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Semisimplicity in symmetric rigid tensor categories

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ABSTRACT

Let λ be a partition of a positive integer n. Let $\mathcal{C}$ be a symmetric rigid tensor category over a field k of characteristic 0 or $char(k) > n$, and let V be an object of $\mathcal{C}$. In our main result (Theorem 4.3) we introduce a finite set of integers $F(\lambda)$ and prove that if the Schur functor $\mathbb{S}_\lambda V$ of V is semisimple and the dimension of V is not in $F(\lambda)$, then V is semisimple. Moreover, we prove that for each $d \in F(\lambda)$ there exist a symmetric rigid tensor category $\mathcal{C}$ over k and a non-semisimple object $V \in \mathcal{C}$ of dimension d such that $\mathbb{S}_\lambda V$ is semisimple (which shows that our result is the best possible). In particular, Theorem 4.3 extends two theorems of Serre for $\mathcal{C} = \mathrm{Rep}(G)$, G is a group, and $\mathbb{S}_\lambda V$ is $\bigwedge^n V$ or $Sym^n V$, and proves a conjecture of Serre (1997) [S2].

1. Introduction

Let G be any group, let k be a field and let $\mathrm{Rep}(G)$ be the category of finite dimensional representations of G over k. A classical result of Chevalley states that in characteristic 0, the tensor product $V \otimes W$ of any two semisimple objects $V, W \in \mathrm{Rep}(G)$ is also semisimple [C]. Later on, Serre proved that this is also the case in positive characteristic p, provided that $\dim V + \dim W < p - 2$ [S1].

In [S2], Serre considered the "converse theorems," and proved that $V \in \mathrm{Rep}(G)$ is semisimple in each one of the following situations: there exists $W \in \mathrm{Rep}(G)$ such that $\dim W \neq 0$ in k and $V \otimes W$ is semisimple [S2, Theorem 2.4], $V^{\otimes n}$ is semisimple for some $n \geqslant 1$ [S2, Theorem 3.4], $\bigwedge^n V$ is semisimple for some $n \geqslant 1$ and $\dim V \neq 2, \ldots, n$ in k [S2, Theorem 5.2.5], or $Sym^n V$ is semisimple for some $n \geqslant 1$ and $\dim V \neq -n, \ldots, -2$ in k [S2, Theorem 5.3.1].

Furthermore, Serre comments that it is easy to check that all the above mentioned results from [S2] extend to categories of linear representations of Lie algebras and restricted Lie algebras (when $p > 0$) [S2, p. 510]. Moreover, Serre explains how to extend his results Theorems 2.4 and 3.4 [S2] to any symmetric rigid tensor category over k, and says on p. 511 [S2]: "I have not

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managed to rewrite the proofs in tensor category style. Still, I feel that Theorem 5.2.5 on $\bigwedge^n V$ and Theorem 5.3.1 on $Sym^n V$ should remain true whenever $n! \neq 0$ in k, i.e., $p = 0$ or $p > n$." This paper originated in an attempt to prove this conjecture of Serre.

A further natural generalization of Serre's results would be to consider any Schur functor $\mathbb{S}_\lambda$, and not only $\bigwedge^n$ and Sym^n. Namely, to look for an extension of Theorems 5.2.5 and 5.3.1 in [S2], where $\mathcal{C}$ is any symmetric rigid tensor category over k, and $V \in \mathcal{C}$ is an object for which $\mathbb{S}_\lambda V$ is semisimple for some partition λ of n. This is precisely the main purpose of this paper.

The paper is organized as follows. In Section 2 we note that in fact Theorem 2.4 from [S2] holds in a much more general situation than the symmetric one. More precisely, let $\mathcal{C}$ be *any* rigid tensor category, and suppose that $W \in \mathcal{C}$ is isomorphic to its double dual W^{**} via an isomorphism i. This allows to define a scalar $\dim_i(W)$ in k, and we show that if $\dim_i(W) \neq 0$ and $V \otimes W$ is semisimple, then V is semisimple (see Theorem 2.3). Examples, other than $\mathcal{C} = \mathrm{Rep}(G)$, are given by braided rigid tensor categories $\mathcal{C}$ and by representation categories $\mathcal{C}$ of Hopf algebras whose squared antipode is inner.

In Section 3 we note that Theorem 3.3 and Corollary 3.4 from [S2] hold in a much more general situation than the symmetric one, as well. More precisely, let $\mathcal{C}$ be *any* rigid tensor category satisfying the *commutativity condition*, and let $V \in \mathcal{C}$. We show that if $V^{\otimes n} \otimes V^{*\otimes m}$ is semisimple for some $m, n \geqslant 0$, not both equal to 0, then V is semisimple. In particular, if $V^{\otimes n}$ is semisimple for some $n \geqslant 1$ then V is semisimple (see Theorem 3.1). Examples, other than $\mathcal{C} = \mathrm{Rep}(G)$, are given by braided rigid tensor categories $\mathcal{C}$.

In Section 4 we state the main result of the paper (Theorem 4.3), and prove various results in preparation for its proof. Our main result extends Theorem 5.2.5 on $\bigwedge^n$ and Theorem 5.3.1 on Sym^n in the group case $\mathcal{C} = \mathrm{Rep}(G)$ [S2], to any symmetric rigid tensor category $\mathcal{C}$ over k and any Schur functor $\mathbb{S}_\lambda$ (so, in particular, it provides a proof to the conjecture of Serre [S2, p. 511]). More precisely, let λ be a partition of a positive integer n, and assume that $char(k) = 0$ or $char(k) > n$. Let $\mathbb{S}_\lambda$ be the associated Schur functor (see [D2]) and let V be an object of $\mathcal{C}$. In Theorem 4.3 we introduce a finite set of integers $F(\lambda)$ and prove that if the dimension of V is not equal in k to an element of $F(\lambda)$ and $\mathbb{S}_\lambda V$ is semisimple, then V is semisimple. Moreover, we prove that for each $d \in F(\lambda)$ there exist a symmetric rigid tensor category $\mathcal{C}$ over k and a non-semisimple object $V \in \mathcal{C}$ of dimension d such that $\mathbb{S}_\lambda V$ is semisimple (which shows that our result is the best possible).

Section 5 is devoted to the proof of Theorem 4.3.

All tensor categories will be assumed to be rigid, k-linear Abelian, with finite dimensional Hom spaces, such that every object has a finite length, and $\mathrm{End}(\mathbf{1}) = k$.

2. From $V \otimes W$ to V in rigid tensor categories

Let $\mathcal{C}$ be a rigid tensor category (see e.g., [BK, Definition 2.1.2]). For an object $V \in \mathcal{C}$ we let

$$coev_V : \mathbf{1} \to V \otimes V^* \quad \text{and} \quad ev_V : V^* \otimes V \to \mathbf{1}$$

denote the coevaluation and evaluation maps associated to V, respectively. Recall that

$$(id_V \otimes ev_V) \circ (coev_V \otimes id_V) = id_V.$$

The following two propositions were proved by Serre for $\mathcal{C} := \mathrm{Rep}(G)$, G any group [S2]. However, it is straightforward to verify that the same proofs work in any rigid tensor category $\mathcal{C}$.

Proposition 2.1. *(See [S2, Proposition 2.1].) Let $V, W \in \mathcal{C}$, and let V' be a sub-object of V. Assume that $coev_W : \mathbf{1} \to W \otimes W^*$ and $V' \otimes W \to V \otimes W$ are split injections. Then $V' \to V$ is a split injection.*

Proposition 2.2. *(See [S2, Proposition 2.3].) Assume $coev_W : \mathbf{1} \to W \otimes W^*$ is a split injection and that $V \otimes W$ is semisimple. Then V is semisimple.*

One instance in which $coev_W : \mathbf{1} \to W \otimes W^*$ is a split injection is the following. Assume that $W \in \mathcal{C}$ is isomorphic to its double dual W^{**}, and fix an isomorphism $i : W \to W^{**}$. This allows us to define the (quantum) dimension $\dim_i(W)$ of W (relative to i) as the composition

$$\dim_i(W) := ev_{W^*} \circ (i \otimes id_{W^*}) \circ coev_W .$$

Note that $\dim_i(W) \in \mathrm{End}(\mathbf{1}) = k$. Now, clearly if $\dim_i(W) \neq 0$ in k, then $coev_W$ is a split injection. (See Remark 2.2 in [S2].)

As a consequence of Propositions 2.1 and 2.2 we have the following theorem, which generalizes Theorem 2.4 in [S2].

Theorem 2.3. *Assume that $W \in \mathcal{C}$ is isomorphic to its double dual W^{**} and let $i : W \to W^{**}$ be an isomorphism. If $V \otimes W$ is semisimple and $\dim_i(W) \neq 0$ in k then V is semisimple.*

Remark 2.4. 1) It is known that if $\mathcal{C}$ is braided, any object W is isomorphic to its double dual W^{**}. So in particular, if H is a quasitriangular (quasi)Hopf algebra over k and $V, W \in \mathrm{Rep}(H)$ such that $V \otimes W$ is semisimple and $\dim W \neq 0$ in k, then V is semisimple. The converse is not true.

2) If H is a Hopf algebra whose squared antipode S^2 is inner (e.g., $S^2 = id$) then any $W \in \mathrm{Rep}(H)$ is isomorphic to W^{**}. Therefore Theorem 2.3 holds for $\mathrm{Rep}(H)$.

3) When $\mathcal{C}$ is symmetric, Serre already pointed out that Theorem 2.4 in [S2] holds for $\mathcal{C}$, with the same proof (see pp. 510–511 in [S2]).

3. From $V^{\otimes n} \otimes V^{*\otimes m}$ to V in rigid tensor categories

The following theorem was proved by Serre for $\mathcal{C} := \mathrm{Rep}(G)$, G any group [S2]. Serre also explains that the same proof works in any symmetric rigid tensor category $\mathcal{C}$. In fact, the symmetry is used only to guarantee that for any $V \in \mathcal{C}$ the morphism

$$id_V \otimes coev_V : V \to V \otimes V \otimes V^*$$

is a split injection. We just note that in fact this is the case in any rigid tensor category $\mathcal{C}$ satisfying the following *commutativity condition*: there exists a functorial isomorphism $c : \otimes \to \otimes^{op}$ such that $c_{V \otimes \mathbf{1}} = c_{\mathbf{1} \otimes V} = id_V$ for any $V \in \mathcal{C}$ (e.g., $\mathcal{C}$ is braided, not necessarily symmetric). Indeed, let $\mathcal{C}$ be a rigid tensor category satisfying the commutativity condition. Then, using the naturality of c, one has

$$(id_V \otimes ev_V) \circ c_{V, V \otimes V^*} \circ (id_V \otimes coev_V) = (id_V \otimes ev_V) \circ (coev_V \otimes id_V) = id_V . \tag{1}$$

Therefore we have the following result, which generalizes Theorem 3.3 and Corollary 3.4 in [S2].

Theorem 3.1. *Let $\mathcal{C}$ be a rigid tensor category satisfying the commutativity condition, and let $V \in \mathcal{C}$. If $V^{\otimes n} \otimes V^{*\otimes m}$ is semisimple for some $m, n \geqslant 0$, not both equal to 0, then V is semisimple. In particular, if $V^{\otimes n}$ is semisimple for some $n \geqslant 1$ then V is semisimple.*

4. From $\mathbb{S}_\lambda V$ to V in symmetric rigid tensor categories

In this section we assume that $\mathcal{C}$ is a *symmetric* rigid tensor category over a field k, with a commutativity constraint c (see e.g., [D1,D2] and [BK, Definition 1.2.7]).

4.1. Schur functors in $\mathcal{C}$

Recall that given an object $X \in \mathcal{C}$ and a nonnegative integer m, the symmetric group S_m acts on $X^{\otimes m}$ via the symmetry c. Let β be a partition of m, and assume that $char(k) > m$ if $char(k) \neq 0$. Let

V_β be the corresponding irreducible representation of S_m and let $c_\beta \in k[S_m]$ be a Young symmetrizer associated with V_β. Then c_β gives rise to a functor

$$c_\beta : \mathcal{C} \to \mathcal{C}, \quad X \mapsto c_\beta\left(X^{\otimes m}\right).$$

Recall that the isomorphism type of the functor c_β does not depend on the choice of c_β. We shall call $\mathbb{S}_\beta X := c_\beta(X^{\otimes m}) \subseteq X^{\otimes m}$ the *Schur functor* of X associated with β.

Schur functors in symmetric rigid tensor categories were introduced (more conceptually) and studied by Deligne in [D2]. Among many other things, it is proved there that for any object $X \in \mathcal{C}$, $(\mathbb{S}_\beta X)^*$ is canonically isomorphic to $\mathbb{S}_\beta X^*$, a fact we shall use often in the sequel.

Example 4.1. Note in particular that $\mathbb{S}_{(0)} X = \mathbf{1}$, $\mathbb{S}_{(1)} X = X$, $\mathbb{S}_{(m)} X = Sym^m X$ and $\mathbb{S}_{(1^m)} X = \bigwedge^m X$.

4.2. The main result

Our goal is to generalize Theorems 5.2.5 and 5.3.1 from [S2] by replacing representations categories $\mathrm{Rep}(G)$ of groups by any symmetric rigid tensor category $\mathcal{C}$, and by replacing the Schur functors $\bigwedge^n$, Sym^n by any Schur functor. More precisely, let λ be a partition of a positive integer n and let $V \in \mathcal{C}$. Our goal is to find out when the semisimplicity of $\mathbb{S}_\lambda V$ implies the semisimplicity of V, in terms of the dimension of V only.

Fix a partition λ of a positive integer n, with $p := p(\lambda)$ rows and $q := q(\lambda)$ columns, and let (i, j) number the row and column of boxes for the Young diagram of λ. Let us introduce some notation.

- Let $R(\lambda)$ denote the integral interval $\{-q, \ldots, p\}$, and let $T(\lambda) \subseteq R(\lambda)$ include 0 if λ is a hook (i.e., $(2,2) \notin \lambda$), 1 if $(3,2) \notin \lambda$, -1 if $(2,3) \notin \lambda$, and $-q, p$ if λ is not a rectangle. Set $F(\lambda) := R(\lambda) \setminus T(\lambda)$.
- Let $G(\lambda)$ denote the set of all values d in k for which there exists a symmetric rigid tensor category $\mathcal{C}$ over k with a non-semisimple object V of dimension d such that $\mathbb{S}_\lambda V$ is semisimple.

Remark 4.2. 1) We have that $F(\lambda) = -F(\lambda^*)$, where λ^* is the conjugate of λ.

2) We have that $G(\lambda) = -G(\lambda^*)$. Indeed, if $(\mathcal{C}, V)$ is a counterexample for (λ, d) (i.e., $\mathcal{C}$ is a symmetric rigid tensor category over k with a non-semisimple object V of dimension d such that $\mathbb{S}_\lambda V$ is semisimple) then $(\mathcal{C} \boxtimes \mathrm{Supervect}, V \otimes \mathbf{1}^{-1})$ is a counterexample for $(\lambda^*, -d)$, where Supervect is the category of finite dimensional super vector spaces over k and $\mathbf{1}^{-1} \in \mathrm{Supervect}$ is the odd 1-dimensional space.

We can now state our main result concisely.

Theorem 4.3. *Let n be a positive integer, $n < char(k)$ in case $char(k) \neq 0$, and let λ be a partition of n. Then the sets $F(\lambda)$ and $G(\lambda)$ coincide (where we view the relevant integers as elements of k in an obvious way).*

Example 4.4. Let $\mathcal{C}$ be a symmetric rigid tensor category over k, and let $V \in \mathcal{C}$.

1) Theorem 4.3 implies for $\lambda = (1^n)$ (respectively, $\lambda = (n)$), that if $\mathbb{S}_\lambda V$ is semisimple and the dimension of V is not equal in k to an integer in the range $2, \ldots, n$ (respectively, $-n, \ldots, -2$), then V is semisimple. For $\mathcal{C} = \mathrm{Rep}(G)$, G is any group, this is Theorem 5.2.5 from [S2] (respectively, Theorem 5.3.1 from [S2]).

2) Theorem 4.3 implies that if $\mathbb{S}_{(2,1)} V$ is semisimple then so is V.

The proof of Theorem 4.3 is given in Section 5. The rest of this section is devoted to preparations for the proof.

4.3. Traces in $\mathcal{C}$

For an object $X \in \mathcal{C}$, let

$$\widetilde{ev}_X := ev_X \circ c_{X,X^*} : X \otimes X^* \to \mathbf{1}.$$

Recall that the dimension $\dim X \in k$ of X is defined by

$$\dim X := \widetilde{ev}_X \circ coev_X : \mathbf{1} \to \mathbf{1}.$$

In [JSV] it is explained that the family of functions

$$Tr^U_{A,B} : \mathrm{Hom}(A \otimes U, B \otimes U) \to \mathrm{Hom}(A, B), \quad A, B, U \in \mathcal{C},$$

defined by

$$Tr^U_{A,B}(f) : A \xrightarrow{id_A \otimes coev_U} A \otimes U \otimes U^* \xrightarrow{f \otimes id_{U^*}} B \otimes U \otimes U^* \xrightarrow{id_B \otimes \widetilde{ev}_U} B, \tag{2}$$

is natural in U, A and B, and satisfies the following property (among other properties)

$$Tr^{U \otimes W}_{A,B}(f) = Tr^U_{A,B}\big(Tr^W_{A \otimes U, B \otimes U}(f)\big). \tag{3}$$

Clearly, $Tr^U_{\mathbf{1},\mathbf{1}}(id_U) = \dim U$.
We have the following two easy lemmas.

Lemma 4.5. *Let $f : A \otimes U \to B \otimes W$ and $g : W \to U$ be morphisms. Then*

$$Tr^U_{A,B}\big((id_B \otimes g) f\big) = Tr^W_{A,B}\big(f(id_A \otimes g)\big).$$

Proof. Follows from the naturality of Tr in U. □

Lemma 4.6. *Let $f : A \otimes U \to B \otimes U$ and $g : W \to W$ be morphisms. Then*

$$Tr^U_{A,B}(f) \otimes Tr^W_{\mathbf{1},\mathbf{1}}(g) = Tr^{U \otimes W}_{A,B}(f \otimes g).$$

Proof. Follows easily from the definition of Tr, and the facts that $(U \otimes W)^* = W^* \otimes U^*$ with

$$coev_{U \otimes W} = (id_U \otimes c_{U^*, W \otimes W^*}) \circ (coev_U \otimes coev_W)$$

and

$$\widetilde{ev}_{U \otimes W} = (\widetilde{ev}_U \otimes \widetilde{ev}_W) \circ (id_U \otimes c_{W \otimes W^*, U^*})$$

(see e.g., [BK]). □

4.4. Traces of permutations

Fix a nonnegative integer m, and an object $X \in \mathcal{C}$. In the sequel we shall identify the symmetric group S_{m-1} with the stabilizer of 1 in S_m.

Lemma 4.7. *For any $\sigma \in S_m$ and $\tau \in S_{m-1}$, $Tr_{X,X}^{X^{\otimes m-1}}(\sigma) = Tr_{X,X}^{X^{\otimes m-1}}(\tau\sigma\tau^{-1})$.*

Proof. Follows easily from Lemma 4.5. □

Lemma 4.8. *We have that $Tr_{X,X}^{X^{\otimes m-1}}((1\cdots m)) = id_X$.*

Proof. For any i let us denote the cycle $(1\cdots i)$ by σ_i. We are going to prove the lemma by induction on m using the relation $\sigma_m = (12)\sigma_{m-1}$. We compute

$$\begin{aligned} Tr_{X,X}^{X^{\otimes m-1}}(\sigma_m) &= Tr_{X,X}^{X}\big(Tr_{X\otimes X,X\otimes X}^{X^{\otimes m-2}}(\sigma_m)\big) \\ &= Tr_{X,X}^{X}\big(Tr_{X\otimes X,X\otimes X}^{X^{\otimes m-2}}\big(\big((12)\otimes id\big)\circ(id\otimes\sigma_{m-1})\big)\big) \\ &= Tr_{X,X}^{X}\big((12)\circ Tr_{X\otimes X,X\otimes X}^{X^{\otimes m-2}}(id\otimes\sigma_{m-1})\big) \\ &= Tr_{X,X}^{X}\big((12)\circ\big(id_X\otimes Tr_{X,X}^{X^{\otimes m-2}}(\sigma_{m-1})\big)\big) \\ &= Tr_{X,X}^{X}\big((12)\circ(id_X\otimes id_X)\big) \\ &= id_X, \end{aligned}$$

where in the first equality we used (3), in the third equality we used the naturality of Tr in $X\otimes X$, in the fifth equality we used the induction assumption, and in the last equality we used (1). □

Lemma 4.9. *Let $\sigma_1\sigma_2\cdots\sigma_N \in S_m$ be a product of disjoint cycles, where reading from left to right the numbers $1,\dots,m$ appear in an increasing order. Then $Tr_{X,X}^{X^{\otimes m-1}}(\sigma_1\sigma_2\cdots\sigma_N) = d^{N-1}id_X$, where d is the dimension of X.*

Proof. Lemma 4.8 is the case $N=1$. Now use Lemma 4.6 to proceed by induction on N. □

Proposition 4.10. *Let $\sigma\in S_m$, and let $N(\sigma)$ denote the number of disjoint cycles in σ. Then*

$$Tr_{X,X}^{X^{\otimes m-1}}(\sigma) = d^{N(\sigma)-1}id_X,$$

where $d := \dim X$.

Proof. It is clear that for any $\sigma\in S_m$ there exists $\tau\in S_{m-1}$ such that $\tau\sigma\tau^{-1}$ decomposes into a product of disjoint cycles $\sigma_1\sigma_2\cdots\sigma_{N(\sigma)}$, where reading from left to right the numbers $1,\dots,m$ are in an increasing order. Now, by Lemma 4.7, $Tr_{X,X}^{X^{\otimes m-1}}(\sigma) = Tr_{X,X}^{X^{\otimes m-1}}(\tau\sigma\tau^{-1})$, and hence the result follows from Lemma 4.9. □

4.5. The morphism $\theta_{X,m,\alpha,\beta}$

Given a partition α of a nonnegative integer $m-1$, let $\alpha+1$ denote the set of partitions of m whose Young diagram is obtained by adding a single box to the Young diagram of α.

Fix an object $X\in\mathcal{C}$ of dimension $d:=\dim X$, and partitions α of $m-1$ and $\beta\in\alpha+1$. We define the morphism

$$\theta_{\alpha,\beta} = \theta_{X,m,\alpha,\beta} : X \to \mathbb{S}_\beta X\otimes\mathbb{S}_\alpha X^*$$

as the following composition:

$$\theta_{\alpha,\beta} : X \xrightarrow{id_X\otimes coev_{\mathbb{S}_\alpha X}} X\otimes\mathbb{S}_\alpha X\otimes\mathbb{S}_\alpha X^* \xrightarrow{c_\beta\otimes c_\alpha} \mathbb{S}_\beta X\otimes\mathbb{S}_\alpha X^*. \tag{4}$$

Consider the morphism

$$P_{\alpha,\beta} = P_{X,m,\alpha,\beta} : X \to X,$$

given as the composition

$$P_{\alpha,\beta} : X \xrightarrow{\theta_{\alpha,\beta}} \mathbb{S}_\beta X \otimes \mathbb{S}_\alpha X^* \hookrightarrow X \otimes X^{\otimes m-1} \otimes X^{*\otimes m-1} \xrightarrow{id_X \otimes \widetilde{ev}_{X^{\otimes m-1}}} X. \tag{5}$$

In what follows we shall see that the morphism $P_{\alpha,\beta}$ is a scalar multiple of the identity morphism id_X by some polynomial $p_{\alpha,\beta}(d)$.

If we identify S_{m-1} with the stabilizer of 1 in S_m, then clearly

$$P_{\alpha,\beta} = Tr^{X^{\otimes m-1}}_{X,X}\big((id_X \otimes c_\alpha) \circ c_\beta \circ (id_X \otimes c_\alpha)\big) : X \to X,$$

and hence, by Lemma 4.5,

$$P_{\alpha,\beta} = Tr^{X^{\otimes m-1}}_{X,X}\big((id_X \otimes c_\alpha) \circ c_\beta\big) : X \to X.$$

As an immediate consequence of Proposition 4.10, we get the following.

Corollary 4.11. *Write $(id_X \otimes c_\alpha) \circ c_\beta \in k[S_m]$ as a k-linear combination of group elements: $(id_X \otimes c_\alpha) \circ c_\beta = \sum_{\sigma \in S_m} f_{\alpha,\beta}(\sigma)\sigma$, and set*

$$p_{\alpha,\beta}(d) := \sum_{\sigma \in S_m} f_{\alpha,\beta}(\sigma) d^{N(\sigma)-1}.$$

Then $P_{\alpha,\beta} = p_{\alpha,\beta}(d) id_X$. In particular, if $p_{\alpha,\beta}(d) \neq 0$ in k then $\theta_{\alpha,\beta}$ is a split injection.

Let χ_β be the character of V_β, and let

$$e_\beta := \frac{\dim V_\beta}{m!} \sum_{\sigma \in S_m} \chi_\beta(\sigma)\sigma$$

be the primitive central idempotent in $k[S_m]$ associated with V_β. Recall that e_β is equal to the sum of all the $(\dim V_\beta)$ Young symmetrizers c_β associated with V_β.

In the following theorem we compute the polynomial $p_{\alpha,\beta}(d)$ explicitly, in terms of χ_β.

Theorem 4.12. *We have that*

$$Tr^{X^{\otimes m-1}}_{X,X}\big((id_X \otimes e_\alpha) \circ e_\beta\big) = \left(\frac{\dim V_\alpha}{m!} \sum_{\sigma \in S_m} \chi_\beta(\sigma) d^{N(\sigma)-1}\right) id_X,$$

and hence

$$p_{\alpha,\beta}(d) = \frac{1}{m! \dim V_\beta} \sum_{\sigma \in S_m} \chi_\beta(\sigma) d^{N(\sigma)-1}.$$

Proof. Clearly,

$$(id_X \otimes e_\alpha) \circ e_\beta = \frac{\dim V_\alpha \dim V_\beta}{(m-1)!m!} \sum_{\sigma \in S_m} \left(\sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta\big(\tau^{-1}\sigma\big) \right) \sigma.$$

Therefore, by Proposition 4.10,

$$\begin{aligned} Tr_{X,X}^{X^{\otimes m-1}}\big((id_X \otimes e_\alpha) \circ e_\beta\big) &= \left(\frac{\dim V_\alpha \dim V_\beta}{(m-1)!m!} \sum_{\sigma \in S_m} \left(\sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta\big(\tau^{-1}\sigma\big) \right) d^{N(\sigma)-1} \right) id_X \\ &= \left(\frac{\dim V_\alpha \dim V_\beta}{(m-1)!m!} \left(\sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta\left(\tau^{-1}\left(\sum_{\sigma \in S_m} d^{N(\sigma)-1}\sigma \right)\right) \right) \right) id_X. \end{aligned}$$

Set $z(d) := \sum_{\sigma \in S_m} d^{N(\sigma)-1}\sigma$. Clearly, $z(d)$ is a central element in $k[S_m]$, hence it acts by the scalar $\chi_\beta(z(d))/\dim V_\beta$ on V_β. In particular, for any $\tau \in S_m$, $\chi_\beta(\tau^{-1}z(d)) = \chi_\beta(\tau^{-1})\chi_\beta(z(d))/\dim V_\beta$. We therefore have

$$Tr_{X,X}^{X^{\otimes m-1}}\big((id_X \otimes e_\alpha) \circ e_\beta\big) = \left(\frac{\dim V_\alpha}{(m-1)!m!} \left(\sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta\big(\tau^{-1}\big) \right) \chi_\beta\big(z(d)\big) \right) id_X.$$

Finally, recall that the multiplicity $[Res_{S_{m-1}}^{S_m} \chi_\beta : \chi_\alpha]$ of V_α in the restriction of V_β from S_m to S_{m-1} is equal to 1 (see e.g. [FH]), i.e.,

$$\frac{1}{(m-1)!} \sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta\big(\tau^{-1}\big) = \big[Res_{S_{m-1}}^{S_m} \chi_\beta : \chi_\alpha\big] = 1.$$

We thus conclude that

$$Tr_{X,X}^{X^{\otimes m-1}}\big((id_X \otimes e_\alpha) \circ e_\beta\big) = \left(\frac{\dim V_\alpha}{m!} \sum_{\sigma \in S_m} \chi_\beta(\sigma) d^{N(\sigma)-1} \right) id_X,$$

as claimed. □

In fact, the polynomial $p_{\alpha,\beta}(d)$ is closely related to a well-known polynomial associated with the partition β. Namely, let $\mathrm{cp}_\beta(d) := \prod_{(i,j)\in\beta}(d+j-i)$ be the content polynomial of β, and recall that the polynomial (in d) $\frac{1}{\dim V_\beta} \sum_{\sigma \in S_m} \chi_\beta(\sigma) d^{N(\sigma)}$ equals $\mathrm{cp}_\beta(d)$ (see e.g. [MacD]). Hence, by Theorem 4.12,

$$p_{\alpha,\beta}(d)d = \frac{1}{m!}\mathrm{cp}_\beta(d). \tag{6}$$

Corollary 4.13. *Let α, β, X and d be as above, and let $p(\beta)$, $q(\beta)$ be the number of rows and columns in the diagram of β, respectively.*

1) *If $d \neq 1-q(\beta), \ldots, p(\beta)-1$ in k then the morphism $\theta_{\alpha,\beta}$ is a split injection.*
2) *Suppose β is a hook. If $d \neq 1-q(\beta), \ldots, -1, 1, \ldots, p(\beta)-1$ in k then the morphism $\theta_{\alpha,\beta}$ is a split injection.*

Proof. 1) Since $d \neq 0$ in k, the result follows from (6) and Theorem 4.12.

2) By Theorem 4.12, $p_{\alpha,\beta}(0) = \frac{1}{m! \dim V_\beta} \sum_\sigma \chi_\beta(\sigma)$, where the sum is taken over all the m-cycles σ in S_m. But it is well known (see e.g. [MacD]) that χ_β vanishes on an m-cycle when β is not a hook, and that $\chi_{(m-s,1^s)}(\sigma) = (-1)^s$ for any $0 \leqslant s \leqslant m$ and m-cycle σ. Therefore $\theta_{\alpha,\beta}$ is a split injection when $d = 0$ as well.

We are done. □

Example 4.14. For the partition $\alpha = (1^{m-1})$, $\mathbb{S}_\alpha X = \bigwedge^{m-1} X$ is the $(m-1)th$ exterior power of X. Hence, by Corollary 4.13, if $\binom{d-1}{m-1} \neq 0$ in k, then the corresponding morphism $\theta_{(1^{m-1}),(1^m)}$ is a split injection. This is a generalization of Lemma 5.1.12 in [S2] in the group case.

4.6. Extensions in $\mathcal{C}$

Let $U, V, W \in \mathcal{C}$ and let $f \in \mathrm{Hom}(V, W)$, $g \in \mathrm{Hom}(W, U)$. We shall denote by f_* and g^* the k-linear maps

$$f_* : \mathrm{Ext}^1(U, V) \to \mathrm{Ext}^1(U, W) \quad \text{and} \quad g^* : \mathrm{Ext}^1(U, V) \to \mathrm{Ext}^1(W, V)$$

induced by f and g, respectively. Namely, given an extension

$$E : 0 \to V \to X \to U \to 0,$$

the extensions

$$f_*(E) : 0 \to W \to Y \to U \to 0 \quad \text{and} \quad g^*(E) : 0 \to V \to Z \to W \to 0$$

are obtained using the pushout

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow \\ W & \dashrightarrow & Y \end{array}$$

and the pullback

$$\begin{array}{ccc} X & \longrightarrow & U \\ \uparrow & & \uparrow{\scriptstyle g} \\ Z & \dashrightarrow & W, \end{array}$$

respectively.

We shall need the following two lemmas.

Lemma 4.15. *For any objects $A, B, X \in \mathcal{C}$, the k-linear spaces $\mathrm{Ext}^1(B, A \otimes X)$ and $\mathrm{Ext}^1(B \otimes X^*, A)$ are canonically isomorphic.*

Proof. One associates to an element

$$0 \to A \otimes X \to W \to B \to 0$$

in $\mathrm{Ext}^1(B, A\otimes X)$ an element in $\mathrm{Ext}^1(B\otimes X^*, A)$ in the following way: since the functor $-\otimes X^*:\mathcal{C}\to\mathcal{C}$ is exact, tensoring our exact sequence with X^* on the right yields the extension

$$E: 0\to A\otimes X\otimes X^*\to W\otimes X^*\to B\otimes X^*\to 0.$$

The corresponding extension

$$0\to A\to \tilde{W}\to B\otimes X^*\to 0$$

in $\mathrm{Ext}^1(B\otimes X^*, A)$ is given by $(id_A\otimes(ev_X\circ c_{X,X^*}))_*(E)$. This assignment defines a k-linear map $\mathrm{Ext}^1(B, A\otimes X)\to \mathrm{Ext}^1(B\otimes X^*, A)$, and it is straightforward to verify that its inverse map is constructed similarly, using the exact functor $-\otimes X$ and the map $(id_B\otimes(c_{X,X^*}\circ coev_X))^*$. □

Lemma 4.16. *Let α be a partition of a nonnegative integer $m-1$, let $\beta\in\alpha+1$ and let $A, B, X\in\mathcal{C}$. Suppose that $\theta_{\alpha,\beta}=\theta_{X,m,\alpha,\beta}$ is a split injection. Then the k-linear map*

$$(id_{B\otimes X}\otimes coev_{\mathbb{S}_\alpha X})_*: \mathrm{Ext}^1(A, B\otimes X)\to \mathrm{Ext}^1\big(A, B\otimes X\otimes \mathbb{S}_\alpha X\otimes \mathbb{S}_\alpha X^*\big)$$

is injective.

Proof. Indeed, since $\theta_{\alpha,\beta}$ is a split injection, we have that

$$(id_B\otimes\theta_{\alpha,\beta})_*: \mathrm{Ext}^1(A, B\otimes X)\to \mathrm{Ext}^1\big(A, B\otimes \mathbb{S}_\beta X\otimes \mathbb{S}_\alpha X^*\big)$$

is injective. But,

$$(id_B\otimes\theta_{\alpha,\beta})_* = (id_B\otimes c_\beta\otimes id_{\mathbb{S}_\alpha X^*})_*\circ(id_{B\otimes X}\otimes coev_{\mathbb{S}_\alpha X})_*.$$

We are done. □

4.7. The filtration on $\mathbb{S}_\lambda V$ defined by a sub-object of V

Fix a sub-object A of V for the rest of the section, and consider the short exact sequence

$$(V): 0\to A\to V\to B\to 0; \tag{7}$$

it is an element in the k-linear space $\mathrm{Ext}^1(B, A)$. Then (V) defines a filtration on $\mathbb{S}_\lambda V$ in the following way. For each $0\leqslant i\leqslant n$ set

$$T_i := \sum_{S\subseteq\{1,\dots,n\},\,|S|=i} V_{S(1)}\otimes\cdots\otimes V_{S(n)},$$

where $V_{S(j)}=V$ if $j\notin S$ and $V_{S(j)}=A$ if $j\in S$. Clearly, the T_i define an S_n-equivariant filtration T_* on $V^{\otimes n}$:

$$V^{\otimes n}=T_0\supseteq T_1\supseteq\cdots\supseteq T_n\supseteq T_{n+1}=0,$$

whose composition factors are

$$T_i/T_{i+1}\cong \bigoplus_{S\subseteq\{1,\dots,n\},\,|S|=i} V_{S,1}\otimes\cdots\otimes V_{S,n},\quad 0\leqslant i\leqslant n,$$

where $V_{S,j}=B$ if $j\notin S$ and $V_{S,j}=A$ if $j\in S$.

The filtration T_* induces a filtration F_* on $\mathbb{S}_\lambda V$:

$$\mathbb{S}_\lambda V = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq F_{n+1} = 0,$$

where $F_i := c_\lambda(T_i)$ is the image of T_i under the Schur functor c_λ. Let

$$V_i := F_i/F_{i+1}, \quad 0 \leqslant i \leqslant n, \tag{8}$$

be the composition factors of F_*, and let

$$V_i^2 := F_{i-1}/F_{i+1}, \quad 1 \leqslant i \leqslant n. \tag{9}$$

Since the filtration T_* is S_n-equivariant, we have

$$V_i \cong c_\lambda(T_i/T_{i+1}) \cong \bigoplus_{\mu \vdash i,\, \nu \vdash n-i} N^\lambda_{\mu,\nu}(\mathbb{S}_\mu A \otimes \mathbb{S}_\nu B), \tag{10}$$

where $N^\lambda_{\mu,\nu} := [Res^{S_n}_{S_i \times S_{n-i}} V_\lambda : V_\mu \otimes V_\nu]$ are the Littlewood–Richardson coefficients (see e.g., [FH]).

For each integer $0 \leqslant i \leqslant n$, let $\lambda - i$ denote the set of all partitions of $n - i$ whose Young diagram is obtained from that of λ after deleting i boxes (by convention, $\lambda - n$ consists of one element (0)). By the Littlewood–Richardson rule (see e.g., [FH]), $N^\lambda_{\mu,\nu} = 0$ if $\mu \notin \lambda - (n-i)$ or $\nu \notin \lambda - i$. Therefore,

$$V_i \cong \bigoplus_{\mu \in \lambda-(n-i),\, \nu \in \lambda - i} N^\lambda_{\mu,\nu}(\mathbb{S}_\mu A \otimes \mathbb{S}_\nu B). \tag{11}$$

(However, $N^\lambda_{\mu,\nu}$ can still equal 0 for some pairs $\mu \in \lambda - (n-i)$, $\nu \in \lambda - i$, e.g., for $\lambda = (2,2)$, $N^{(2,2)}_{(1^2),(2)} = 0$.)

Observe also that for any $\mu' \in \lambda - (n-i+1)$, $\mu \in \lambda - (n-i)$ and $\nu \in \lambda - i$, c_μ defines a morphism

$$c_\mu \otimes id_{\mathbb{S}_\nu B} : \mathbb{S}_{\mu'} A \otimes V \otimes \mathbb{S}_\nu B \to V_i^2.$$

Since V_i^2 is a subquotient of $\mathbb{S}_\lambda V$, the following lemma is clear.

Lemma 4.17. *If $\mathbb{S}_\lambda V$ is semisimple then the exact sequence*

$$(V_i^2) : 0 \to V_i \to V_i^2 \to V_{i-1} \to 0 \tag{12}$$

splits for any $1 \leqslant i \leqslant n$.

4.8. The semisimplicity of V

Let $1 \leqslant i \leqslant n$ be an integer, $\mu' \in \lambda - (n-i+1)$ and $\nu \in \lambda - i$. Tensoring our exact sequence (V) by $\mathbb{S}_{\mu'} A$ on the left yields the extension

$$E_1 : 0 \to \mathbb{S}_{\mu'} A \otimes A \to \mathbb{S}_{\mu'} A \otimes V \to \mathbb{S}_{\mu'} A \otimes B \to 0. \tag{13}$$

Tensoring E_1 by $\mathbb{S}_\nu B$ on the right yields the extension

$$E_2 : 0 \to \mathbb{S}_{\mu'} A \otimes A \otimes \mathbb{S}_\nu B \to \mathbb{S}_{\mu'} A \otimes V \otimes \mathbb{S}_\nu B \to \mathbb{S}_{\mu'} A \otimes B \otimes \mathbb{S}_\nu B \to 0. \tag{14}$$

Set

$$\mu'_+ := \{\mu \in \mu' + 1 \mid N^{\lambda}_{\mu,\nu} \neq 0\}, \qquad \nu_+ := \{\nu' \in \nu + 1 \mid N^{\lambda}_{\mu',\nu'} \neq 0\}. \tag{15}$$

The following lemma is clear.

Lemma 4.18. *Let $1 \leqslant i \leqslant n$ be an integer, and let $\mu' \in \lambda - (n - i + 1)$, $\nu \in \lambda - i$. Then for any $\mu \in \mu'_+$ and $\nu' \in \nu_+$, the triple $(c_\mu \otimes id_{\mathbb{S}_\nu B}, c_\mu \otimes id_{\mathbb{S}_\nu B}, id_{\mathbb{S}_{\mu'} A} \otimes c_{\nu'})$ defines a morphism of extensions $E_2 \to (V_i^2)$:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{S}_{\mu'} A \otimes A \otimes \mathbb{S}_\nu B & \longrightarrow & \mathbb{S}_{\mu'} A \otimes V \otimes \mathbb{S}_\nu B & \longrightarrow & \mathbb{S}_{\mu'} A \otimes B \otimes \mathbb{S}_\nu B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_\mu \otimes id_{\mathbb{S}_\nu B}} & & \downarrow{\scriptstyle c_\mu \otimes id_{\mathbb{S}_\nu B}} & & \downarrow{\scriptstyle id_{\mathbb{S}_{\mu'} A} \otimes c_{\nu'}} & & \\
0 & \longrightarrow & V_i & \longrightarrow & V_i^2 & \longrightarrow & V_{i-1} & \longrightarrow & 0.
\end{array}$$

Fix an integer $1 \leqslant i \leqslant n$, and $\mu' \in \lambda - (n - i + 1)$, $\nu \in \lambda - i$. For any $\mu \in \mu'_+$ and $\nu' \in \nu_+$, define the following two subsets of the ground field k:

$$A_i(\mu', \mu, \nu) := \{d \mid p_{\mu',\mu}(d) = 0\} \subseteq k \tag{16}$$

and

$$B_i(\mu', \nu, \nu') := \{d \mid p_{\nu,\nu'}(d) = 0\} \subseteq k. \tag{17}$$

Example 4.19. By convention, $\lambda - n = \{(0)\}$. Therefore, for any $\mu', \nu \in \lambda - 1$, we have that $A_1((0), (1), \nu) = B_n(\mu', (0), (1)) = \emptyset$. On the other extreme, by Corollary 4.13, $B_1((0), \nu, \lambda) = A_n(\mu', \lambda, (0)) = \{1 - q(\lambda), \ldots, p(\lambda) - 1\}$ if λ is not a hook, and $B_1((0), \nu, \lambda) = A_n(\mu', \lambda, (0)) = \{1 - q(\lambda), \ldots, -1, 1, \ldots, p(\lambda) - 1\}$ if λ is a hook.

Set $a := \dim A$, $b := \dim B$ for the rest of the paper.

Lemma 4.20. *Let $1 \leqslant i \leqslant n$ be an integer, and let $\mu' \in \lambda - (n - i + 1)$, $\nu \in \lambda - i$. Let $\mu \in \mu'_+$, and let $\nu' \in \nu_+$ be such that $b \notin B_i(\mu', \nu, \nu')$. Then $(c_\mu)_*(E_1) = 0$ in $\mathrm{Ext}^1(\mathbb{S}_{\mu'} A \otimes B, \mathbb{S}_\mu A)$.*

Proof. By Lemma 4.18 and a standard fact on extensions (see e.g., [MacL]), we have that

$$(c_\mu \otimes id)_*(E_2) = (id \otimes c_{\nu'})^*(V_i^2).$$

Since by Lemma 4.17, $(V_i^2) = 0$, we have that $(c_\mu \otimes id)_*(E_2) = 0$ in $\mathrm{Ext}^1(\mathbb{S}_{\mu'} A \otimes B \otimes \mathbb{S}_\nu B, \mathbb{S}_\mu A \otimes \mathbb{S}_\nu B)$.

Let

$$f : \mathrm{Ext}^1(\mathbb{S}_{\mu'} A \otimes B, \mathbb{S}_\mu A) \to \mathrm{Ext}^1(\mathbb{S}_{\mu'} A, \mathbb{S}_\mu A \otimes B^*)$$

be the isomorphism given by Lemma 4.15, let

$$\mathrm{Ext}^1(\mathbb{S}_{\mu'} A, \mathbb{S}_\mu A \otimes B^*) \xrightarrow{(id \otimes coev_{\mathbb{S}_\nu B^*})_*} \mathrm{Ext}^1(\mathbb{S}_{\mu'} A, \mathbb{S}_\mu A \otimes B^* \otimes \mathbb{S}_\nu B^* \otimes \mathbb{S}_\nu B),$$

and let

$$g : \mathrm{Ext}^1(\mathbb{S}_{\mu'} A, \mathbb{S}_\mu A \otimes B^* \otimes \mathbb{S}_\nu B^* \otimes \mathbb{S}_\nu B) \to \mathrm{Ext}^1(\mathbb{S}_{\mu'} A \otimes B \otimes \mathbb{S}_\nu B, \mathbb{S}_\mu A \otimes \mathbb{S}_\nu B)$$

be the isomorphism given by Lemma 4.15 (composed with the appropriate commutativity constraints). Then, it is straightforward to verify that

$$0 = (c_\mu \otimes id)_*(E_2) = \big(g \circ (id \otimes coev_{\mathbb{S}_\nu B^*})_* \circ f\big)\big((c_\mu)_*(E_1)\big).$$

Now, by our assumption on b and Theorem 4.12, the morphism $\theta_{B^*,n-i+1,\nu,\nu'}$ is a split injection. Therefore, by Lemma 4.16, $(id \otimes coev_{\mathbb{S}_\nu B^*})_*$ is injective, and the result follows. □

We are now ready to prove the key proposition for the proof of Theorem 4.3.

Proposition 4.21. *Assume there exist an integer $1 \leqslant i \leqslant n$, a pair of partitions $\mu' \in \lambda - (n-i+1)$, $\nu \in \lambda - i$ and a pair of partitions $\mu \in \mu'_+$, $\nu' \in \nu_+$, such that $a \notin A_i(\mu', \mu, \nu)$ and $b \notin B_i(\mu', \nu, \nu')$. Then $(V) = 0$ in* $\mathrm{Ext}^1(B, A)$.

Proof. By Theorem 4.12, the morphisms $\theta_{A,i-1,\mu',\mu}$ and $\theta_{B,n-i+1,\nu,\nu'}$ are split injections. Consider now the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Ext}^1(B,A) & \xrightarrow{f} & \mathrm{Ext}^1(B, A \otimes \mathbb{S}_{\mu'}A \otimes \mathbb{S}_{\mu'}A^*) & \xrightarrow{g} & \mathrm{Ext}^1(B, \mathbb{S}_\mu A \otimes \mathbb{S}_{\mu'}A^*) \\ & & \downarrow \cong & & \downarrow \cong \\ & & \mathrm{Ext}^1(B \otimes \mathbb{S}_{\mu'}A, A \otimes \mathbb{S}_{\mu'}A) & \xrightarrow{(c_\mu)_*} & \mathrm{Ext}^1(B \otimes \mathbb{S}_{\mu'}A, \mathbb{S}_\mu A), \end{array}$$

where $f := (id_A \otimes coev_{\mathbb{S}_{\mu'}A})_*$, $g := (c_\mu \otimes id_{\mathbb{S}_{\mu'}A^*})_*$, and the two vertical isomorphisms are given by Lemma 4.15. Observe that $gf = (\theta_{A,i-1,\mu',\mu})_*$ is injective. It is now clear that the proposition follows from Lemmas 4.16 and 4.20. □

5. The proof of Theorem 4.3

5.1. $F(\lambda) \subseteq G(\lambda)$

We have to show that if $d \in F(\lambda)$ then $d \in G(\lambda)$, i.e., that there exists a symmetric rigid tensor category $\mathcal{C}$ over k with a non-semisimple object V of dimension d such that $\mathbb{S}_\lambda V$ is semisimple. This follows from the following two observations.

1) Let r, s be nonnegative integers such that $r+s \geqslant 2$. One can introduce on the superspace $V := \mathbb{C}^{r|s}$ a structure of a non-semisimple representation of some supergroup (e.g., the supergroup of upper triangular matrices). On the other hand, if λ contains a box $(r+1, s+1)$ then $\mathbb{S}_\lambda V = 0$ (see e.g., [D2]), so $\mathbb{S}_\lambda V$ is automatically semisimple while V is not.
2) Suppose $\lambda = (q^p)$ is a rectangle. If V is a non-semisimple group representation of dimension $p > 1$, then $\mathbb{S}_\lambda V = (\bigwedge^p V)^{\otimes q}$ is 1-dimensional, so is automatically semisimple, while V is not. Finally, for $-q$, use now Remark 4.2.

We are done.

5.2. $G(\lambda) \subseteq F(\lambda)$

Let $\mathcal{C}$ be any symmetric rigid tensor category over k, and let $V \in \mathcal{C}$ be an object of $\mathcal{C}$. We have to show that if $\dim V \notin F(\lambda)$ and $\mathbb{S}_\lambda V$ is semisimple then so is V (i.e., $\dim V \notin G(\lambda)$). To this end, it is enough to show that if $\dim V \notin F(\lambda)$ then there exist an integer $1 \leqslant i \leqslant n$, a pair of partitions $\mu' \in \lambda - (n-i+1)$, $\nu \in \lambda - i$, and a pair of partitions $\mu \in \mu'_+$, $\nu' \in \nu_+$, satisfying the conditions of Proposition 4.21.

Let λ^* denote the conjugate of λ. Write

$$\lambda = (\lambda_1, \ldots, \lambda_p) \quad \text{and} \quad \lambda^* = \big(\lambda'_1, \ldots, \lambda'_q\big),$$

where $q = \lambda_1 \geqslant \cdots \geqslant \lambda_p \geqslant 1$ and $p = \lambda'_1 \geqslant \cdots \geqslant \lambda'_q \geqslant 1$.

5.2.1. The general case

In this subsection we prove that if $\dim V \notin R(\lambda)$ then the exact sequence (V) splits.

If $i = 1$, $A_1((0), (1), \nu) = \emptyset$ for any $\nu \in \lambda - 1$ (see Example 4.19), so there is no condition on a. Therefore, if b *is not* equal in k to an element of $B_1((0), (1), \nu)$ for some $\nu \in \lambda - 1$, we are done. So suppose b is equal in k to some element of

$$B_1\big((0), (1), \nu\big) = \{1 - q, \ldots, p - 1\},$$

which we shall continue to denote by b (so now $b \in \mathbb{Z}$).

Subcase 1. Suppose that $b > 0$, and set $i := p - b + 1$; then $2 \leqslant i \leqslant p$. Let $\mu' := (\lambda_{p-i+1}, \ldots, \lambda_p - 1)$ be the last i rows of λ without the last box, and let $\nu := (\lambda_1, \ldots, \lambda_{p-i})$ be the first $p - i$ rows of λ. Let $\mu := (\lambda_{p-i+1}, \ldots, \lambda_p)$ and let $\nu' := (\lambda_1, \ldots, \lambda_{p-i}, 1)$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$.

We now use (6) to find out that

$$A_{n_i}\big(\mu', \mu, \nu\big) = \big\{d \in k \;\big|\; p_{\mu',\mu}(d) = 0\big\} = \{1 - \lambda_{p-i+1}, \ldots, i - 1\} \tag{18}$$

and

$$B_{n_i}\big(\mu', \nu, \nu'\big) = \big\{d \in k \;\big|\; p_{\nu,\nu'}(d) = 0\big\} = \{1 - q, \ldots, p - i\}, \tag{19}$$

where $n_i := \sum_{j=p-i+1}^{p} \lambda_j$. We therefore see that $b = p - i + 1 \notin B_{n_i}(\mu', \nu, \nu')$. Now, if $a \notin A_{n_i}(\mu', \mu, \nu)$, we are done. Otherwise, we are done by our assumption on $\dim V$ (since $\dim V = a + b$).

Subcase 2. Suppose that $b = 0$. Then $0 \notin B_n(\mu', (0), (1)) = \emptyset$ for any $\mu' \in \lambda - 1$. Now, if $a \notin A_n(\mu', (0), (1))$, we are done. Otherwise, we are done by our assumption on $\dim V$.

Subcase 3. Suppose that $b < 0$, and set $i := q + b + 1$; then $2 \leqslant i \leqslant q$. Let $\mu' := (\lambda'_{q-i+1}, \ldots, \lambda'_q - 1)$ be the last i columns of λ without the last box, and let $\nu := (\lambda'_1, \ldots, \lambda'_{q-i})$ be the first $q - i$ columns of λ. Let $\mu := (\lambda'_{q-i+1}, \ldots, \lambda'_q)$ and let $\nu' := (\lambda'_1, \ldots, \lambda'_{q-i}, 1)$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$.

We now use (6) to find out that

$$A_{n_i}\big(\mu', \mu, \nu\big) = \big\{d \in k \;\big|\; p_{\mu',\mu}(d) = 0\big\} = \big\{1 - i, \ldots, \lambda'_{q-i+1} - 1\big\} \tag{20}$$

and

$$B_{n_i}\big(\mu', \nu, \nu'\big) = \big\{d \in k \;\big|\; p_{\nu,\nu'}(d) = 0\big\} = \{i - q, \ldots, p - 1\}, \tag{21}$$

where $n_i := \sum_{j=q-i+1}^{q} \lambda'_j$. We therefore see that $b = i - q - 1 \notin B_{n_i}(\mu', \nu, \nu')$. Now, if $a \notin A_{n_i}(\mu', \mu, \nu)$, we are done. Otherwise, we are done by our assumption on $\dim V$.

5.2.2. The non-rectangle case

Assume λ is not a rectangle. We have to show that $\dim V = p$ and $\dim V = -q$ are allowed. Let b, i be as in Subcase 1 of 5.2.1.

Let $\mu' := (\lambda_{p-i+2}, \ldots, \lambda_p)$ and $\nu := (\lambda_1, \ldots, \lambda_{p-i+1} - 1)$ be the last $i - 1$ rows of λ and the first $p - i + 1$ rows of λ without the last box, respectively. Choose $\mu \in \mu'_+$ with $i - 1$ rows (it exists since λ is not a rectangle!) and let $\nu' := (\lambda_1, \ldots, \lambda_{p-i+1})$. It follows easily from the Littlewood–Richardson rule that $\nu' \in \nu_+$. Moreover, we now have that $A_{n_i}(\mu', \mu, \nu) = \{1 - \lambda_{p-i+2}, \ldots, i - 2\}$, where $n_i := 1 + \sum_{j=p-i+2}^{p} \lambda_j$. Hence, $i - 1 \notin A_{n_i}(\mu', \mu, \nu)$ and $b = p - i + 1 \notin B_{n_i}(\mu', \nu, \nu')$. We thus conclude that $\dim V = p$ is allowed in this case, as claimed.

The claim that $\dim V = -q$ is allowed follows now from Remark 4.2.

5.2.3. The case $(3,2) \notin \lambda$ *or* $(2,3) \notin \lambda$

Suppose $(3,2) \notin \lambda$. We have to show that $\dim V = 1$ is allowed.

Subcase 1. Let b, i be as in Subcase 1 of 5.2.1.

First note that for $2 \leqslant i \leqslant p - 2$, $\lambda_{p-i+1} = 1$. Hence $b \notin B_{n_i}(\mu', \mu, \nu)$ and $1 - b \notin A_{n_i}(\mu', \mu, \nu)$ (see (18), (19)).

Now, for $i = p - 1$ (so $b = 2$), take $\mu' := (1^{p-1})$, $\mu := (1^p)$, $\nu := (q - 1, \lambda_2 - 1)$ and $\nu' := (q, \lambda_2 - 1)$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$. Moreover, $-1 \notin A_p(\mu', \mu, \nu)$ and $2 \notin B_p(\mu', \nu, \nu')$.

For $i = p$ (so $b = 1$), take $\mu' := (q, 1^{p-2})$, $\mu := (q, 1^{p-1})$, $\nu := (\lambda_2 - 1)$ and $\nu' := (\lambda_2)$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$. Moreover, $0 \notin A_{p+q-1}(\mu', \mu, \nu)$ and $1 \notin B_{p+q-1}(\mu', \nu, \nu')$.

Subcase 2. Let b, i be as in Subcase 2 of 5.2.1.

Take $\mu' := (\lambda_2 - 1)$, $\mu := (\lambda_2)$, $\nu := (q, 1^{p-2})$ and $\nu' := (q, 1^{p-1})$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$. Moreover, $1 \notin A_{\lambda_2}(\mu', \mu, \nu)$ and $0 \notin B_{\lambda_2}(\mu', \nu, \nu')$.

Subcase 3. Let b, i be as in Subcase 3 of 5.2.1.

First note that for $2 \leqslant i \leqslant q - 2$, $\lambda'_{q-i+1} = 1$. Hence $b \notin B_{n_i}(\mu', \mu, \nu)$ and $1 - b \notin A_{n_i}(\mu', \mu, \nu)$ (see (20), (21)).

Now, for $i = q - 1, q$ (so $b = -2, -1$), take $\mu' := (\lambda_1, \lambda_2 - 1)$, $\mu := (\lambda_1, \lambda_2)$, $\nu := (1^{p-2})$ and $\nu' := (1^{p-1})$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$. Moreover, $2, 3 \notin A_{n+2-p}(\mu', \mu, \nu)$ and $-1, -2 \notin B_{n+2-p}(\mu', \nu, \nu')$.

We therefore conclude that $\dim V = 1$ is allowed in this case, as claimed.

Finally, the claim that $\dim V = -1$ is allowed in the case $(2,3) \notin \lambda$ follows now from Remark 4.2.

5.2.4. The hook case

Assume λ is a hook. We have to show that $\dim V = 0$ is allowed.

Subcase 1. Let b, i be as in Subcase 1 of 5.2.1.

Since $\lambda_{p-i+1} = 1$ for $i < p$, we get from (18) that $-b \notin A_{n_i}(\mu', \mu, \nu)$. On the other hand, for $i = p$ (so $b = 1$), take $\mu' := (1^{p-1})$, $\mu := (1^p)$, $\nu := (q - 1)$ and $\nu' := (q)$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$. Moreover, $-1 \notin A_p(\mu', \mu, \nu)$ and $1 \notin B_p(\mu', \nu, \nu')$.

Subcase 2. Let b, i be as in Subcase 3 of 5.2.1.

Since $\lambda'_{q-i+1} = 1$ for $i < q$, we get from (20) that $-b \notin A_{n_i}(\mu', \mu, \nu)$. On the other hand, for $i = q$ (so $b = -1$), take $\mu' := (q - 1)$, $\mu := (q)$, $\nu := (1^{p-1})$ and $\nu' := (1^p)$. It follows easily from the Littlewood–Richardson rule that $\mu \in \mu'_+$ and $\nu' \in \nu_+$. Moreover, $1 \notin A_q(\mu', \mu, \nu)$ and $-1 \notin B_q(\mu', \nu, \nu')$.

We therefore conclude that $\dim V = 0$ is allowed in this case, as claimed.

This concludes the proof of the theorem.

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