hilbert space X depends continuously on a parameter w then its orthogonal subspace depends continuously on w. Here the continuous dependence on the parameter can be taken in several senses corresponding to various topologies on the set of orthogonal projectors of X; in finite dimensions these topologies coincide. In the infinite-dimensional case more care is needed and a convergence introduced by U. Mosco [27] has been proved to be of fundamental importance (see [2, 25, 28, 35, 39] for instance). Here we extend this convergence to a nonsequential setting. This is not made just for the sake of generality. In several problems one must consider functions f depending on two variables (w, x) and the (partial) Legendre–Fenchel transform f_w^* of $f_w: x \to f(w, x)$ depends on the parameter w which appears to be in a space topologized by a nonmetrizable weak topology

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(see [3] for instance). The interest of using topologies instead of sequential convergences also appears when one deals with the convergence of convex sets.

Here our main purpose consists in showing that a simple result about continuous dependence of polar cones can serve to prove in a simple unified geometric way several results in convex analysis and variational convergence. This idea is certainly not new: in [25] it is attributed to Rockafellar (see [33] in this connection); it is worked out in the pioneering works of Wijsman [40, 41] and Walkup and Wets [38], where it is used for a different kind of convergence not considered here; see also [9, 16, 39]. Here we deal systematically with the geometrical constructions involved in this process as they are considered to be of general use. For instance an application to the study of the approximate subdifferential of a convex function relying on the results of this paper will appear elsewhere.

The fact that we deal with convergences instead of topologies should not obscure the simplicity of the results concerning the operations of taking conical hulls and intersections with hyperplanes which are the main ingredients of the constructions we use. In order to keep close to the usual topological framework we use nets instead of filters so that in a first reading of Section 3 one may suppose the spaces are just topological spaces. The required notions about convergences are recalled in Section 1 where an interpretation of pseudo-upper semicontinuity in terms of closedness with respect to an appropriate convergence is presented. This is not the sole motivation for dealing with convergences instead of topologies: the theory of distributions, the study of order structures and differentiability theory [14] show that convergence structures can be very convenient. In measure theory a.e. convergence is quite usual; more recently convergence with respect to a function (written $x' \xrightarrow{f} x$) appears to be an indispensable tool in nonsmooth analysis. The continuity of the Fenchel–Legendre–Young correspondence is dealt with in Section 4.

After a lecture in Limoges about the results of this paper it was kindly pointed out to one of the authors by S. Dolecki that close connections exist with his results [11] and with the paper by Back [5] and the thesis of Joly [18]. Although these two authors remain in the sequential case they do use a similar convergence. Moreover they compare this convergence with the familiar sequential Mosco convergence; their methods are analytical and quite different. They specialize their results to some classes of topological vector space (tvs) as metrizable spaces and barreled spaces, a topic not discussed here as our general thrust is directed toward (not necessarily reflexive) Banach spaces although we also deal with pairs of tvs in duality. In particular we show the usefulness of the classical bounded weak* topology [12, 15] in variational convergence, a fact which seems to deserve some attention.

1. Characterization of Pseudo-Upper Semicontinuity

Recall that a multifunction $F: W \to X$ between two topological spaces is *upper semicontinuous* (usc) at $w_0 \in W$ if for each open subset V of X containing $F(w_0)$ there exists a neighborhood U of w_0 such that $F(U) := \bigcup_{u \in U} F(u) \subset V$. It has been recognized for several decades that this is a very restrictive condition. Therefore some authors prefer to use closedness at w_0, where F is said to be *closed* at w_0 if for any $x_0 \in X \backslash F(w_0)$ there exist neighborhoods U of w_0, V of x_0 in W such that $F(U) \cap V = \varnothing$. From the very definition of the limit superior as

$$\limsup_{w \to w_0} F(w) := \bigcap_{U \in \mathcal{N}(w_0)} \overline{F(U)},$$

where $\mathcal{N}(w_0)$ is the set of neighborhoods of w_0 and $\bar{A}$ is the closure of A in X, we have that F is closed at w_0 iff $\limsup_{w \to w_0} F(w) = F(w_0)$.

Here we use the following weakening of upper semicontinuity.

1.1. Definition (see [6]). Let W and X be two topological spaces and let $\mathcal{K}$ be a subfamily of the family $\mathcal{K}_X$ of compact subsets of X. A multifunction $F: W \to X$ is said to be $\mathcal{K}$-usc at $w_0 \in W$ if for each $K \in \mathcal{K}$ the multifunction $F_K: W \to X$ given by $F_K(w) = F(w) \cap K$ is usc at w_0. When $\mathcal{K} = \mathcal{K}_X$, F is said to be pseudo-upper continuous at w_0 (puc at w_0).

Obviously F is $\mathcal{K}$-usc at w_0 iff F is usc at w_0 for the topology on X generated by the subbase $\mathcal{O}(\mathcal{K}) = \{X \backslash K : K \in \mathcal{K}\}$; when $\mathcal{K}$ is stable under finite unions and arbitrary intersections, $\mathcal{O}(\mathcal{K})$ itself is a topology.

For X locally compact and $\mathcal{K} = \mathcal{K}_X$ this topology has been introduced by Fell [13]. As we intend to apply the preceding notion to the case X is a dual vector space endowed with its weak* topology, it is of interest to dispose of choices for $\mathcal{K}$ other than $\mathcal{K}_X$ itself. For instance one may take convex members of $\mathcal{K}_X$, bounded members of $\mathcal{K}_X$, or closed equicontinuous subsets of X. When X is the dual of a Banach space X_*, endowed with its weak* topology these distinctions are irrelevant; this is not so when X is just a normed vector space (nvs) or a locally convex topological vector space (lcs).

The following results show the interest of the notion introduced above. Their (simple) proofs are left to the reader for the sake of brevity.

1.2. Lemma. *If a multifunction $F: W \to X$ is usc at w_0 or closed at w_0 then for any subfamily $\mathcal{K}$ of $\mathcal{K}_X$, F is $\mathcal{K}$-usc at w_0. Conversely if $\mathcal{K}$ contains the family of compact metrizable subsets of X and if F is $\mathcal{K}$-usc at w_0 then F is sequentially closed at w_0.*

1.3. Lemma. *Suppose d is a distance on X for which the closed balls belong to a subfamily $\mathscr{K}$ of $\mathscr{K}_X$. Then, if $F: W \to X$ is $\mathscr{K}$-usc at w_0, for each $x \in X$ the function $w \to d(x, F(w)) := \inf\{d(x, x') : x' \in F(w)\}$ is lsc at w_0.*

1.4. Lemma (compare with [7, Theorem II.20]). *Suppose X is a lcs endowed with its weak topology. A multifunction $F: W \to X$ such that $F(w_0)$ is closed and convex is puc at w_0 iff for any continuous linear functional x^* on X and any $K \in \mathscr{K}_X$ the function $w \to h(x^*, F(w) \cap K) := \sup\{\langle x^*, x\rangle : x \in F(w) \cap K\}$ is usc at w_0.*

We intend to give an interpretation of pseudo-upper semicontinuity in terms of the notion of limit superior for some convergence. Some preliminaries are in order.

In several important instances in analysis the use of topologies is awkward or impossible whereas simple and natural convergences such as test function and distribution convergences, order convergences, a.e. convergence, continuous convergence on spaces of mappings, and convergence of subsets of a topological space are at hand. Here we mainly use a variant of the weak and weak* convergence which seems to be more convenient than the bounded weak star convergence.

The axioms usually adopted for defining a convergence space use filters [8, 10, 14], and hence may be of little attractiveness to analysists. Here we introduce an equivalent definition in terms of nets similar to the axioms of $\mathscr{L}$-spaces introduced and used by Urysohn [37], Kuratowski [23], and Kisynski [21].

Let us recall that a *net* N on a set X is a mapping $N: I \to X$ from a directed set $(I, \leqslant)$ into X; recall that a set I is directed by a preorder $\leqslant$ on I (i.e., a transitive, reflexive relation) if for any i, j in I there exists $k \in I$ with $i \leqslant k$, $j \leqslant k$, so that $\mathscr{I} = \{I_j : j \in I\}$ is a filter base in I, where $I_j = \{i \in I : i \geqslant j\}$.

A *subnet* of N is a net $P: J \to X$ such that for each $i \in I$ there exists $j \in J$ with $P(J_j) \subset N(I_i)$ (i.e., the filtered family associated to P is finer than the filtered family associated to N in the sense that its image filter base is finer than the image of $\mathscr{I}$ by I). This notion introduced by Aarnes and Andenaes [1] is less restrictive than the more usual concept of strict subnet. A net $P: J \to X$ is a *strict subnet* of N if there exists a filtering mapping $S: J \to I$ with $P = N \circ S$, where S is said to be *filtering* if for each $i \in I$ there exists $j \in J$ with $S(J_j) \subset I_i$. As shown below each subnet P of N has a subnet Q which is a strict subnet of N, so that for our purposes these two notions can be used equivalently.

For simplicity we consider only convergences with uniqueness of limits, as defined by the following axioms.

1.5. DEFINITION. A convergence c on a set X is a relation between nets of X and points of X denoted by $(x_i)_{i \in I} \xrightarrow{c} x$ verifying the following rules:

(C_1) the constant net with value x converges to x;

(C_2) if $P = (x_j)_{j \in J}$ is a subnet of a net $N = (x_i)_{i \in I}$ with $(x_i)_{i \in I} \xrightarrow{c} x$ then $(x_j)_{j \in J} \xrightarrow{c} x$;

(C_3) if $x \in X$ and a net $N = (x_i)_{i \in I}$ of X are such that any subnet

$$P = (x_j)_{j \in J} \text{ of } N \text{ has a subnet } Q = (x_k)_{k \in K} \xrightarrow{c} x \text{ then } (x_i)_{i \in I} \xrightarrow{c} x.$$

It follows from (C_2) that the convergence of a net $N: I \to X$ in fact depends on its image filter base $\mathscr{B}_N = \{N(I_i) : i \in I\}$: for if P and Q are two equivalent nets (in this sense that P (resp. Q) is a subnet of Q (resp. P) or if the filters generated by $\mathscr{B}_P$ and $\mathscr{B}_Q$ coincide) then P converges iff Q converges.

Given two convergence spaces (X, c_X) and (Y, c_Y), a mapping $f: X \to Y$ is said to be *continuous* at $x \in X$ if $f \circ N$ converges to $f(x)$ whenever N is a net with limit x in X; f is said to be continuous if f is continuous at each point of X. It is well known that when the convergences c_X and c_Y are convergences associated with topologies, this notion reduces to usual continuity. A *convergential vector space* is a vector space endowed with a convergence for which the vector operations are continuous.

With any convergence c on a set X one can associate a topology $\tau(c)$: a subset C of X is declared to be closed if it contains the limits of its converging nets. The convergence $c(\tau(c))$ associated with $\tau(c)$ is coarser than c (i.e., $\mathrm{Id} : (X, c) \to (X, c(\tau(c)))$ is continuous); if c is the convergence associated with a topology σ on X then $\tau(c) = \sigma$ and $c(\tau(c)) = c$.

Given a topological space (X, σ) (or even a convergence space) and a covering $\mathscr{K}$ of X we define a new convergence $c = c(\mathscr{K})$ on X by setting $(x_i)_{i \in I} \xrightarrow{c} x$ iff $(x_i)_{i \in I} \xrightarrow{\sigma} x$ and for each cofinal subset H of I there exists a cofinal subset J of H and an element K of $\mathscr{K}$ such that $x_j \in K$ for each $j \in J$. The axioms of convergence spaces are readily verified. In general the convergence c associated to σ and $\mathscr{K}$ is finer than the convergence of σ and the topology $\tau(c)$ corresponding to c is finer than σ.

When $\mathscr{K}$ is a family of compact subsets, the topology $\tau(c)$ is easy to describe and corresponds to a familiar construction. When (X, σ) is a k-space (as a metric space or a locally compact space) and $\mathscr{K} = \mathscr{K}_X$, $\tau(\sigma)$ coincides with σ by what follows.

1.6. LEMMA. *Given a subfamily $\mathscr{K}$ of the family $\mathscr{K}_X$ of compact subsets of a topological space (X, σ) the topology $\tau(c)$ associated with the convergence c defined above is the inductive topology associated with the family $\{K : K \in \mathscr{K}\}$ of subspaces of X, i.e., the strongest topology on X inducing on the members of $\mathscr{K}$ their relative topology $\sigma \mid K$.*

Proof. It suffices to prove that the closed subsets of $(X, \tau(c))$ are the subsets A of X such that $A \cap K$ is closed in $(K, \sigma \mid K)$ for each $K \in \mathscr{K}$.

Obviously any subset A of X verifying this property is closed for $\tau(c)$. Conversely if A is closed for $\tau(c)$ and if $K \in \mathscr{K}$, for any converging net $(x_i)_{i \in I}$ contained in $A \cap K$, $(x_i)_{i \in I}$ converges in c; hence its limit belongs to A and K, so that $A \cap K$ is closed in $(K, \sigma \mid K)$. ∎

In particular when (X, σ) is a weak*-dual Banach space the topology associated with the preceding convergence, with $\mathscr{K} = \mathscr{K}_X$ or with the family of bounded subsets of X, is the classical bounded weak* topology $\beta(\sigma^*)$ ([12, 15] for instance). Note however that the preceding result does not mean that c is the convergence associated with this topology, although c and this topology have the same converging sequences.

We are now ready for the characterization of pseudo-upper semicontinuity.

1.7. PROPOSITION. *Let c be the convergence on X associated with $\mathscr{K}$ as above.*

(a) *If $F: W \to X$ is closed at w_0 for c then F is $\mathscr{K}$-usc at w_0 and $F(w_0) \cap K$ is closed for each $K \in \mathscr{K}$.*

(b) *Conversely if (X, σ) is Hausdorff, $F(w_0) \cap K$ is closed for each $K \in \mathscr{K}$ and F is $\mathscr{K}$-usc at w_0 then F is closed at w_0 for c.*

(c) *Suppose that $\mathscr{K}$ is hereditary in the sense that for any $K \in \mathscr{K}$ any closed subset of K is in $\mathscr{K}$. Then F is $\mathscr{K}$-usc at w_0 for c iff F is continuous at w_0 as a mapping from W into the space 2^X of subsets of X endowed with the topology generated by the family $\{S(X \backslash K) : K \in \mathscr{K}\}$, where $S(G) = \{A \in 2^X : A \subset G\}$.*

Proof. (a) Suppose $\limsup_{w \to w_0} F(w) \subset F(w_0)$. Let $K \in \mathscr{K}$ and let G be an open subset of (X, σ) containing $F(w_0) \cap K$. If, for each neighborhood V of w_0 in W, $F(V) \cap K$ meets $X \backslash G$ we can find a net $(w_i)_{i \in I}$ in W and a net $(x_i)_{i \in I}$ in $X \backslash G$ with $x_i \in F(w_i) \cap K$ for each $i \in I$. As K is compact we can find a converging subnet $(x_j)_{j \in J}$ in $(K, \sigma \mid K)$. Then $(x_j)_{j \in J}$ converges in c and its limit x_0 belongs to $F(w_0)$ by our assumption, and in $X \backslash G$ which is closed for σ. This is a contradiction; therefore there exists a neighborhood V of w_0 with $F(V) \cap K \subset G$ and F is $\mathscr{K}$-usc at w_0. The closedness of $F(w_0) \cap K$ for each $K \in \mathscr{K}$ is obvious.

(b) It suffices to prove that if $(w_i)_{i \in I}$ is a net with limit w_0 in W and if $(x_i)_{i \in I}$ is a net with c-limit x_0 in X with $x_i \in F(w_i)$ for each $i \in I$ then $x_0 \in F(w_0)$. Taking subnets if necessary we may assume $(x_i)_{i \in I}$ is contained in some $K \in \mathscr{K}$. As $(K, \sigma \mid K)$ is Hausdorff, $F_K(w_0) = F(w_0) \cap K$ is closed in K and F_K is usc at w_0, F_K is closed at w_0. Hence $x_0 \in F_K(w_0) \subset F(w_0)$.

(c) Let us suppose F is $\mathscr{K}$-usc and let $K \in \mathscr{K}$ be such that $F(w_0) \in S(X \setminus K)$ or $F(w_0) \cap K = \emptyset$. Then, as F_K is usc we have $F_K(w) = \emptyset$ or $F(w) \in S(X \setminus K)$ for w close to w_0. Conversely if F is continuous for the topology generated by the sets $S(X \setminus K)$ for $K \in \mathscr{K}$, for any open subset G of X with $F(w_0) \cap K \subset G$ we have $F(w_0) \subset G \cup (X \setminus K) = X \setminus ((X \setminus G) \cap K)$. As $K' = (X \setminus G) \cap K \in \mathscr{K}$ we have $F(w) \subset X \setminus K'$, hence $F(w) \cap K \subset G$, for w close to w_0. ∎

Now let us define convergence structures on the set $\mathscr{P}(X)$ of subsets of a convergence space (X, c).

It is convenient to call a net $(x_j)_{j \in J}$ of (X, c) *subordinated* to a net $(F_i)_{i \in I}$ of $\mathscr{P}(X)$ if for each $i_0 \in I$ there exists $j_0 \in J$ such that

$$\{x_j : j \geqslant j_0\} \subset \bigcup_{i \geqslant i_0} F_i.$$

When the F_i's are singletons, $(x_j)_{j \in J}$ is subordinated to $(F_i)_{i \in I}$ iff $(x_j)_{j \in J}$ is a subnet of $(f_i)_{i \in I}$, where $F_i = \{f_i\}$. When there exists a filtering mapping $p: J \to I$ such that $x_j \in F_{p(j)}$ for each $j \in J$ the net $(x_j)_{j \in J}$ is said to be strongly subordinated to $(F_i)_{i \in I}$. This is the most usual way of getting subordinated nets.

1.8. Definition. Let (X, c) be a convergence space and let $(F_i)_{i \in I}$ be a net of $\mathscr{P}(X)$. The limit superior of $(F_i)_{i \in I}$ is the set $\limsup_{i \in I} F_i$ of limits of converging nets $(x_j)_{j \in J}$ which are subordinate to $(F_i)_{i \in I}$.

In the preceding definition one could use strongly subordinated nets instead of subordinated nets without changing $\limsup_{i \in I} F_i$. This follows from the fact that any net $(x_j)_{j \in J}$ subordinated to $(F_i)_{i \in I}$ has a subnet $(x_k)_{k \in K}$ which is strongly subordinated to $(F_i)_{i \in I}$, taking K as

$$K = \{(i, j) \in I \times J : x_j \in F_i\},$$

with the product preorder, and $p: (i, j) \to i$. It is easy to see that when c is the convergence structure associated with a topology σ the preceding definition of $\limsup_{i \in I} F_i$ coincides with the usual one,

$$x \in \limsup_{i \in I} F_i \Leftrightarrow \forall V \in \mathscr{N}_\sigma(x) \qquad \forall i \in I, \exists k \in I, k \geqslant i, V \cap F_k \neq \emptyset,$$

$\mathscr{N}_\sigma(x)$ denoting the filter of neighborhoods of x in (X, σ).

1.9. Definition. The limit inferior of a net $F: I \to \mathscr{P}(X)$ of subsets of a convergence space (X, c) is the set $\liminf_{i \in I} F_i$ of $x \in X$ such that for each cofinal subset J of I there exists a net $(x_k)_{k \in K}$ subordinated to $(F_j)_{j \in J}$ which converges to x,

$$\liminf_{i \in I} F_i = \bigcap_{J \in \mathscr{G}(I)} \limsup_{j \in J} F_j$$

where $\mathscr{G}(I)$, the grill of $(I, \leqslant)$, is the family of all cofinal subsets of I.

The preceding definitions can be extended to families of subsets $(F_w)_{w \in W}$ of (X, c) parametrized by a topological space W. Here and in the sequel we suppose W is a subspace of a topological space Ω and ω is a point in $\Omega \setminus W$. We set $W^\circ = W \cup \{\omega\}$. In this case we define $\limsup_{w \to \omega} F(w)$ (standing for $\limsup_{w \to \omega, w \in W} F(w)$) as the set of $x \in X$ such that $x \in \limsup_{i \in I} F(w_i)$ for some net $(w_i)_{i \in I}$ in W with limit ω. When $W^\circ = I^\circ$, where $(I, \leqslant)$ is a directed set, with the topology generated by $\{I_i \cup \{\omega\} : i \in I\}$ we recover the preceding definition, as shown by the discussion following Definition 1.8. We define $\liminf_{w \to \omega} F(w)$ as the set of $x \in W$ such that for each net $(w_i)_{i \in I}$ in W with limit ω, x belongs to $\liminf_{i \in I} F(w_i)$. We call F: $W \to X$ lower semicontinuous at $w_0 \in W$ if $F(w_0) \subset \liminf_{w \to w_0} F(w)$.

2. Continuity of Polarities

In the sequel w_0 is a point of a topological space W embedded in some topological space Ω; we set $W^\circ = W \cup \{\omega\}$ with $\omega \in \Omega \setminus W$ and we keep our preceding convention about $\lim_{w \to \omega}$. We denote by (X, Y) a pair of topological vector spaces in duality. Unless otherwise stated the topology σ of X is the weak topology $\sigma(X, Y)$; the topology τ of Y is stronger than the weak topology $\sigma(Y, X)$. We denote by $\mathscr{K}$ a subfamily of the family $\mathscr{K}_X$ of weakly compact subsets of X. It generates a new convergence c on X as described before Lemma 1.6; it also gives rise to the topology on Y of uniform convergence on the members of $\mathscr{K}$ we call the $\mathscr{K}$-topology. The family $\mathscr{K}_c$ of circled convex elements of $\mathscr{K}$ is said to be *saturating* in $\mathscr{K}$ if for any K, L in $\mathscr{K}$ there exists $M \in \mathscr{K}_c$ with $M \supset K$, $M \supset L$.

Let Q: $W \to Y$ be a multifunction whose values are cones in Y; we denote by P: $W \to X$ the multifunction given by $P(w) = Q(w)^\circ$, where the polar S° of a subset S of Y is given by

$$S^\circ = \{x \in X : \forall y \in S \ \langle x, y \rangle \leqslant 1\},$$

so that S° coincides with $S := \{x \in Y : \forall y \in S \ \langle x, y \rangle \leqslant 0\}$ when S is a cone in Y. The following simple result is the key point of our study. Although it does not seem to have appeared in the following form several weaker versions of it are known (see [17, 30] for instance).

2.1. THEOREM. (a) *Suppose the topology* τ *on* Y *is stronger than the* $\mathscr{K}$*-topology. Then if* Q *is lsc at* w_0 *its polar multifunction* $P = Q^\circ$ *is* $\mathscr{K}$*-usc at* w_0.

(b) *Suppose* τ *is weaker than the* $\mathscr{K}$*-topology and* $\mathscr{K}_c$ *is saturating in* $\mathscr{K}$. *Suppose the values of* Q *are closed convex cones. Then, if* $P = Q^\circ$ *is* $\mathscr{K}$*-usc at* w_0, Q *is lsc at* w_0.

Taking into account the characterization of lower semicontinuity of Q at w_0 by $\liminf_{w \to w_0} Q(w) \supset Q(w_0)$ and the characterization of $\mathscr{K}$-upper semicontinuity of P obtained in Proposition 1.7 the preceding result is a consequence of the following theorem in which the closed convex hull of A in (X, σ) is denoted by $\overline{\text{co}}(A)$.

2.2. THEOREM. (a) *Suppose* τ *is stronger than the* $\mathscr{K}$*-topology on* Y. *Then, for any cone-valued multifunction* $Q: W \to Y$

$$c\text{-}\limsup_{w \to \omega} Q(w)^\circ \subset (\liminf_{w \to \omega} Q(w))^\circ.$$

(b) *Suppose* τ *is weaker than the* $\mathscr{K}$*-topology on* Y *and* $\mathscr{K}_c$ *is saturating in* $\mathscr{K}$. *Then for any multifunction* $P: W \to X$ *whose values are closed convex cones*

$$(c\text{-}\limsup_{w \to \omega} P(w))^\circ \subset \liminf_{w \to \omega} P(w)^\circ,$$

(c) *When* τ *is the* $\mathscr{K}$*-topology on* Y *and* $\mathscr{K}_c$ *is saturating in* $\mathscr{K}$, *for any cone-valued multifunction* $Q: W \to Y$ *and* $P = Q^\circ$

$$\overline{\text{co}}(c\text{-}\limsup_{w \to \omega} Q(w)) = (\liminf_{w \to \omega} Q(w))^\circ$$

$$(c\text{-}\limsup_{w \to \omega} P(w))^\circ = \liminf_{w \to \omega} P(w)^\circ.$$

Let us observe that parts (a) and (b) of Theorem 2.2 are in fact just rephrasings of the corresponding assertions of Theorem 2.1: extending P and Q to $W \cup \{\omega\}$ by setting $\hat{Q}(\omega) = \liminf_{w \to \omega} Q(\omega)$, $\hat{P}(\omega) = \overline{\text{co}}(\limsup_{w \to \omega} P(\omega))$ we see that the corresponding assumptions and conclusions about P and Q on one hand, and $\hat{P}$ and $\hat{Q}$ on the other hand, correspond. Now the first equality of Theorem 2.2(c) is a consequence of parts (a) and (b), taking polars in (b) so that

$$c\text{-}\limsup_{w \to \omega} P(w) \subset (\liminf_{w \to \omega} Q(w))^\circ \subset (c\text{-}\limsup_{w \to \omega} P(w))^{\circ\circ}$$

since for $A = c - \limsup_{w \to \omega} P(w)$ we have $A^{\circ\circ} = \overline{\text{co}}(A)$. The second one follows similarly by replacing Q by P° in (a) and taking polars there, as $P(w)^{\circ\circ}=P(w)$:

$$(c\text{-}\limsup_{w\to\omega} P(w))^\circ \subset \liminf_{w\to\omega} P(w)^\circ \subset (\liminf_{w\to\omega} P(w)^\circ)^{\circ\circ} \subset (c\text{-}\limsup_{w\to\omega} P(w))^\circ.$$

Proof. (a) Let us suppose τ is stronger than the $\mathscr{K}$-topology on Y and let $\hat{x}\in c\text{-}\limsup_{w\to\omega} P(w)$, with $P(w)=Q(w)^\circ$: there exist nets $(w_i)_{i\in I}\to\omega$ in W, $(x_i)_{i\in I}\xrightarrow{c}\hat{x}$ in (X, c) with $x_i\in P(w_i)$ for each $i\in I$. Taking a subnet if necessary we may suppose $(x_i)_{i\in I}$ is contained in some $K\in\mathscr{K}$. Now for any $\hat{y}\in\liminf Q(w)$ we can find a subnet $(w_j)_{j\in J}$ of $(w_i)_{i\in I}$ and a net $(y_j)_{j\in J}$ with limit $\hat{y}$ such that $y_j\in Q(w_j)$ for each $j\in J$. As the topology τ is stronger than uniform convergence on K we get $\lim_{j\in J}\langle x_j, y_j-\hat{y}\rangle=0$ and

$$\langle \hat{x}, \hat{y}\rangle=\lim_{j\in J}\langle x_j, \hat{y}\rangle=\lim\langle x_j, y_j\rangle\leqslant 0.$$

This proves that $\hat{x}\in(\liminf_{w\to\omega} Q(w))^\circ$.

(b) Setting $Q(w)=P(w)^\circ$ we have $P(w)=Q(w)^\circ$ as $P(w)$ is a closed convex cone in (X,σ). Let $\hat{y}\in Y\backslash\liminf_{w\to\omega} Q(w)$; we want to prove that $\hat{y}\in Y\backslash(c\text{-}\limsup_{w\to\omega} P(w))^\circ$. By assumption there exists a neighborhood V of 0 in (Y,τ) and a net $(w_i)_{i\in I}$ in W with limit ω such that $Q(w_i)\cap(\hat{y}-V)=\varnothing$ for each $i\in I$. As τ is weaker than the $\mathscr{K}$-topology we may suppose

$$V=\{y\in Y : \sup_{x\in K_1}|\langle y, x\rangle|\leqslant\varepsilon, ..., \sup_{x\in K_1}|\langle y, x\rangle|\leqslant\varepsilon\}$$

for some $\varepsilon>0$ and some $K_1, ..., K_n$ in $\mathscr{K}$. As $\mathscr{K}_c$ is saturating in $\mathscr{K}$ we can find $K\in\mathscr{K}_c$ with $K_1\cup\cdots\cup K_n\subset K$ so that $U:=(\varepsilon^{-1}K)^\circ$ is contained in V. Thus the $\mathscr{K}_c$-topology coincides with the $\mathscr{K}$-topology and is stronger than τ. The Mackey–Arens theorem guaranteeing that X is the dual of (Y,τ), using the Hahn–Banach theorem, for each $i\in I$ we can find $x_i\in X$ and $r_i\in\mathbb{R}$ with

$$\langle x_i, y\rangle\leqslant r_i<\langle x_i, \hat{y}-v\rangle \qquad \text{for any} \quad y\in Q(w_i),\ v\in V.$$

As $Q(w_i)$ is a cone we have $r_i\geqslant 0$, hence $\langle x_i, v\rangle<s_i:=\langle x_i,\hat{y}\rangle$ for each $i\in I$. Thus $s_i>0$, and $\hat{x}_i:=\varepsilon s_i^{-1}x_i\in\varepsilon V^\circ\subset\varepsilon U^\circ=K$. As K is compact in (X,σ) we may suppose without loss of generality that $(\hat{x}_i)_{i\in I}$ converges in K, hence in c. By assumption its limit $\hat{x}$ belongs to $c\text{-}\limsup_{w\to\omega} P(w)$ as $\hat{x}_i=\varepsilon s_i^{-1}x_i\in P(w_i)$ for each $i\in I$. Now $\langle\hat{x}_i,\hat{y}\rangle=\varepsilon s_i^{-1}\langle x_i,\hat{y}\rangle=\varepsilon$, hence $\langle\hat{x},\hat{y}\rangle=\varepsilon$, so that $\hat{y}\in Y\backslash(c\text{-}\limsup_{w\to\omega} P(w))^\circ$. ∎

2.3. COROLLARY. *Let (X,σ) be the weak* dual of a lcs space (Y,τ) and let $P: W\to X$, $Q: W\to Y$ be multifunctions whose values are closed convex* *cones. Then if $\mathscr{E}$ is the family of closed convex circled equicontinuous subsets of (X, σ), P is $\mathscr{E}$-usc iff Q is lsc.*

This follows from the facts that τ coincides with the $\mathscr{E}$-topology, the elements of $\mathscr{E}$ are compact in (X, σ), and $\mathscr{E}_c$ is saturating in $\mathscr{E}$.

2.4. COROLLARY. *Let (X, Y) be a pair of vector spaces in duality, X being endowed with the weak topology $\sigma(X, Y)$ and Y with the Mackey topology $\tau(Y, X)$. Suppose that any weakly compact subset of X is contained in a convex weakly compact subset of X. Let $P: W \to X$ and $Q: W \to Y$ be multifunctions whose values at $w \in W$ are mutually polar convex cones. Then P is pseudo-usc at w_0 iff Q is lsc at w_0.*

Proof. This follows from the fact that the Mackey topology in this case coincides with the $\mathscr{K}_X$-topology. ∎

We are especially interested in the following case.

2.5. COROLLARY. *Let X be the weak* dual of a Banach space Y. Then if $P: W \to X$ and $Q: W \to Y$ are multifunctions whose values are mutually polar cones, P is pseudo-ucs at w_0 iff Q is lsc at w_0.*

By our previous results this is equivalent to c-closedness of P at w_0; here this property can be written in the following way: for any net $(w_i)_{i \in I}$ with limit w_0 in W, for any weak*-convergent bounded net $(x_i)_{i \in I}$ with $x_i \in F(w_i)$ for each $i \in I$ $\lim x_i \in P(w_0)$.

For a sequence of weak* closed convex cones we can replace c-closedness by bw^*-closedness. Although the converging sequences for c are exactly the converging sequences for the bw^*-topology this fact does not follow immediately from what precedes.

2.6. THEOREM. *Suppose X is the dual space of a Banach space Y and (Q_n) is a sequence of cones in Y. Then, endowing X with the bw^*-topology, and Y with its strong topology*

$$\overline{\text{co}}(bw^*\text{-}\limsup_n Q_n^\circ) = (\liminf_n Q_n)^\circ$$

$$(bw^*\text{-}\limsup_n Q_n^\circ)^\circ = \liminf_n Q_n^{\circ\circ}.$$

Proof. The second equality is a consequence of the first one in which Q_n is replaced by $Q_n^{\circ\circ}$, taking polars and using the fact that the limit inferior of a sequence of closed convex cones in Y is a closed convex cone, and hence coincides with its bipolar.

Since c-convergence implies bw^*-convergence we have

$$\overline{\text{co}}(bw^*\text{-}\limsup_n Q_n^\circ) \supset \overline{\text{co}}(c\text{-}\limsup_n Q_n^\circ) = (\liminf Q_n)^\circ$$

by Theorem 2.2(c). Suppose there exists $\hat{x} \in bw^*\text{-}\limsup Q_n^\circ$ with $\hat{x} \in X \backslash (\liminf Q_n)^\circ$. Then we can find $\hat{y} \in \liminf Q_n$ and $r > 0$ with $\langle \hat{x}, y \rangle \geqslant 3r$. Let (y_n) be a sequence with limit $\hat{y}$ such that for some $m \in \mathbb{N}$ and any $n \geqslant m$, $y_n \in Q_n$. Let

$$K = \{0, r^{-1}y\} \cup \{r^{-1}(y_n - \hat{y}) : n \in \mathbb{N}\}.$$

By [12] or [15] K° is a neighborhood of 0 in the bw^*-topology on X. Let $k \geqslant m$ be such that $\|\hat{x}\| \, \|y_n - \hat{y}\| < r$ for $n \geqslant k$.

We can find $n \geqslant k$ such that $(\hat{x} - K^\circ) \cap Q_n^\circ \neq \emptyset$. Let $x_n \in (\hat{x} - K^\circ) \cap Q_n^\circ$, so that $\langle x_n - \hat{x}, y_n - \hat{y} \rangle \geqslant -r$, $\langle x_n - \hat{x}, \hat{y} \rangle \geqslant -r$,

$$\langle x_n, y_n \rangle = \langle x_n - \hat{x}, y_n - \hat{y} \rangle + \langle x_n - \hat{x}, \hat{y} \rangle + \langle \hat{x}, y_n - \hat{y} \rangle + \langle \hat{x}, \hat{y} \rangle < 0,$$

a contradiction with $x_n \in Q_n^\circ$. Since $(\liminf Q_n)^\circ$ is closed convex we get $(\liminf Q_n)^\circ \supset \overline{\text{co}}(bw^*\text{-}\limsup Q_n)$ and equality holds. ∎

Given a topology ρ on a set X and a convergence c on X a multifunction $P: W \to X$ is said to *converge* as $w \to \omega$ if

$$c\text{-}\limsup_{w \to \omega} P(w) \subset \liminf_{w \to \omega} P(w),$$

where the limit inferior is taken with respect to ρ. This happens when $\omega \in W$ and P is lsc at ω and c-closed at ω. This definition is a natural extension of Mosco's convergence [27]. Its full interest occurs when one can guarantee that

$$\liminf_{w \to \omega} P(w) \subset c\text{-}\limsup_{w \to \omega} P(w)$$

since then there exists a unique subset between these two limits.

When c is the convergence associated with a topology σ on X weaker than ρ and a family $\mathscr{K}$ of compact subsets of (X, σ) this occurs under each one of the following assumptions:

(a) Any point of (X, ρ) has a neighborhood belonging to $\mathscr{K}$.

(b) Any compact subset of (X, ρ) is contained in some element of $\mathscr{K}$, ρ is metrizable, and the filter of neighborhoods of ω has a countable basis.

Assumption (a) is satisfied when (X, σ) is a dual Banach space with its weak* topology and its norm topology ρ.

2.7. PROPOSITION. *Suppose (X, Y) is a dual pair, X and Y being endowed with topologies ρ and τ, respectively, which are stronger than the weak topologies and provided with families $\mathscr{K}$ and $\mathscr{L}$ of weakly compact sets and the associated convergences c. Suppose ρ is stronger than the $\mathscr{L}$-topology, τ*

is weaker than the $\mathscr{K}$-topology, and $\mathscr{K}_c$ is saturating in $\mathscr{K}$. If $P: W \to X$, $Q: W \to Y$ are multifunctions whose values are mutually polar convex cones and if P converges as $w \to \omega$ then Q converges as $w \to \omega$.

Proof. Using Theorem 2.2(a) with the roles of $(X, \rho, \mathscr{K}, P)$ and $(Y, \tau, \mathscr{L}, Q)$ interchanged we have

$$c\text{-}\limsup_{w \to \omega} Q(w) \subset (\liminf_{w \to \omega} P(w))^\circ;$$

using our assumption on P and Theorem 2.2(b) we get

$$(\liminf_{w \to \omega} P(w))^\circ \subset (c\text{-}\limsup_{w \to \omega} P(w))^\circ \subset \liminf_{w \to \omega} Q(w). \quad \blacksquare$$

2.8. COROLLARY. *Let (X, Y) be a dual pair, X and Y being endowed with the Mackey topologies and provided with the families $\mathscr{M}_X$ and $\mathscr{M}_Y$ of convex weakly compact subsets of X and Y. Then for mutually polar multifunctions $P: W \to X$, $Q: W \to Y$ with closed convex cones values P converges as $w \to \omega$ iff Q converges as $w \to \omega$.*

Thus the convergences we introduced lead to a completely symmetric situation even when one deals with nonreflexive Banach spaces.

3. CONTINUITY OF INTERSECTIONS AND CONICAL HULLS

It is easy to see that when $F: W \to X$ is a multifunction and H is a subset of X the multifunction $G: W \to X$ given by $G(w) = F(w) \cap H$ is not necessarily lower semicontinuous when F is lower semicontinuous. However, we need only a special case of this situation. For the general case see [4, 31].

3.1. LEMMA. *Let (E, c) be a convergence vector space and let $\hat{E} = E \times \mathbb{R}$, $H = E \times \{1\}$. Let $Q: W \to \hat{E}$ be a cone-valued multifunction and let $C: W \to E$ be given by $C(w) \times \{1\} = Q(w) \cap H$. Then for any $w_0 \in W$*

(a) *if* $\limsup_{w \to w_0} Q(w) \subset Q(w_0)$ *then* $\limsup_{w \to w_0} C(w) \subset C(w_0)$;

(b) *if* $\limsup_{w \to w_0} Q(w) \supset Q(w_0)$ *then* $\limsup_{w \to w_0} C(w) \supset C(w_0)$;

(c) *if* $Q(w_0) \subset \liminf_{w \to w_0} Q(w)$ *then* $C(w_0) \subset \liminf_{w \to w_0} C(w)$;

(d) *if* $Q(w_0) \supset \liminf_{w \to w_0} Q(w)$ *then* $C(w_0) \supset \liminf_{w \to w_0} C(w)$.

Proof. Assertion (a) is trivial. Let us show assertion (b). Given $x \in C(w_0)$ we have $(x, 1) \in Q(w_0)$; hence there exists a net $(w_i)_{i \in I}$ in W with limit w_0 and a net $(\hat{x}_i)_{i \in I}$ in $\hat{E}$ with limit $(x, 1)$ such that $\hat{x}_i \in Q(w_i)$ for each $i \in I$. Writing $\hat{x}_i = (x_i, r_i)$ we have $(r_i) \to 1$, hence $r_i > 0$ for i large enough and $x = \lim_i r_i^{-1} x_i$ with $r_i^{-1} x_i \in C(w_i)$ for i large enough, so that $x \in \limsup_{w \to w_0} C(w)$. The proof of assertions (c) and (d) is similar. ∎

3.2. PROPOSITION. *Let H be a closed affine hyperplane of a tvs X. Suppose $0 \in X \backslash H$. Let $Q: W \to X$ be a cone-valued multifunction and let $C: W \to X$ be given by $C(w) = Q(w) \cap H$. Then for any $w_0 \in W$*

(a) *if Q is closed at w_0 then C is closed at w_0;*

(b) *if Q is lower semicontinuous at w_0 then C is lower semicontinuous at w_0.*

Proof. If $h \in H$ and if H_0 denotes the hyperplane $H_0 = H - h$ parallel to H and passing through the origin, we can identify X with $H_0 \times \mathbb{R}$, H with $H_0 \times \{1\}$ and the result follows from Lemma 3.1. ∎

Let us now tackle the inverse process. Namely, given a multifunction $F: W \to X$, under what conditions can we transfer continuity properties of F to continuity properties of the multifunction $\hat{F}: W \to X$ given by $\hat{F}(w) = \mathbb{R}_+ F(w)$, the conical hull of $F(w)$? This problem was considered in [9, 16, 40, 41] in the finite-dimensional case and our results are only slight extensions of the results of [16, 41]. Lower continuity properties are easily transferred.

3.3. PROPOSITION. *If $F: W \to X$ is a multifunction from W into a convergence vector space (X, c), if $\hat{F}: W \to X$ is given by $\hat{F}(w) = \mathbb{R}_+ F(w)$, and if $F(w_0) \subset \liminf_{w \to w_0} F(w)$ then*

$$\hat{F}(w_0) \subset \liminf_{w \to w_0} \hat{F}(w).$$

Proof. Let $\hat{x} \in \hat{F}(w_0)$. Either $\hat{x} = 0$ and then $\hat{x} \in \liminf_{w \to w_0} \hat{F}(w)$, else $\hat{x} = r\bar{x}$ with $r > 0$, $\bar{x} \in F(w_0)$. Then, by assumption, for each net $(w_i)_{i \in I}$ in W with limit w_0 and each cofinal subset J of I we can find a subnet $(w_k)_{k \in K}$ of $(w_j)_{j \in J}$ and a net $(x_k)_{k \in K}$ in X with $(x_k)_{k \in K} \xrightarrow{c} \bar{x}$ and $x_k \in F(w_k)$ for each $k \in K$. Then $(rx_k) \xrightarrow{c} r\bar{x} = \hat{x}$ and $\hat{x} \in \liminf_{w \to w_0} \hat{F}(w)$. ∎

We cannot expect to transfer so easily closure properties from F to $\hat{F}$. In fact, as a multifunction closed at a point w_0 has a closed value at this point, the relation $\hat{F}$ may be nonclosed at w_0 even if F is closed at w_0 and constant. For instance, taking W arbitrary, $X = \mathbb{R}^2$,

$$F(w) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geqslant 1,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$$

we have $\hat{F}(w) = \{(0, 0)\} \cup]0, +\infty[\times]0, +\infty[$ and $\hat{F}$ cannot by closed. However, if we add to $\hat{F}$ the recession cone of $F(w)$ we get a closed multifunction. This prompts us to introduce the following definition.

3.4. DEFINITION. The asymptotic cone $K_\infty C$ to a subset C of a convergence vector space (X, c) is the set of vectors $v \in X$ such that $(t_i^{-1}x_i) \xrightarrow{c} v$ for some net $(t_i)_{i\in I}$ in $]0, +\infty[$ with limit $+\infty$ and some net $(x_i)_{i\in I}$ of C:

$$K_\infty C = \limsup_{t\to\infty} t^{-1}C.$$

It is easy to see that if $\hat{C} = \mathbb{R}_+(C \times \{1\})$ is the cone of $\hat{X} = X \times \mathbb{R}$ generated by $C \times \{1\}$ then

$$K_\infty C \times \{0\} = \text{cl}(\hat{C}) \cap X \times \{0\},$$

where cl A denotes the closure of a subset A of $\hat{X}$. In fact

$$\text{cl}(\hat{C}) = \mathbb{R}_+(\text{cl}\, C \times \{1\}) \cup K_\infty C \times \{0\}.$$

Our notation stems from the fact that when X is a Banach space and when $\sigma: X \to S$ denotes the stereographic projection of X into the unit sphere of $\hat{X} = X \times \mathbb{R}$, $K_\infty C \times \{0\}$ is the contingent cone to the image of C by σ at $(0, 1)$.

3.5. *Remark.* When C is closed and star shaped with respect to some point $x_0 \in C$ (i.e., $x_0 + [0, 1](C - x_0) \subset C$) it is easy to see that $K_\infty C$ coincides with the recession cone to C

$$0^+C = \{v \in X : \forall r \in \mathbb{R}_+ \; x_0 + rv \in C\}.$$

As is well known, this cone does not depend on the choice of x_0 in st C, the set of points with respect to which C is starshaped.

The following lemma is the key step for the following more concrete criteria.

3.6. LEMMA. *Suppose $F: W \to X$ is a multifunction with values in a convergence vector space X. Suppose*

(a) $0 \in X \backslash \limsup_{w\to w_0} F(w)$;

(b) $\limsup_{w\to w_0} F(w) \subset F(w_0)$

(c) $\limsup_{(t,w)\to(+\infty, w_0)} t^{-1}F(w) \subset K_\infty F(w_0)$.

Then $\limsup_{w\to w_0} \hat{F}(w) \subset \mathbb{R}_+ F(w_0) \cup K_\infty F(w_0)$.

Proof. Let $\hat{x} \in \limsup_{w\to w_0} \hat{F}(w)$ and let $(w_i)_{i\in I} \to w_0$, $(\hat{x}_i)_{i\in I} \to \hat{x}$ with $\hat{x}_i \in \hat{F}(w_i)$ for each $i \in I$. Let us write $\hat{x}_i = r_i x_i$ with $r_i \in \mathbb{R}_+$, $x_i \in F(w_i)$. Taking a subnet if necessary, we may suppose $(r_i)_{i\in I}$ has a limit r in $[0, +\infty]$. We cannot have $r = +\infty$ as otherwise we would have $0 = \lim r_i^{-1}\hat{x}_i = \lim_{i\in I} x_i$, a contradiction with assumption (a). If $r \in]0, +\infty[$ we get that $(x_i) = (r_i^{-1}\hat{x}_i) \to r^{-1}\hat{x}$, hence $\hat{x} \in \mathbb{R}_+ F(w_0)$. Finally if $r = 0$, taking $t_i = r_i^{-1}$ we get by assumption (c) that $\hat{x} = \lim t_i^{-1} x_i \in K_\infty F(w_0)$. ∎

Let us give an instance in which closedness can be transferred from F to $\hat{F}$ or rather to F^h given by

$$F^h(w) := \mathbb{R}_+ F(w) \cup K_\infty F(w).$$

Note that when C is a closed subset of X not containing 0, $C^h = \mathbb{R}_+ C \cup K_\infty C$ is the closed cone generated by C.

3.7. Proposition. *Suppose $F: W \to X$ is a multifunction with closed convex values in a convergence vector space X. Suppose*

(a) $0 \in X \backslash \limsup_{w \to w_0} F(w)$;

(b) *F is closed at w_0;*

(c) *for each net $(w_i)_{i \in I}$ with limit w_0 in W, $\limsup_{i \in I} F(w_i)$ is nonempty.*

Then the multifunction F^h defined above is closed at w_0.

Assumption (c) is satisfied if $\liminf_{w \to w_0} F(w)$ is nonempty. It is also satisfied if F is subcompact at w_0. Using the the terminology of [29], (c) can be rephrased as: F is lsc at (w_0, X).

Let us consider first the asymptotic part of the preceding result.

3.8. Lemma. *With the assumptions and the notations of the preceding proposition one has*

$$\limsup_{w \to w_0} K_\infty F(w) \subset K_\infty F(w_0).$$

Proof. Let $v \in \limsup_{w \to w_0} K_\infty F(w)$: we can write $v = \lim v_i$ with $v_i \in K_\infty F(w_i)$, where $(w_i)_{i \in I}$ is a net in W with limit w_0. Using assumption (c) we can find a subnet $(w_j)_{j \in J}$ of $(w_i)_{i \in I}$, a subnet $(v_j)_{j \in J}$ of $(v_i)_{i \in I}$, and a converging net $(z_j)_{j \in J}$ with $z_j \in F(w_j)$ for each $j \in J$. Let $z = \lim_{j \in J} z_j$. Then for each $j \in J$, using Remark 3.5 and the fact that $F(w_j)$ is closed convex, we have for any $r \in \mathbb{R}$, $z_j + rv_j \in F(w_j)$. Thus $z + rv = \lim_j (z_j + rv_j) \in F(w_0)$ by assumption (b). This proves that $v = \lim_n n^{-1}(z + nv)$ belongs to $K_\infty F(w_0)$. ∎

3.9. Lemma. *Let $F: W \to X$ be a multifunction such that* $\limsup_{w \to w_0} F(w) \subset F(w_0)$. *Suppose $(t_j)_{j \in J} \to +\infty$ in $\mathbb{R}_+$, $(w_j)_{j \in J} \to w_0$ in W, $(\hat{x}_j)_{j \in J} \to \hat{x}$ with $x_j := t_j \hat{x}_j \in F(w_j)$ for each $j \in J$ and $\hat{x} \neq 0$. Let $z \in X$ be such that for each*

$r \in \mathbb{R}_+$ there exists a cofinal subset K of J, a net $(z_k)_{k \in K}$ with limit z in X, a net $(s_k)_{k \in K}$ with limit $s \in [r, +\infty[$ in $\mathbb{R}$ with

$$(1 - t_k^{-1} s_k)\, z_k + t_k^{-1} s_k x_k \in F(w_k).$$

Then $\hat{x} \in K_\infty F(w_0)$.

When F has convex values we can take $s_j = r$ for each $j \in J$ and set $K = \{k \in J : rt_j^{-1} \leqslant 1\}$, a cofinal subset of J. Then the assumption $(1 - t_k^{-1} s_k)\, z_k + t_k^{-1} s_k x_k \in F(w_k)$ is fulfilled if $z_k \in F(w_k)$.

Proof. Let us prove that $\hat{x} \in K_\infty F(w_0)$ by showing that for each $r \in \mathbb{R}_+$ there exists $s \geqslant r$ with $z + s\hat{x} \in F(w_0)$. We take K and $(s_k)_{k \in K}$ as in the statement. Then $s = \lim s_k$ belongs to $[r, +\infty[$ and

$$y_k := (1 - t_k^{-1} s_k)\, z_k + t_k^{-1} s_k x_k \to z + s\hat{x}.$$

As $y_k \in F(w_k)$ we get that $z + s\hat{x} \in \limsup_{k \in K} F(w_k) \subset F(w_0)$. ∎

End of the proof of Proposition 3.7. Using assumption (c) of Proposition 3.7 we see that if $(t_i)_{i \in I}$, $(w_i)_{i \in I}$, $(\hat{x}_i)_{i \in I}$ are nets with limits $+\infty$, w_0, x, respectively, such that $t_j \hat{x}_j \in F(w_j)$, we can find subnets $(t_j)_{j \in J}$, $(\hat{x}_j)_{j \in J}$, and a net $(z_j)_{j \in J}$ with limit $z \in \limsup_{i \in I} F(w_i)$, $z_j \in F(w_j)$ for each $j \in J$. Applying the preceding lemma with $s_k = r$ for k in the cofinal subset K of $j \in J$ with $t_j \geqslant r$ we conclude that condition (c) of Lemma 3.6 is fulfilled whenever $\hat{x}$ is nonnull while the conclusion $\hat{x} \in K_\infty F(w_0)$ is trivial if $\hat{x} = 0$.

3.10. *Remark.* We gave Lemmas 3.6 and 3.9 as separate statements although they seem to be rather technical. The reason lies in the fact that they can be applied to nonconvex situations. Even more nonconvexities can be considered with the following variant of Lemma 3.9.

3.11. LEMMA. *Let X be a normed vector space endowed with a convergence c weaker than the norm convergence. Let $F: W \to X$ be a multifunction such that $\limsup_{w \to w_0} F(w) \subset F(w_0)$. Suppose $(t_i)_{i \in I} \to +\infty$ in $\mathbb{R}_+$, $(w_i)_{i \in I} \to w_0$ in W, $x_i \in F(w_i)$ with $(t_i^{-1} x_i) \to \hat{x} \neq 0$. Suppose that there exists $m > 0$ such that for each $r \in \mathbb{R}_+$ there exists a cofinal subset J of I, a c-converging net $(z_j)_{j \in J}$ with $|z| \leqslant m$ for $z = \lim_{j \in J} z_j$, and a converging net $(s_j)_{j \in J}$ in $[r, \infty[$ such that*

$$(1 - s_j t_j^{-1})\, z_j + s_j t_j^{-1} x_j \in F(w_j) \qquad \textit{for each} \quad j \in J.$$

Then $\hat{x} \in K_\infty F(w_0)$.

Proof. Let us show that for each $r \in \mathbb{R}_+$ we can find $z_r \in X$ with $|z_r| \leqslant m$ and $s_r \in [r, +\infty[$ such that $y_r := z_r + s_r \hat{x} \in F(w_0)$. Then we have $\hat{x} = s_r^{-1}(y_r - z_z)$, $y_r \in F(w_0)$, and as $|s_r^{-1} z_r| \leqslant r^{-1} m \to 0$ as $r \to \infty$ we also have $(s_r^{-1} z_r) \xrightarrow{c} 0$, $(s_r^{-1} y_r) \xrightarrow{c} \hat{x}$, so that $\hat{x} \in K_\infty F(w_0)$.

Given $(t_i)_{i\in I}$, $(w_i)_{i\in I}$, $(x_i)_{i\in I}$ as in the statement and $r\in\mathbb{R}_+$ we consider the nets $(z_j)_{j\in J}$ and $(s_j)_{j\in J}$ and we denote by z_r and s_r their respective limits. By assumption $|z_r|\leqslant m$, $s_r\geqslant r$ and $y_j\in F(w_j)$ for $y_j=(1-s_jt_j^{-1})z_j+s_jt_j^{-1}x_j$. Then $(y_j)\to z_r+s_r\hat{x}$, so that $z_r+s_r\hat{x}\in F(w_0)$. ∎

3.12. EXAMPLE. Let $X=\mathbb{R}^2$, $W=[0,1]$, $F\colon W\to X$ being given by

$$F(w)=\{x=(x^1,x^2)\in X : x^1\geqslant 0,\ x^2\geqslant 0,\ x^1w\leqslant 1,\ x^2w\leqslant 1,\ x^1x^2=1\}.$$

Clearly $\lim_{w\to 0_+} F(w)=F(0)$. Let us check the assumptions of the preceding lemma. Given $(t_i)_{i\in I}\to+\infty$, $(w_i)_{i\in I}\to 0$, $x_i\in F(w_i)$ with $(t_i^{-1}x_i)\to\hat{x}=(u,v)\neq(0,0)$ we may suppose $u>0$, $v=0$ (the case $u=0$, $v>0$ being analogous). Then we take $s_i=s=\max(r,1)$ and, for $(u_i,v_i)=t_i^{-1}x_i$,

$$z_i=(0,(1-st_i^{-1})^{-1}(s^{-1}u_i^{-1}-sv_i)).$$

Then we have $(z_i)\to(0,s^{-1}u^{-1})$ and we can take $m=u^{-1}$ since

$$(1-st_i^{-1})z_i+st_i^{-1}x_i\in F(w_i).$$

3.13. COROLLARY. *Let (X,σ) and (Y,τ) be a pair of tvs in duality and let c be the convergence on X associated with σ and a family $\mathscr{K}$ of compact subsets of (X,σ) such that $\mathscr{K}_c$ is saturating in $\mathscr{K}$ and τ is weaker than the $\mathscr{K}$-topology. Let $F\colon W\to X$ be a closed convex-valued multifunction such that*

(a) *F is closed at w_0;*

(b) *for each net $(w_i)_{i\in I}$ with limit w_0 in W, $\limsup_{i\in I}F(w_i)$ is nonempty.*

Then the polar multifunction F° is lower semicontinuous at w_0.

Proof. We have

$$F^\circ(w)=\{y\in Y:\forall x\in F(w)\ \langle x,y\rangle\leqslant 1\}$$
$$=\{y\in Y:(y,1)\in[(F(w)\times\{1\})^h]^\circ\},$$

where $X\times\mathbb{R}$ and $Y\times\mathbb{R}$ are paired through the coupling functional $\langle(x,r),(y,s)\rangle=\langle x,y\rangle-rs$. Proposition 3.7 shows that P given by $P(w)=(F(w)\times\{1\})^h$ is closed at w_0. It follows from Theorem 2.2(b) and Proposition 3.2(b) that F° is lsc at w_0. ∎

3.14. PROPOSITION. *Suppose (X,Y) is a dual pair as in Proposition 2.7. If $F\colon W\to X$ is a closed convex-valued multifunction which converges as $w\to w_0$, then $G\colon W\to Y$ given by $G(w)=F(w)^\circ$ converges as $w\to w_0$.*

Proof. This follows from Proposition 2.7, setting $P(w)=(F(w)\times\{1\})^h$, $Q(w)=P(w)^\circ$ and taking Lemma 3.1 into account. ∎

4. Application to the Fenchel Conjugacy

In this section we consider the following problem. Given a net $(f_i)_{i \in I}$ of closed nonimproper convex functions on X, to what extent can we assert that the net $(f_i^*)_{i \in I}$ of Fenchel conjugates converges if the net $(f_i)_{i \in I}$ converges? This problem has been considered by many authors and is treated in detail in the book [2] in the case of sequences. Here we consider the case of a family of functions $(f_w)_{w \in W}$ parametrized by a topological space W embedded in some topological space Ω with $\omega \in \operatorname{cl} W$ in Ω (this point of view is equivalent to the point of view of nets or filters of functions).

Using epigraphs of functions one can define natural and important concepts of convergence for functions on a topological space X, or more generally on a convergence space (X, c). Recall that the epigraph of $f: X \to \bar{\mathbb{R}}$ is the set

$$E(f) = E_f = \{(x, r) \in X \times \mathbb{R} : f(x) \leqslant r\}.$$

4.1. Definition. The epilimit superior of the family $(f_w)_{w \in W}$ of functions on the convergence space (X, c) is the function f denoted by $\operatorname{els}_{w \in \omega} f_w$ whose epigraph is the limit inferior of the epigraphs of f_w as $w \to \omega$:

$$E(\operatorname*{els}_{w \to \omega} f_w) = \liminf_{w \to \omega} E(f_w).$$

The epilimit inferior of $(f_w)_{w \in W}$ is the function denoted by $\operatorname{eli}_{w \to \omega} f_w$ whose epigraph is the limit superior of the epigraphs of f_w as $w \to \omega$:

$$E(\operatorname*{eli}_{w \to \omega} f_w) = \limsup_{w \to \omega} E(f_w).$$

Both limits can be characterized in terms of (f_w) and nets, and when X is a topological space, in terms of neighborhoods (see [2]) but we do not use these characterizations here.

Let us recall that the conjugate functional associated $f: X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is the functional $f^*: Y \to \bar{\mathbb{R}}$ given by

$$f^*(y) = \sup_{x \in X} [\langle x, y \rangle - f(x)].$$

A similar definition holds for $g : Y \to \bar{\mathbb{R}}$. In the sequel we suppose f is *nonimproper* (i.e., f is not identically $+\infty$ and does not take the value $-\infty$), so that f^* is nonimproper too.

Our study relies on the following geometrical interpretation of this conjugacy correspondence. Let us associate to $f: X \to \mathbb{R}^\bullet = \mathbb{R} \cup \{+\infty\}$ the cone generated by $E(f) \times \{1\}$ in $X \times \mathbb{R} \times \mathbb{R}$:

$$P(f) = \mathbb{R}_+ (E(f) \times \{1\}).$$

This set is the image by the involution S: $(x, r, s) \to (x, s, r)$ of the epigraph of the mapping $\hat{f}$: $X \times \mathbb{R} \to \mathbb{R}^{\bullet}$ obtained by homogenizing f: $\hat{f}(0, 0) = 0$ and $\hat{f}(x, r) = rf(r^{-1}x)$ for $(x, r) \in X \times]0, +\infty[$, $\hat{f}(x, r) = +\infty$ for $(x, r) \in X \times]-\infty, 0]$ with $(x, r) \neq (0, 0)$. Similarly, if h: $Y \to \mathbb{R}^{\circ}$ we set

$$Q(g) = \mathbb{R}_+(E(g) \times \{1\}).$$

Now let us define a coupling functional $\langle\ ,\ \rangle$ on $X \times \mathbb{R}^2 \times Y \times \mathbb{R}^2$ by

$$\langle (x, r, s), (y, t, u) \rangle = \langle x, y \rangle - ru - st.$$

This coupling functional is obviously strongly nondegenerated. It yields a striking interpretation of the conjugacy correspondence similar to [33, Theorem 14.4]. We include a proof for completeness as the arguments of the proof of [33, Theorem 14.4] are valid for finite-dimensional spaces only.

4.2. LEMMA. *For any* $f: X \to \bar{\mathbb{R}}$ *one has*

$$E(f^*) \times \{1\} = P(f)^{\circ} \cap Y \times \mathbb{R} \times \{1\}.$$

If moreover f *is convex lsc and nonimproper then* $P(f)^{\circ}$ *is the closure of the convex cone* $Q(f^*)$ *generated by* $E(f^*) \times \{1\}$.

Proof. By our definitions

$$\begin{aligned} E(f^*) &= \{(y, t) \in Y \times \mathbb{R} : \forall x \in X\ t \geqslant \langle x, y \rangle - f(x)\} \\ &= \{(y, t) \in Y \times \mathbb{R} : \forall (x, s) \in E(f)\ s + t \geqslant \langle x, y \rangle\} \\ &= \{(y, t) \in Y \times \mathbb{R} : \forall (x, s) \in E(f)\ \langle (x, s, 1), (y, t, 1) \rangle \leqslant 0\}. \end{aligned}$$

Therefore $(y, t) \in E(f^*)$ iff $(y, t, 1) \in P(f)^{\circ}$.

Now let us suppose that f is convex lsc and nonimproper. Let us first show that $(0, -1, 0)$ does not belong to cl $P(f)$. Otherwise we could find a net $((x_i, s_i, t_i))_{i \in I}$ in $P(f)$ with limit $(0, -1, 0)$, and, as $P(f)$ is a convex cone, for any $r \in \mathbb{R}_+$ and any $(x_0, s_0) \in E(f)$ we would have

$$(x_0, s_0, 1) + r(x_i, s_i, t_i) = (1 + rt_i)\left(\frac{x_0 + rx_i}{1 + rt_i}, \frac{s_0 + rs_i}{1 + rt_i}, 1\right) \in P(f)$$

for i so large that $1 + rt_i > 0$

$$(x_0, s_0 - r) = \lim\left(\frac{x_0 + rx_i}{1 + rt_i}, \frac{s_0 + rs_i}{1 + rt_i}\right) \in E(f)$$

as $E(f)$ is closed. As r is arbitrary in $\mathbb{R}_+$ this would imply $f(x_0) = -\infty$, a contradiction.

As $(0, -1, 0)$ does not belong to cl $P(f)$ we can apply the Hahn–Banach theorem which yields $(y, t, u) \in P(f)^\circ$ with

$$u = \langle (0, -1, 0), (y, t, u) \rangle > 0.$$

Let $D = Y \times \mathbb{R} \times \mathbb{R}_+$. As $P(f)^\circ \subset D$, since $P(f) + \{0\} \times \mathbb{R}_+ \times \{0\} \subset P(f)$ and as $(y, t, u) \in P(f)^\circ \cap \operatorname{int} D$ we have

$$P(f)^\circ = P(f)^\circ \cap D = \operatorname{cl}(P(f)^\circ \cap \operatorname{int}(D))$$

by a well-known fact about convex sets. Therefore $P(f)^\circ$ is the closed cone generated by $P(f)^\circ \cap Y \times \mathbb{R} \times \{1\} = E(f^*) \times \{1\}$. ∎

4.3. PROPOSITION. *For any family $(g_w)_{w \in W}$ of extended real-valued functions on Y one has, if τ is stronger than the $\mathcal{K}$-topology and if X is equipped with the convergence c associated to σ and $\mathcal{K}$,*

$$(\operatorname*{els}_{w \to \omega} g_w)^* \leqslant \operatorname*{eli}_{w \to \omega} g_w^*.$$

As observed by Dolecki [11], this inequality can be shown directly whenever X and Y are endowed with convergence or topologies making the pairing $X \times Y \to \mathbb{R}$ jointly lsc.

Proof. Let $g = \operatorname{els} g_w$, $Q(\omega) = Q(g)$, $Q(w) = Q(g_w)$ for $w \in W$. Since by Proposition 3.3

$$Q(\omega) \subset \liminf_{w \to \omega} Q(w)$$

Theorem 2.2(a) implies that

$$c\text{-}\limsup_{w \to \omega} Q(w)^\circ \subset Q(\omega)^\circ.$$

Using Lemmas 3.1 and 4.2 in which the roles of X and Y are interchanged, taking the intersection with the affine subspace $X \times \mathbb{R} \times \{1\}$ of $X \times \mathbb{R}^2$ we get the result. ∎

4.4. PROPOSITION. *For any family $(f_w)_{w \in W}$ of closed nonimproper convex functions on X satisfying the assumption*

for each net $(w_i)_{i \in I}$ in W with limit ω there exists $K \in \mathcal{K}$, a subnet $(w_j)_{j \in J}$ of $(w_i)_{i \in I}$ and a net $(x_j)_{j \in J}$ in K with $\limsup_{j \in J} f_{w_j}(x_j) < +\infty$, (H) *one has, provided that τ is weaker than the $\mathscr{K}$-topology on Y and $\mathscr{K}_c$ is saturating in $\mathscr{K}$,*

$$(\operatorname*{eli}_{w\to\omega} f_w)^* \geqslant \operatorname*{els}_{w\to\omega} f_w^*.$$

Proof. Let $f = (\text{eli}_{w\to\omega} f_w)^{**}$ so that $E(f) \supset \limsup_{w\to\omega} E(f_w)$ when $X \times \mathbb{R}$ is endowed with the product convergence of c and of the natural convergence of $\mathbb{R}$. Assumption (H) can be rephrased as: for any net $(w_i)_{i\in I}$ in W with limit ω lim sup $E(f_{w_i})$ is nonempty. Thus f is not the constant function with value $+\infty$ and without loss of generality we may suppose f does not assume the value $-\infty$ as the inequality is obvious when $f^* = +\infty$. Proposition 3.7 ensures that

$$\limsup_{w\to\omega} P(f_w) \subset \text{cl } P(f),$$

so that, by Theorem 2.2(b)

$$P(f)^\circ \subset \liminf_{w\to\omega} P(f_w)^\circ.$$

Then Lemmas 3.1(c) and 4.2 yield the result

$$E(f^*) \subset \liminf_{w\to\omega} E(f_w^*). \quad ∎$$

Although the following corollary could be deduced from Theorem 2.2(c) by similar methods, we deduce it from the preceding two propositions.

4.5. COROLLARY. *Suppose the topology τ on Y is the $\mathscr{K}$-topology and $\mathscr{K}_c$ is saturating in $\mathscr{K}$. Then for any family $(f_w)_{w\in W}$ of closed nonimproper convex functions on X satisfying the condition* (H) *one has*

$$(\operatorname*{eli}_{w\to\omega} f_w)^* = \operatorname*{els}_{w\to\omega} f_w^*.$$

Proof. Using Proposition 4.4, putting $g_w = f_w^*$, and taking the conjugates in Proposition 4.3, one has, since lim inf $E(f_w^*)$ is closed and convex,

$$\operatorname*{els}_{w\to\omega} f_w^* = (\operatorname*{els}_{w\to\omega} f_w^*)^{**} \geqslant (\operatorname*{eli}_{w\to\omega} f_w)^* \geqslant \operatorname*{els}_{w\to\omega} f_w^*. \quad ∎$$

In the following statement, given a dual pair (X, Y), a family $(f_w)_{w\in W}$ of extended real-valued functions on X is said to *epiconverge* to $f: X \to \overline{\mathbb{R}} := [-\infty, +\infty]$ if

$$\operatorname*{els}_{w\to\omega} f_w \leqslant f \leqslant \operatorname*{eli}_{w\to\omega} f_w,$$

where the epilimit superior is taken with respect to the Mackey topology and the epilimit inferior is taken with respect to the convergence c associated with $\sigma(X, Y)$ and the family $\mathscr{W}_X$ of convex weakly compact subsets of X. Then $e\text{-}\lim_{w \to \omega} f_w$ stands for $\text{els}_{w \to \omega} f_w$ and $\text{eli}_{\omega \to \omega} f_w$. The following corollary extends the famous Joly–Mosco result.

4.6. COROLLARY. *Let (X, Y) be a dual pair of tvs and let $(f_w)_{w \in W}$ be a family of closed nonimproper convex functions on X. Then $(f_w)_{w \in W}$ epiconverges to a closed nonimproper convex function f on X iff $(f_w^*)_{w \in W}$ epiconverges to f^*.*

Proof. This follows from the symmetry of the statement and from Corollary 4.5 in as much as condition (H) is satisfied when $\liminf_{w \to \omega} E(f_w) = E(f)$ is nonempty. ∎

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