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Cesàro Summability of Hardy Spaces on the Ring of Integers in a Local Field

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Let $\mathcal{O}$ be the ring of integers in a local field K. We solve an open problem due to M. H. Taibleson (1975, "Math. Notes," Vol. 15, Princeton Univ. Press, Princeton, NJ): Suppose $f \in L^1(\mathcal{O})$. Does the Cesàro means of f converge to f almost everywhere if K has characteristic zero? To this end we study the (H^p, L^p) boundedness of the associated maximal operator σ^* to get the corresponding interpolation result on Hardy–Lorentz spaces; in particular we obtain that σ^* is of weak type $(1, 1)$. The proof mainly depends on certain estimates for the oscillatory Dirichlet kernels, which are refinements of those obtained earlier by the author (1997, *J. Math. Anal. Appl.* **208**, 528–552).

Key Words: local field; Cesàro means; atomic Hardy spaces; interpolation.

1. INTRODUCTION

In 1955, Fine [3] proved that, similar to the case of trigonometric Fourier series, the Walsh–Fourier series of a function f in $L^1([0, 1])$ is Cesàro summable almost everywhere; Fine's proof was based on a crucial lemma [3, Lemma 3].

In 1967, Taibleson [11] extended the pointwise convergence result for $(C, 1)$ means of L^1-functions defined on the ring of integers in a p-series field K (char $K = p$, p a prime), using a p-series variant of Fine's lemma (it is known that the dyadic group, or the Walsh–Paley group can be identified with the ring of integers in a 2-series field). Later, he asked in

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[12, Chap. II, p. 114] whether or not the conclusion holds for K of characteristic zero.

From the late 1970s to the 1990s many authors [4, 6, 8, 15] have studied the Cesàro summability of Hardy spaces on the Walsh–Paley group, dyadic martingales, or even higher dimensional generalizations of the Walsh system, but little progress was made for the above stated question on a p-adic field.

In this paper we study the $(C,1)$ summability for Hardy spaces $H^{\epsilon}(\mathscr{O})$. We prove the ϵ-quasi-locality of σ^* by a close analysis of the behavior of the Dirichlet kernels $D_{q^k}(x)$ on $\mathscr{P}^{-1}$ (Main Lemma). Thus σ^* maps H^{ϵ} boundedly into L^{ϵ}, $\frac{1}{2} < \epsilon \leq 1$, and it follows by interpolation that σ^* maps $H^{\epsilon,q}$ boundedly into $L^{\epsilon,q}$, $0 < q \leq \infty$. As a corollary we obtain that σ^* is of weak type $(1,1)$ and hence the conclusion of the almost everywhere convergence of $\sigma_n f$ for L^1 functions.

The Main Lemma reads as follows (we postpone the proof to Section 4).

MAIN LEMMA. (i) *Suppose* char $K = p$. *Then* $D_{q^k}(x) \geq 0$ *for* $x \in \mathscr{P}^{-1}$ *and*

$$D_{q^k}(x) = \begin{cases} q^k & \text{if } x \in \alpha_{-1}\beta^{-1} + \mathscr{P}^k,\ \alpha_{-1} \in GF(q), \\ 0 & \text{if otherwise in } \mathscr{P}^{-1}. \end{cases}$$

(ii) *Suppose* char $K = 0$. *If* $x = (x_{-1}, x_0, \dots, x_{k-1})$, *the sum*

$$\begin{aligned} I(x) := \sum_{(\alpha_k, \dots, \alpha_{\ell-1})} \Big| D_{q^\ell}\big(x_{-1}\beta^{-1} + \cdots + x_{k-1}\beta^{k-1} \\ + \alpha_k \beta^k + \cdots + \alpha_{\ell-1}\beta^{\ell-1}\big)\Big| \\ \leq C_q (\ell - k)^c q^{\ell-k} p^{\|\Theta(x)\|}; \end{aligned}$$

moreover,

$$\sum_{x=(x_{-1},\dots,x_{k-1})} p^{\|\Theta(x)\|} \leq C_q k^c q^k,$$

where x_i, $\alpha_j \in GF(q)$, $c = \log_p q$. *The mapping* $\Theta : x \mapsto (Y^0, Y^1, \dots, Y^{c-1}) := \prod_{j=0}^{c-1} Y^j$ *is defined as in Claim* 1 *of Section* 4 *with* $Y^j = (y^j_{-1,0}, y^j_{0,0}, \dots, y^j_{k-1,0})$, $y^j_{i,0} \in GF(p)$; *and the "norm"* $\|\cdot\|$ *is given by* $\|\Theta(x)\| = \sum_{j=0}^{c-1} \|Y^j\|$ *with* $\|Y^j\| = 0$ *if* $Y^j \in$ *Case* A; s *if* $Y^j \in$ *Case* B_s *or* C_s $(1 \leq s \leq k)$; k *if* $Y^j \in$ *Case* D; *and* $-\infty$ *if* $Y^j \in$ *Case* E (*consult Section* 4 *or* [17, *Sect.* 4] *for descriptions of the cases* A, B_s, C_s, D, E).

Part (i) of the Main Lemma can be derived directly from the proof of Lemma 5(i) in [17, p. 543]. Part (ii) is a refinement of the crucial estimate for I_k contained in the proof of Lemma 5(ii) in [17, Sect. 4, p. 549], where we note that the estimate for I_k in [17] is a special case of the Main Lemma (ii).

Here we remark that one can also use a p-series variant of Fine's lemma (see Yano [16]) to directly get the weak type $(1,1)$ result when char $K = p$. In this connection we refer to, in passing, [9, Chap. 1, Sect. 2.3] for a related result on the integral of Marcinkiewicz. But the similar method does not seem to apply for the p-adic cases (char $K = 0$) due to the unavailability of a p-adic version of the decomposition formula for K_{2^n}, the Walsh–Fejér kernels, given, e.g., in [8; 15, (8), (9)].

2. DEFINITIONS AND NOTATION

Let K be a locally compact, totally disconnected, nondiscrete (complete) field [12]. If K is of finite characteristic, it is a field of formal Laurent series over $GF(q)$, the Galois field, $q = p^c$ (p a prime). If K is of characteristic zero then K is either a p-adic number field or a finite algebraic extension of such a field.

Denote by $\mathscr{O}$ the ring of integers in K and β a prime element in K, $|\beta| = q^{-1}$. Then $\mathscr{O} = \{x \in K : |x| \le 1\}$ and the fractional ideals $\mathscr{P}^k = \{x \in K : |x| \le q^{-k}\}$, whose characteristic function is denoted by Φ_k, $k \in \mathbb{Z}$. Since $\mathscr{O}/\mathscr{P}$ is isomorphic to $GF(q)$ (the prime ideal $\mathscr{P} := \mathscr{P}^1 = \beta\mathscr{O}$), $GF(q)$ will be identified with $\{\alpha_k\}_{k=0}^{q-1}$, a fixed full set of coset representatives of $\mathscr{P}$ in $\mathscr{O}$. With this identification, one can write $\alpha_k = \sum_{j=0}^{c-1} \alpha_{k,j} \epsilon_j$ (mod $\mathscr{P}$), $\alpha_{k,j} \in GF(p)$, and $\{\epsilon_j\}_{j=0}^{c-1}$ being a basis of $GF(q)$ over $GF(p)$. Thus every $x \in \mathscr{P}^s = \beta^s \mathscr{O}$, $s \in \mathbb{Z}$, has a unique representation [12, p. 10; 17]

$$x = \sum_{\ell=s}^{\infty} x_\ell \beta^\ell, \qquad x_\ell \in GF(q).$$

Let χ be a (fixed) character of K that is trivial on $\mathscr{O}$ but nontrivial on $\mathscr{P}^{-1}$. If $\{u(k)\}_{k=0}^{\infty}$ is a complete set of coset representatives of $\mathscr{O}$ in K (with the natural ordering [12, p. 84]), $\{\chi_{u(k)}\}_{k=0}^{\infty}$ forms a complete orthonormal system on $\mathscr{O}$. Write $\chi_k = \chi_{u(k)}$.

As in [17, 19], choose the base values of the character χ as follows:

if char $K = p$,

$$\chi\left(\epsilon_\mu \beta^{-j}\right) = \begin{cases} \exp\dfrac{2\pi i}{p}, & \mu = 0,\ j = 1 \\ 1, & \mu = 1,\dots,c-1 \text{ or } j \neq 1; \end{cases}$$

if char $K = 0$,

$$\chi\left(\epsilon_\mu \beta^{-j}\right) = \begin{cases} \exp\dfrac{2\pi i}{p^j}, & \mu = 0,\ j \in \mathbb{N} \\ 1, & \mu = 1,\dots,c-1,\ j \in \mathbb{N}. \end{cases} \tag{1}$$

Since any character χ' that is trivial on $\mathscr{O}$ but nontrivial on $\mathscr{P}^{-1}$ can be expressed as $\chi'(x) = \chi_\xi(x)$ for some $\xi \in \mathscr{O} \setminus \mathscr{P}$, it is easy to see that results of this paper are independent of our choice of χ.

Define the $(C, 1)$ kernels to be

$$K_n(x) = \frac{1}{n} \sum_{k=1}^{n} D_k(x),$$

where $D_k(x)$ are the Dirichlet kernels given by $D_n = \sum_{k=0}^{n-1} \chi_k$.

The *Cesàro means of f* are the sums

$$\sigma_n f = K_n * f = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{f}(u(k))\,\chi_k.$$

The *maximal operator associated with Cesàro means* is

$$\sigma^* f(x) = \sup_{n \geq 0} \left|\sigma_n f(x)\right|.$$

In the remaining part of this section we need certain concepts of Hardy spaces and an interpolation result, in the local field setting.

Let us briefly recall the definition of atomic Hardy spaces [7]. For $0 < \epsilon \leq 1$, a function $a : \mathscr{O} \to \mathbb{C}$ is called a *regular ϵ-atom* if there exists a sphere $I = x_0 + \mathscr{P}^N$ such that

(i) $\operatorname{supp} a \subset I$

(ii) $\|a\|_\infty \leq |I|^{-1/\epsilon}$

(iii) $\int_{\mathscr{O}} a(x)\,dx = 0.$

The *exceptional atom* is the constant function $\Phi_0(x) \equiv 1$.

DEFINITION 1. The *Hardy space* $H^\epsilon := H^\epsilon(\mathcal{O})$ consists of those tempered distributions $f \in \mathcal{S}' := \mathcal{S}'(\mathcal{O})$ admitting an atomic decomposition

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where a_k are ϵ-atoms (regular or exceptional) with $\sum|\lambda_k|^\epsilon < \infty$. Moreover, the norm $\|f\|_{H^\epsilon}$ is the infimum of the numbers $(\sum|\lambda_k|^\epsilon)^{1/\epsilon}$ taken over all such representations of f.

The (atomic) H^ϵ spaces can also be characterized in terms of the "Poisson" maximal function $D^* f = \sup_{k \geq 0}|f * D_{q^k}|$, with

$$D_{q^k} = q^k \Phi_k. \tag{2}$$

DEFINITION 2. Let $0 < \epsilon \leq 1$. A distribution $f \in \mathcal{S}'$ belongs to H_*^ϵ if and only if $D^* f \in L^\epsilon$, with the norm $\|f\|_{H_*^\epsilon} := \|D^* f\|_\epsilon < \infty$. Moreover, we have $\|f\|_{H_*^\epsilon} \approx \|f\|_{H^\epsilon}$ (see [7] for its locally compact version on Vilenkin groups).

If we let $1 < \epsilon \leq \infty$ in Definition 2, we will have $H^\epsilon = L^\epsilon$, $1 < \epsilon \leq \infty$, as is known.

The facts in Definition 2 are also included in Definition 3, below.

Recall that the *Lorentz space* $L^{\epsilon,q}(\mathcal{O})$, $0 < \epsilon < \infty$, $0 < q \leq \infty$ is the set of all measurable functions f, with the quasi-norm

$$\|f\|_{\epsilon,q} := \left(\int_0^\infty \left[t^{1/\epsilon}\tilde{f}(t)\right]^q \frac{dt}{t}\right)^{1/q} < \infty$$

and for $q = \infty$,

$$\|f\|_{\epsilon,\infty} := \sup_{t>0} t^{1/\epsilon}\tilde{f}(t) < \infty,$$

where $\tilde{f}$ denotes the nonincreasing rearrangement of f [1, 15].

DEFINITION 3. For $0 < \epsilon < \infty$, $0 < q \leq \infty$, the *Hardy–Lorentz space* $H^{\epsilon,q} := H^{\epsilon,q}(\mathcal{O}) \subset \mathcal{S}'$ consists of all distributions satisfying

$$\|f\|_{H^{\epsilon,q}} := \|D^* f\|_{\epsilon,q} < \infty.$$

It is easy to see that $L^{\epsilon,\epsilon} = L^\epsilon$, $L^{\epsilon,\infty} = L_*^\epsilon$ (the weak-L^ϵ space), $0 < \epsilon \leq \infty$. Therefore, $H^{\epsilon,\epsilon} = H_*^\epsilon$, $0 < \epsilon \leq \infty$, and L^1 is continuously embedded into $H^{1,\infty}$ (the weak-H^1 space), because D^* maps L^1 boundedly into weak-L^1. For $\epsilon > 1$, we also have $H^{\epsilon,q} = L^{\epsilon,q}$.

The dyadic martingale version of these facts is contained in [13, Chap. 5; 15] while we notice that $\{f * D_{q^k}\}_{k \geq 0}$ is a regular martingale on $\mathscr{O}$ for $f \in \mathscr{S}'$, where $f * D_{q^k}(x) = \langle f, D_{q^k}(x - \cdot)\rangle \in \mathscr{S} \cap \mathscr{S}', \mathscr{S} = \mathscr{S}(\mathscr{O})$ being the test function class (see [12, Chaps. IV, III] for its locally compact version). So, saying a distribution $f \in H_*^\epsilon$ $(0 < \epsilon \leq 1)$ is equivalent to saying the martingale $\{f * D_{q^k}\}_{k \geq 0} \in H_*^\epsilon = H^\epsilon$, as called in [15]. The same is true for $H^{\epsilon, q}$.

Borrowing the definition from [15], we say an operator T is *ϵ-quasi-local* if for any ϵ-atom a with support I,

$$\int_{\mathscr{O} \setminus I} |Ta|^\epsilon \, dx \leq C_\epsilon. \tag{3}$$

Note that if T commutes with translations, then in order to show T is ϵ-quasi-local it is enough to show T satisfies the above inequality for any ϵ-atom a with support $I = \mathscr{P}^N$.

We will need the next local field analog of an interpolation theorem due to Weisz (combining Theorem 1 and Theorem B in [15]).

THEOREM W. *Let $0 < \epsilon_0 \leq 1$. Suppose a sublinear T is ϵ_0-quasi-local and bounded on L^∞. Then*

(i) *T is bounded from H^{ϵ_0} to L^{ϵ_0};*

furthermore, we have by interpolation

(ii) *T is bounded from $H^{\epsilon, q}$ to $L^{\epsilon, q}$ for every $\epsilon_0 < \epsilon < \infty$ and $0 < q \leq \infty$. In particular, T is of weak type $(1,1)$ whenever $0 < \epsilon_0 < 1$.*

We omit the proof; it is strictly parallel to Weisz's proof in the context of dyadic martingale Hardy spaces (see [13–15]); part (ii) is also the analog of the Euclidean case (see [1, 2]). Here we remark that under the same conditions of Theorem W, with $0 < \epsilon_0 < 1$, one obtains that T is bounded from H^{ϵ_0} to L^{ϵ_0}. Then, from this one may directly claim that *T has to be of weak type* $(1,1)$ (without recourse to part (ii) of Theorem W). See [5, p. 312] for a similar remark on $\mathbb{R}^n$ settings (of course, though, Theorem W is more general).

3. THE MAXIMAL OPERATOR σ^*

From [12, p. 97; 17] we know that

$$\|K_n\|_1 \leq A,$$

which implies that σ^* is bounded on L^∞. If we can show that σ^* is ϵ-quasi-local $(\frac{1}{2} < \epsilon \leq 1)$, we will have the following main theorem by Theorem W.

THEOREM. *Let* $\frac{1}{2} < \epsilon \leq 1$. *Then* σ^* *is well defined on* H^ϵ *and satisfies*

(i) $\|\sigma^* f\|_\epsilon \leq C_\epsilon \|f\|_{H^\epsilon}$.

Moreover,

(ii) *if* $\frac{1}{2} < \epsilon < \infty$ *and* $0 < q \leq \infty$, *then* σ^* *is well defined on* $H^{\epsilon, q}$ *and satisfies*

$$\|\sigma^* f\|_{\epsilon, q} \leq C_{\epsilon, q} \|f\|_{H^{\epsilon, q}}.$$

In particular, σ^* *is of weak type* $(1, 1)$, *i.e.*,

$$|\{\sigma^* f > \lambda\}| \leq C\lambda^{-1} \|f\|_1.$$

Remark. For $\epsilon = q > 1$, the theorem implies that σ^* is bounded on L^ϵ, which was proved in [18, Theorem 6] as a special case for the Riesz means with $R^*_{1,1} = \sigma^*$.

Our main concern will be to prove the result for char $K = 0$. We would rather, however, treat both cases char $K = p$ and char $K = 0$ in a comparable way, through which it is hoped that the proof for the case char $K = p$ may somehow illustrate our method used to deal with the more complicated case char $K = 0$.

Since σ^* commutes with translations, it is enough to show σ^* satisfies (3) for any regular ϵ-atom a centered at 0 with support $\mathscr{P}^N$, in view of the remark before Theorem W. Henceforth fix a to be such an atom.

The condition supp $a \subset \mathscr{P}^N$ implies that $\hat{a}$ is constant on cosets of $\mathscr{P}^{-N}$, so $\hat{a}(u(k)) = \hat{a}(0) = \int_{\mathscr{O}} a = 0$ whenever $|u(k)| \leq q^N$.

Let $n = \sum_{i=0}^{\ell} r_i q^i$, $0 \leq r_i < q$, $r_\ell \neq 0$. Then if $n \leq q^N$,

$$\sigma_n a(x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{a}(u(k))\, \chi_k(x) = 0,$$

recalling that $|u(k)| = q^j$ if and only if $q^{j-1} \leq k < q^j$, $j \geq 1$. We have

$$\sigma^* a(x) = \sup_{n > q^N} |\sigma_n a(x)|. \tag{4}$$

We begin with the following lemma. Set $f = |a|$ throughout this section. Define

$$K^* f(x) = \sup_{\ell \geq N} |K_{q^\ell}| * f(x)$$

$$\mathscr{L} f(x) = q^{-N} \sum_{j > s}^{N} q^j |K_{q^j}| * f(x).$$

LEMMA 1. *If* $x \notin \mathscr{P}^N$, $|x| = q^{-s}$ $(0 \le s < N)$, *we have*

$$\sigma^* a(x) \le C\big(K^* f(x) + \mathscr{L} f(x) + q^{2(s-N)+N/\epsilon}\big).$$

Assuming Lemma 1, we apply Jensen's inequality to get for $0 < \epsilon \le 1$, $|x| = q^{-s}$ $(0 \le s < N)$

$$(\sigma^* a)^\epsilon(x) \le C\big(K^* f^\epsilon(x) + \mathscr{L} f^\epsilon(x) + q^{2(s-N)\epsilon + N}\big).$$

Observe that the function $R(x) := q^{N(1/\epsilon - 2)}|x|^{-2}(1 - \Phi_N) \in L^\epsilon$ with $\|R\|_\epsilon \le C_\epsilon (\frac{1}{2} < \epsilon \le 1)$. In view of Lemma 1, to show that $\sigma^* f$ satisfies (3) it suffices to show the following two lemmas.

LEMMA 2. *Suppose* (i) char $K = p$, $0 < \epsilon \le 1$ *or* (ii) char $K = 0$, $\frac{1}{2} < \epsilon \le 1$. *Then*

$$\int_{x \notin \mathscr{P}^N} (K^* f)^\epsilon \, dx \le C_\epsilon.$$

LEMMA 3. *Suppose* $\frac{1}{2} < \epsilon \le 1$. *Then*

$$\int_{x \notin \mathscr{P}^N} (\mathscr{L} f)^\epsilon \, dx \le C_\epsilon.$$

Proof of Lemma 1. Let $n = rq^k + t$, $r \ge 0$, $k \ge 0$, $0 \le t < q^k$. From [17, Lemma 2], we have the recurrence formula

$$\begin{aligned} nK_n(x) &= q^k D_{q^k}(x)(r-1)K_{r-1}(\beta^{-k}x) + tD_{q^k}(t)D_r(\beta^{-k}x) \\ &\quad + D_r(\beta^{-k}x)q^k K_{q^k}(x) + \chi_r(\beta^{-k}x)tK_t(x) \end{aligned} \tag{5}$$

(cf. [12, p. 99, (6.29)] which should read as above; there was a minor misprint).

Let $n = \sum_{j=0}^{\ell} r_j q^j$, $0 \le r_j < q$, $r_\ell \ne 0$, and $t_k = \sum_{j=0}^{k-1} r_j q^j$ $(1 \le k \le \ell + 1)$. Then (5) gives

$$\begin{aligned} nK_n(x) = & \sum_{j=1}^{\ell} q^j D_{q^j}(x)(r_j - 1)K_{r_j - 1}(\beta^{-j}x)\chi_{n - t_{j+1}}(x) \\ & + \sum_{j=1}^{\ell} t_j D_{q^j}(x) D_{r_j}(\beta^{-j}x)\chi_{n - t_{j+1}}(x) \\ & + \left(\sum_{j=1}^{\ell} D_{r_j}(\beta^{-j}x) q^j K_{q^j}(x)\chi_{n - t_{j+1}}(x) + r_0 K_{r_0}(x)\chi_{n - r_0}(x)\right) \\ := & \sum_1 + \sum_2 + \sum_3, \end{aligned}$$

using $\chi_{\rho q^k+\tau}(x) = \chi_\rho(\beta^{-k}x)\chi_\tau(x)$, $\rho, k \geq 0$, $0 \leq \tau < q^k$ [17, Lemma 1]. It follows that

$$|K_n * a(x)| \leq n^{-1}\left(\left|\sum_1\right| + \left|\sum_2\right| + \left|\sum_3\right|\right) * f(x). \tag{6}$$

Let $\ell \geq N$, $|x| = q^{-s}$ $(0 \leq s < N)$. For the first term on the RHS of (6),

$$\begin{aligned} n^{-1}\left|\sum_1\right| * f(x) &\leq q^2 n^{-1}\left(\sum_{j>s}^{\ell} q^j D_{q^j} * f(x) + \sum_{j=1}^{s} q^j D_{q^j} * f(x)\right) \\ &\leq 0 + Cn^{-1}\left(\sum_{j=1}^{s} q^{2j}\right)\|f\|_1 \\ &\leq Cq^{2(s-N)+N/\epsilon}, \end{aligned}$$

where we have used (2) and the conditions supp $f \subset \mathscr{P}^N$ and $\|f\|_\infty \leq q^{N/\epsilon}$. Similarly,

$$n^{-1}\left|\sum_2\right| * f(x) \leq Cq^{2(s-N)+N/\epsilon}.$$

For the last term on the RHS of (6),

$$\begin{aligned} n^{-1}\left|\sum_3\right| * f(x) &\leq Cn^{-1}\left(\sum_{j>N}^{\ell} q^j |K_{q^j}| * f(x) + \sum_{j>s}^{N} q^j |K_{q^j}| * f(x)\right. \\ &\qquad \left. + \sum_{j=0}^{s} q^j K_{q^j} * f(x)\right) \\ &\leq Cn^{-1}\left(\sum_{j>N}^{\ell} q^j\right) K^* f(x) + \mathscr{L}f(x) + Cn^{-1}\left(\sum_{j=1}^{s} q^{2j}\right)\|f\|_1 \\ &\leq C\left(K^* f(x) + \mathscr{L}(x) + q^{2(2-N)+N/\epsilon}\right). \end{aligned}$$

Combining the above inequalities proves Lemma 1.

Proof of Lemma 2. Like in the proof of Lemma 6 of [17], write, with $r = q^{\ell-k}$,

$$\begin{aligned} K_{q^\ell}(t) &= \sum_{\rho=0}^{r-1} \sum_{\tau=0}^{q^k-1} \left(1 - \frac{\rho q^k + \tau}{q^\ell}\right) \chi_\rho(\beta^{-k}t)\chi_\tau(t) \\ &= D_{q^k}(t) \sum_{\rho=0}^{r-1}\left(1 - \frac{\rho q^k}{q^\ell}\right)\chi_\rho(\beta^{-k}t) - D_r(\beta^{-k}t)\sum_{\tau=0}^{q^k-1} \frac{\tau}{q^\ell}\chi_\tau(t). \end{aligned}$$

If $|t| = q^{-k+1}$ $(1 \le k \le \ell)$, use (2) to get

$$\left|K_{q^\ell}(t)\right| \le q^{2k-\ell}\left|D_r(\beta^{-k}t)\right|. \tag{7}$$

Let $\ell > N$ and $x \in x_s\beta^s + x_{s+1}\beta^{s+1} + \cdots + \chi_{N-1}\beta^{N-1} + \mathscr{P}^N$, $x_s \ne 0$, $0 \le s < N$. Then the condition $\operatorname{supp} f \subset \mathscr{P}^N$ implies that, using (7),

$$\begin{aligned} |K_{q^\ell}| * f(x) &= \int_{|t|=q^{-s}} f(x-t)\left|K_{q^\ell}(t)\right| dt \\ &\le q^{2s-\ell+2}\int_{|t|=q^{-s}} f(x-t)\left|D_{q^{\ell-s-1}}(\beta^{-s-1}t)\right| dt \\ &= q^{s-\ell+1}\int_{|u|=q} f(x-\beta^{s+1}u)\left|D_{q^{\ell-s-1}}(u)\right| du := I^\ell(x). \end{aligned}$$

We consider the cases char $K = p$ and 0 separately.

Case I. char $K = p$. By the Main Lemma (i),

$$\begin{aligned} I^\ell(x) &\le q^{s-\ell+1}\sum_{\alpha_{-1}\in GF(q)^*} q^{\ell-s-1}\int_{\alpha_{-1}\beta^{-1}+\mathscr{P}^{\ell-s-1}} f(x-\beta^{s+1}u)\,du \\ &= q^{s+1}\sum_{\alpha_{-1}\ne 0}\int_{\alpha_{-1}\beta^s+\mathscr{P}^\ell} f(x-t)\,dt, \end{aligned}$$

where the sum is taken over the set $GF(q)^* := GF(q)\setminus\{0\}$.

Notice that for $t \in \alpha_{-1}\beta^s + \mathscr{P}^\ell$, $x - t \in \mathscr{P}^N$ provided

$$(x_s,\ldots,x_{N-1}) = \Big(\alpha_{-1}\,\overbrace{0,\ldots,0}^{N-s-1}\Big) \in \overbrace{GF(q)\times\cdots\times G(q)}^{N-s} := G(q)^{N-s}.$$

One finds, by the condition $\operatorname{supp} f \subset \mathscr{P}^N$, that the sum equals $\int_{\mathscr{P}^\ell} f(u)\,du$ if x is in $\alpha_{-1}\beta^s + \mathscr{P}^N$ for some $\alpha_{-1} \in GF(q)^*$; 0 if x is otherwise in $\mathscr{P}^s \setminus \mathscr{P}^{s+1}$.

Thus we obtain the pointwise estimates: let $|x| = q^{-s}$ $(0 \le s < N)$. Then

$$I^\ell(x) \le \begin{cases} Cq^{s-\ell+N/\epsilon} & \text{if } x \in \alpha_{-1}\beta^s + \mathscr{P}^N,\ \alpha_{-1} \ne 0 \\ 0 & \text{if otherwise in } \mathscr{P}^s \setminus \mathscr{P}^{s+1}. \end{cases}$$

It follows that for $|x| = q^{-s}$,

$$K^*f(x) \le \sup_{\ell > N} I^\ell(x) \le Cq^{s-N+N/\epsilon}\Phi_{\cup\,\alpha_{-1}\ne 0}\left\{\alpha_{-1}\beta^s + \mathscr{P}^N\right\},$$

where Φ_E denotes the characteristic function of the set E.

Therefore if $0 < \epsilon \leq 1$,

$$\int_{x \notin \mathscr{P}^N} (K^* f)^{\epsilon}\, dx = \sum_{s=0}^{N-1} \int_{|x| = q^{-s}} (K^* f)^{\epsilon}\, dx$$
$$\leq C \sum_{s=0}^{N-1} q^{(s-N)\epsilon + N} q^{-N} (q-1) \leq C \sum_{j=1}^{\infty} q^{-j\epsilon} := C_{\epsilon},$$

which proves Lemma 2 for char $K = p$.

Case II. char $K = 0$. Since D_{q^k} is constant on cosets of $\mathscr{P}^k$,

$$I^{\ell}(x) \leq q^{s-\ell+1} \sum_{\substack{(\alpha_{-1}, \alpha_0, \dots, \alpha_{\ell-s-2}) \in GF(q)^{\ell-s} \\ \alpha_{-1} \neq 0}} \left| D_{q^{\ell-s-1}}\left(\alpha_{-1} \beta^{-1} + \alpha_0 \beta^0 + \cdots + \alpha_{\ell-s-2} \beta^{\ell-s-2} \right) \right|$$
$$\cdot \int_{\alpha_{-1} \beta^{-1} + \cdots + \alpha_{\ell-s-2} \beta^{\ell-s-2} + \mathscr{P}^{\ell-s-1}} f\left(x - \beta^{s+1} u\right) du$$
$$= q^{2s-\ell+2} \sum_{\substack{(\alpha_{-1}, \dots, \alpha_{\ell-s-2}) \\ \alpha_{-1} \neq 0}} \left| D_{q^{\ell-s-1}}\left(\alpha_{-1} \beta^{-1} + \cdots + \alpha_{\ell-s-2} \beta^{\ell-s-2} \right) \right|$$
$$\cdot \int_{\alpha_{-1} \beta^{s} + \cdots + \alpha_{\ell-s-2} \beta^{\ell-1} + \mathscr{P}^{\ell}} f(x-t)\, dt.$$

Again $\operatorname{supp} f \subset \mathscr{P}^N$ implies that if $(\alpha_{-1}, \alpha_0, \dots, \alpha_{N-s-2}) \neq (x_s, x_{s+1}, \dots, x_{N-1})$ in $GF(q)^{N-s}$, then $f(x-t) = 0$ for $t \in \alpha_{-1} \beta^{s} + \cdots + \alpha_{N-s-2} \beta^{N-1} + \mathscr{P}^N$; and so

$$I^{\ell}(x) \leq q^{2s-\ell+2}$$
$$\cdot \sum_{\substack{(\alpha_{N-s-1}, \dots, \alpha_{\ell-s-2}) \\ \in GF(q)^{\ell-N}}} \left| D_{q^{\ell-s-1}}\left(x_s \beta^{-1} + \cdots + x_{N-1} \beta^{N-s-2} + \alpha_{N-s-1} \beta^{N-s-1} + \cdots + \alpha_{\ell-s-2} \beta^{\ell-s-2} \right) \right|$$
$$\cdot \int_{x_s \beta^{s} + \cdots + x_{N-1} \beta^{N-1} + \alpha_{N-s-1} \beta^{N} + \cdots + \alpha_{\ell-s-2} \beta^{\ell-1} + \mathscr{P}^{\ell}} f(x-t)\, dt$$
$$\leq C \|f\|_{\infty} \cdot q^{2(s-\ell)}$$
$$\cdot \sum_{(\alpha_{N-s-1}, \dots, \alpha_{\ell-s-2})} \left| D_{q^{\ell-s-1}}\left(x_s \beta^{-1} + \cdots + x_{N-1} \beta^{N-s-2} + \alpha_{N-s-1} \beta^{N-s-1} + \cdots + \alpha_{\ell-s-2} \beta^{\ell-s-2} \right) \right|.$$

Now substitute $\ell - s - 1, N - s - 1$ into ℓ, k, respectively, in the Main Lemma (ii) to establish that for $\ell > N$,

$$
\begin{aligned}
I^{\ell}(x) &\le C\|f\|_{\infty} q^{2(s-\ell)} (\ell - N)^{c} q^{\ell - N} p^{\|\Theta(x)\|} \\
&\le C(\ell - N)^{c} q^{N-\ell} q^{2(s-N)+N/\epsilon} p^{\|\Theta(x)\|}.
\end{aligned}
$$

Hence if $x \in x_s \beta^s + \cdots + x_{N-1}\beta^{N-1} + \mathscr{P}^N$ $(x_s \neq 0)$, $0 \le s < N$ we obtain the pointwise estimates

$$
K^* f(x) \le C q^{2(s-N)+N/\epsilon} p^{\|\Theta(x)\|}.
$$

We evaluate the integral for $0 \le s < N$, $0 < \epsilon \le 1$

$$
\begin{aligned}
\int_{|x|=q^{-s}} (K^* f)^{\epsilon}\, dx &= \sum_{\substack{(x_s, \dots, x_{N-1}) \\ x_s \neq 0}} \int_{x_s\beta^s + \cdots + x_{N-1}\beta^{N-1} + \mathscr{P}^N} (K^* f)^{\epsilon}\, dx \\
&\le C q^{-N} \sum_{\substack{(x_s, \dots, x_{N-1}) \\ x_s \neq 0}} q^{2(s-N)\epsilon + N} p^{\|\Theta(x)\|\epsilon} \\
&\le C q^{2(s-N)\epsilon} \sum_{\substack{(x_s, \dots, x_{N-1}) \\ x_s \neq 0}} p^{\|\Theta(x)\|} \\
&\le C q^{2(s-N)\epsilon} (N-s)^{c} q^{N-s} \\
&= C (N-s)^{c} q^{(2\epsilon - 1)(s-N)}
\end{aligned}
$$

by the second inequality in the Main Lemma (ii).

Therefore if $\frac{1}{2} < \epsilon \le 1$,

$$
\begin{aligned}
\int_{x \notin \mathscr{P}^N} (K^* f)^{\epsilon}\, dx &= \sum_{s=0}^{N-1} \int_{|x|=q^{-s}} (K^* f)^{\epsilon}\, dx \\
&\le C \sum_{s=0}^{N-1} (N-s)^{c} q^{(2\epsilon - 1)(s-N)} \\
&\le C \sum_{j=1}^{\infty} j^{c} q^{-(2\epsilon - 1)j} := C_{\epsilon},
\end{aligned}
$$

which proves Lemma 2 for char $K = 0$. The proof is complete.

Proof of Lemma 3. First we establish the following estimates.
For $|x| = q^{-s}$, $0 \leq s < N$

$$\mathscr{L}f(x) \leq \begin{cases} q^{s-2N+N/\epsilon} \sum_{j>s}^{N} q^j \Phi_{\cup_{\alpha_{-1} \neq 0}\{\alpha_{-1}\beta^s + \mathscr{P}^j\}}(x) & \text{if char } K = p, \\ q^{2(s-N)+N/\epsilon} \sum_{j>s}^{N} \left| D_{q^{j-s-1}}(x\beta^{-s-1}) \right| & \text{if char } K = 0. \end{cases} \tag{8}$$

As in the proof of Lemma 2, we have for $|x| = q^{-s}$ $(0 \leq s < N)$, with ℓ replaced by j $(s < j \leq N)$

$$|K_{q^j}| * f(x) \leq q^{s-j+1} \int_{|u|=q} f(x - \beta^{s+1}u) \left| D_{q^{j-s-1}}(u) \right| du := I^j(x).$$

Consider the two cases char $K = p$ and 0.

Case I. char $K = p$. By the Main Lemma (i),

$$I^j(x) = q^{s-j+1} \sum_{\alpha_{-1} \in GF(q)^*} q^{j-s-1} \int_{\alpha_{-1}\beta^{-1} + \mathscr{P}^{j-s-1}} f(x - \beta^{s+1}u)\, du$$

$$= q^{s+1} \sum_{\alpha_{-1} \neq 0} \int_{\alpha_{-1}\beta^s + \mathscr{P}^j} f(x - t)\, dt.$$

Notice that for $t \in \alpha_{-1}\beta^s + \mathscr{P}^j$, $x - t \notin \mathscr{P}^j$ if and only if $x_s\beta^s + x_{s+1}\beta^{s+1} + \cdots + x_{j-1}\beta^{j-1} \neq \alpha_{-1}\beta^s$. We get that if $x \in \alpha_{-1}\beta^s + \mathscr{P}^j$ for some $\alpha_{-1} \in GF(q)^*$, then

$$I^j(x) = q^{s+1} \int_{\mathscr{P}^j} f(u)\, du = q^{s+1} \int_{\mathscr{P}^N} f(u)\, du \leq C q^{s-N+N/\epsilon};$$

if x is otherwise in $\mathscr{P}^s \setminus \mathscr{P}^{s+1}$, then $I^j(x) = 0$.
Thus for $|x| = q^{-s}$ $(0 \leq s < N)$

$$\mathscr{L}f(x) = q^{-N} \sum_{j>s}^{N} q^j |K_{q^j}| * f(x) \leq q^{-N} \sum_{j>s}^{N} q^j I^j(x)$$

$$\leq C q^{s-2N+N/\epsilon} \sum_{j>s}^{N} q^j \Phi_{\cup_{\alpha_{-1} \neq 0}\{\alpha_{-1}\beta^s + \mathscr{P}^j\}}.$$

This is the first inequality in (8).

Case II. char $K = 0$.

For $s < j \le N$, $x \in x_s \beta^s + \cdots + x_{j-1}\beta^{j-1} + \mathscr{P}^j$, $x_s \ne 0$

$$I^j(x) = q^{s-j+1} \sum_{\substack{(\alpha_{-1}, \dots, \alpha_{j-s-2}) \\ \alpha_{-1} \ne 0}} \left| D_{q^{j-s-1}}\left(\alpha_{-1}\beta^{-1} + \cdots + \alpha_{j-s-2}\beta^{j-s-2} \right) \right|$$

$$\cdot \int_{\alpha_{-1}\beta^{-1} + \cdots + \alpha_{j-s-2}\beta^{j-s-2} + \mathscr{P}^{j-s-1}} f\left(x - \beta^{s+1}u\right) du$$

$$= q^{2s-j+2} \sum_{(\alpha_{-1}, \dots, \alpha_{j-s-2})} \left| D_{q^{j-s-1}}\left(\alpha_{-1}\beta^{-1} + \cdots + \alpha_{j-s-2}\beta^{j-s-2} \right) \right|$$

$$\cdot \int_{\alpha_{-1}\beta^{s} + \alpha_0 \beta^{s+1} + \cdots + \alpha_{j-s-2}\beta^{j-1} + \mathscr{P}^{j}} f(x-t)\, dt.$$

For $t \in \alpha_{-1}\beta^s + \cdots + \alpha_{j-s-2}\beta^{j-1} + \mathscr{P}^j$, we note that $x - t \notin \mathscr{P}^j \ (\supset \mathscr{P}^N)$ provided $(\alpha_{-1}, \alpha_0, \dots, \alpha_{j-s-2}) \ne (x_s, x_{s+1}, \dots, x_{j-1})$. So

$$I^j(x) = q^{2s-j+2} \left| D_{q^{j-s-1}}\left(x_s\beta^{-1} + x_{s+1}\beta^0 + \cdots + x_{j-1}\beta^{j-s-2} \right) \right|$$

$$\cdot \int_{x_s\beta^s + \cdots + x_{j-1}\beta^{j-1} + \mathscr{P}^j} f(x-t)\, dt$$

$$= q^{2s-j+2} \left| D_{q^{j-s-1}}\left(x\beta^{-s-1} \right) \right| \left| \int_{\mathscr{P}^N} f(u)\, du \right.$$

$$\le C q^{2s-j-N+N/\epsilon} \left| D_{q^{j-s-1}}\left(x\beta^{-s-1} \right) \right|.$$

Thus for $|x| = q^{-s}$ $(0 \le s < N)$

$$\mathscr{L}f(x) \le q^{-N} \sum_{j>s}^{N} q^j I^j(x) \le C q^{2(s-N)+N/\epsilon} \sum_{j>s}^{N} \left| D_{q^{j-s-1}}\left(x\beta^{-s-1} \right) \right|,$$

which is the second inequality in (8).

As a result of (8), we get that if char $K = p$, $\frac{1}{2} < \epsilon \le 1$, using Jensen's inequality,

$$\int_{x \notin \mathscr{P}^N} (\mathscr{L}f)^{\epsilon}(x)\, dx$$

$$\le C q^{N(1-\epsilon)} \sum_{s=0}^{N-1} q^{(s-N)\epsilon} \int_{|x| = q^{-s}} \sum_{j>s}^{N} q^{j\epsilon} \Phi_{\cup_{\alpha_{-1} \ne 0} \{\alpha_{-1}\beta^s + \mathscr{P}^j\}}\ dx$$

$$
\begin{aligned}
&= Cq^{N(1-\epsilon)} \sum_{s=0}^{N-1} q^{(s-N)\epsilon}(q-1) \sum_{j>s}^{N} q^{j(\epsilon-1)} \\
&\le Cq^{N(1-\epsilon)} \sum_{s=0}^{N-1} q^{(s-N)\epsilon} \begin{cases} N-s, & \epsilon = 1 \\ q^{(\epsilon-1)s}, & \epsilon < 1 \end{cases} \\
&= C \begin{cases} \sum_{s=0}^{N-1} (N-s) q^{s-N}, & \epsilon = 1 \\ \sum_{s=0}^{N-1} q^{(2\epsilon-1)(s-N)}, & \frac{1}{2} < \epsilon < 1 \end{cases} \\
&:= C_\epsilon .
\end{aligned}
$$

Also, we get from (8) that if char $K = 0$, $\frac{1}{2} < \epsilon \le 1$,

$$
\begin{aligned}
&\int_{x \notin \mathscr{P}^N} (\mathscr{L} f)^{\epsilon}(x)\, dx \\
&\quad \le C \sum_{s=0}^{N-1} q^{2(s-N)\epsilon + N} \sum_{j>s}^{N} \int_{|x| = q^{-s}} \left| D_{q^{j-s-1}}\left(x\beta^{-s-1}\right)\right|^{\epsilon} dx \\
&\quad \le C \sum_{s=0}^{N-1} q^{2(s-N)\epsilon + N - s} \sum_{j>s}^{N} \int_{|u| = q} \left| D_{q^{j-s-1}}(u)\right|^{\epsilon} du \\
&\quad \le C \sum_{s=0}^{N-1} q^{(2\epsilon-1)(s-N)} \sum_{j>s}^{N} \left(\int_{|u|=q} \left| D_{q^{j-s-1}}(u) \right| du \right)^{\epsilon} (q-1)^{1-\epsilon} \\
&\qquad\qquad \text{(by Hölder's inequality)} \\
&\quad \le C \sum_{s=0}^{N-1} q^{(2\epsilon-1)(s-N)} \sum_{j>s}^{N} (j-s)^{c\epsilon} \le C \sum_{s=0}^{N-1} (N-s)^{c\epsilon+1} q^{(2\epsilon-1)(s-N)} \\
&\quad := C_\epsilon < \infty,
\end{aligned}
$$

where we have used the estimate from [17, Lemma 5(ii)],

$$
\int_{|u|=q} \left| D_{q^k}(u) \right| du = O(k^c), \qquad c = \log_p q .
$$

The proof is complete.

As a final remark, we give a corollary concerning the maximal operator $\tilde{\sigma}$ associated with $(C, 1)$ means of power orders, for the p-series case. Define

$$
\tilde{\sigma} f(x) = \sup_{\ell \ge 0} \left| \sigma_{q^\ell} f(x) \right| = \sup_{\ell \ge 0} \left| K_{q^\ell} * f(x) \right| .
$$

Note that if a is an ϵ atom satisfying conditions (i)–(iii) in Section 2 with $I = \mathscr{P}^N$,

$$\tilde{\sigma} a(x) = \sup_{\ell > N} \left| \sigma_{q^\ell} a(x) \right|,$$

then Lemma 2 implies that $\tilde{\sigma}$ is ϵ_0-quasi-local if char $K = p$ and $0 < \epsilon_0 \leq 1$. Thus we have a better estimate for $\tilde{\sigma}$ than that for σ^* on the p-series case.

COROLLARY. *Suppose* char $K = p$ *and* $0 < \epsilon_0 \leq 1$. *Then* $\tilde{\sigma}$ *is bounded from* H^{ϵ_0} *to* L^{ϵ_0}. *Moreover,* $\tilde{\sigma}$ *is bounded from* $H^{\epsilon, q}$ *to* $L^{\epsilon, q}$ *for every* $0 < \epsilon < \infty$ *and* $0 < q \leq \infty$.

4. PROOF OF THE MAIN LEMMA

We have only to prove part (ii) of the Main Lemma. We will follow the spirit of the proof of Lemma 5 in [17]. It is to be noted that for the case $q = p$ $(c = 1)$ the same proof goes more straightforwardly.

Let K be a finite algebraic extension of a p-adic field (char $K = 0$, $q = p^c$). Since every $x \in \mathscr{P}^s$, $s \in \mathbb{Z}$ has a unique expression

$$x = \sum_{\ell = s}^{\infty} x_\ell \beta^\ell,$$

$x_\ell \in GF(q)$, ($GF(q)$ being identified with a fixed set of coset representatives of $\mathscr{P}$ in $\mathscr{O}$), we will write $x = (x_s, x_{s+1}, \dots)$ when convenient.

Recall that $\{\epsilon_j\}_{j=0}^{c-1} \subset \mathscr{O} \setminus \mathscr{P}$ is a basis of $GF(q)$ over $GF(p) = \{0, 1, \dots, p-1\}$. Let $x = x_{-1}\beta^{-1} + x_0 \beta^0 + \cdots + x_{k-1}\beta^{k-1}$, $\alpha = \alpha_k \beta^k + \cdots + \alpha_{\ell-1}\beta^{\ell-1}$. Then simple induction shows that for each j, there exist unique $y_{-1}^j, \dots, y_{k-1}^j$ and $\gamma_k^j, \dots, \gamma_{\ell-1}^j$ in $GF(q)$ satisfying the following equations (mod $\mathscr{O}$) over K,

$$\begin{cases} x_{-1}\beta^{-1}\epsilon_j = y_{-1}^j \beta^{-1} \\ \left(x_{-1}\beta^{-2} + x_0 \beta^{-1}\right)\epsilon_j = y_{-1}^j \beta^{-2} + y_0^j \beta^{-1} \\ \vdots \\ \left(x_{-1}\beta^{-k-1} + x_0\beta^{-k} + \cdots + x_{k-1}\beta^{-1}\right)\epsilon_j \\ \qquad = y_{-1}^j \beta^{-k-1} + y_0^j \beta^{-k} + \cdots + y_{k-1}^j \beta^{-1} \end{cases} \tag{9}$$

and

$$\begin{cases} \left(x_{-1}\beta^{-k-2} + \cdots + x_{k-1}\beta^{-2} + \alpha_k\beta^{-1}\right)\epsilon_j \\ \qquad = y^j_{-1}\beta^{-k-2} + \cdots + y^j_{k-1}\beta^{-2} + \gamma^j_k\beta^{-1} \\ \vdots \\ \left(x_{-1}\beta^{-\ell-1} + \cdots + x_{k-1}\beta^{-\ell+k-1} + \alpha_k\beta^{-\ell+k} + \cdots + \alpha_{\ell-1}\beta^{-1}\right)\epsilon_j \\ \qquad = y^j_{-1}\beta^{-\ell-1} + \cdots + y^j_{k-1}\beta^{-\ell+k-1} \\ \qquad\quad + \gamma^j_k\beta^{-\ell+k} + \cdots + \gamma^j_{\ell-1}\beta^{-1}, \end{cases} \tag{10}$$

where $y^j_n = \sum_{k=0}^{c-1} y_{n,k}\epsilon_k$, $\gamma^j_m = \sum_{k=0}^{c-1}\gamma^j_{m,k}\epsilon_k$, $y^j_{n,k}\gamma^j_{m,k} \in GF(\rho)$.

From [17, Lemma 3] we have

$$D_{q^\ell}(x+\alpha) = \prod_{j=0}^{c-1}\prod_{s=1}^{\ell-1}\left|D_p\left(\epsilon_j\beta^{-s}(x+\alpha)\right)\right| := d(x,\alpha).$$

Also, from the formula for $D_p(\epsilon_j\beta^{-s}x)$ in [17, p. 545] we derive, by the expression (1) of χ and the above Eqs. (9), (10), that if $y^j_{-1,0} \neq 0$,

$$\prod_{s=0}^{\ell-1}\left|D_p\left(\epsilon_j\beta^{-s}(x+\alpha)\right)\right|$$

$$= \frac{\left|1 - \exp\left((2\pi i/p)y^j_{-1,0}\right)\right|}{\left|1 - \exp\left(2\pi i/p^{\ell+1}\right)\left(y^j_{-1,0} + \cdots + p^k y^j_{k-1,0} + p^{k+1}\gamma^j_{k,0} + \cdots + p^\ell\gamma^j_{\ell-1,0}\right)\right|}$$

$$= \frac{\sin\left((\pi/p)y^j_{-1,0}\right)}{\sin\left(\pi/p^{\ell+1}\right)\left(y^j_{-1,0} + \cdots + p^k y^j_{k-1,0} + p^{k+1}\gamma^j_{k,0} + \cdots + p^\ell\gamma^j_{\ell-1,0}\right)}.$$

Let $H(Y^j,\Gamma^j) = \prod_{s=0}^{\ell-1}|D_p(\epsilon_j\beta^{-s}(x+\alpha))|$. Then

$$d(x,\alpha) = \prod_{j=0}^{c-1} H(Y^j,\Gamma^j) := \prod_{j=0}^{c-1} H_j,$$

where

$$H_j = \begin{cases} \dfrac{\sin\left((\pi/p)y^j_{-1,0}\right)}{\sin\left(\pi/p^{\ell+1}\right)\left(y^j_{-1,0} + \cdots + p^k y^j_{k-1,0} + p^{k+1}\gamma^j_{k,0} + \cdots + p^\ell\gamma^j_{\ell-1,0}\right)} \\ \qquad \text{if } y^j_{-1,0} \neq 0, \\ p^\ell \qquad \text{if } y^j_{-1,0} = \cdots = y^j_{k-1,0} \\ \qquad\qquad = \gamma^j_{k,0} = \cdots = \gamma^j_{\ell-1,0} = 0, \\ 0 \qquad \text{otherwise}. \end{cases} \tag{11}$$

Using (11) and the relation $1 \leq \frac{x}{\sin x} \leq \frac{\pi}{2}$, $x \in (0, \frac{\pi}{2}]$ we shall evaluate H_j by the following cases A–E, which appears rather tedious.

Put $Y^j = (y^j_{-1,0}, \dots, y^j_{k-1,0})$, $\Gamma^j = (\gamma^j_{k,0}, \dots, \gamma^j_{\ell-1,0})$. Hereby case $A(\sigma)$ means that Y^j belongs to Case A and Γ^j belongs to subcase (σ) (denoted by $Y^j \in A$ and $\Gamma^j \in (\sigma)$, resp.; or simply $(Y^j, \Gamma^j) \in A(\sigma)$). More precisely, in the subcases below, (a) means the condition $0 < \gamma^j_{\ell-1,0} < p-1$; (b_i), $1 \leq i < \ell - k$ means $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{\ell-i,0} = 0$ but $\gamma^j_{\ell-i-1,0} \neq 0$, and $(b_{\ell-k})$ means $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0} = 0$. Furthermore, (c_i), $1 \leq i < \ell - k$ means $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{\ell-i,0} = p-1$ but $\gamma^j_{\ell-i-1,0} < p-1$ and $(c_{\ell-k})$ means $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0} = p-1$. Similar notations apply for cases $B_s(\sigma)$, $C_s(\sigma)$, $D(\sigma)$.

Case A. $y^j_{-1,0} \neq 0$, $0 < y^j_{k-1,0} < p-1$ (there are $(p-1)\overbrace{p \cdots p}^{k-1}(p-2)$ tuples in $Y^j \in A$). Consider the subcases of A.

$A(a)$ If $0 < \gamma^j_{\ell-1,0} < p-1$, by (11) there are $\overbrace{p \cdots p}^{\ell-k-1}(p-2)$ tuples in $\Gamma^j \ni$ ($\ni$ means "such that" for a moment)

$$H_j \approx p;$$

here and below $\alpha \approx \beta$ denotes the relation $c_1 \beta \leq \alpha \leq \beta$, c_1 a positive constant.

$A(b_1)$ If $\gamma^j_{\ell-1,0} = 0$ but $\gamma^j_{\ell-2,0} \neq 0$, there are $\overbrace{p \cdots p}^{\ell-k-2}(p-1)$ tuples in $\Gamma^j \ni$

$$H_j \approx p^2.$$

$A(b_2)$ If $\gamma^j_{\ell-1,0} = \gamma^j_{\ell-2,0} = 0$ but $\gamma^j_{\ell-3,0} \neq 0$, there are $\overbrace{p \cdots p}^{\ell-k-3}(p-1)$ tuples in $\Gamma^j \ni$

$$H_j \approx p^3.$$

…

$A(b_{\ell-k})$ If $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0} = 0$ (note $y^j_{k-1,0} \neq 0$), there is 1 tuple in $\Gamma^j \ni$

$$H_j \approx p^{\ell-k+1}.$$

$A(c_1)$ If $\gamma^j_{\ell-1,0} = p-1$ but $\gamma^j_{\ell-2,0} < p-1$, there are $\overbrace{p \cdots p}^{\ell-k-2}(p-1)$ tuples in $\Gamma^j \ni$

$$H_j \approx p^2.$$

$A(c_2)$ If $\gamma^j_{\ell-1,0} = \gamma^j_{\ell-2,0} = p-1$ but $\gamma^j_{\ell-3,0} < p-1$, there are $\overbrace{p\cdots p}^{\ell-k-3}(p-1)$ tuples in $\Gamma^j \ni$

$$H_j \approx p^3.$$

…

$A(c_{\ell-k})$ If $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0} = p-1$ (note $y^j_{k-1,0} < p-1$), there is 1 tuple in $\Gamma^j \ni$

$$H_j \approx p^{\ell-k+1}.$$

Case B_1. $y^j_{-1,0} \neq 0$, $y^j_{k-1,0} = 0$ but $y^j_{k-2,0} \neq 0$ (there are $(p-1)\overbrace{p\cdots p}^{k-2}(p-1)$ tuples in $Y^j \in B_1$).

Consider the subcases of B_1.

$B_1(a)$ If $0 < \gamma^j_{\ell-1,0} < p-1$, there are $\overbrace{p\cdots p}^{\ell-k-1}(p-2)$ tuples in $\Gamma^j \ni$

$$H_j \approx p.$$

$B_1(b_1)$ If $\gamma^j_{\ell-1,0} = 0$ but $\gamma^j_{\ell-2,0} \neq 0$ there are $\overbrace{p\cdots p}^{\ell-k-2}(p-1)$ tuples in $\Gamma^j \ni$

$$H_j \approx p^2.$$

…

$B_1(b_{\ell-k-1})$ If $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k+1,0} = 0$ but $\gamma^j_{k,0} \neq 0$, there are $p-1$ tuples in $\Gamma^j \ni$

$$H_j \approx p^{\ell-k}.$$

$B_1(b_{\ell-k})$ If $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0} = 0$ (note $y^j_{k-1,0} = 0$ but $y^j_{k-2,0} \neq 0$), there is 1 tuple in $\Gamma^j \ni$

$$H_j \approx p^{\ell-k+2}.$$

For subcases $B_1(c_1)$–$B_1(c_{\ell-k})$, the same estimates as in $A(c_1)$–$A(c_{\ell-k})$ apply, respectively.

In general, we obtain, via a similar argument, that for case B_s $(1 \le s \le k)$: $y^j_{-1,0} \neq 0$, $y^j_{k-1,0} = \cdots = y^j_{k-s,0} = 0$ but $y^j_{k-s-1,0} \neq 0$ (there are $(p-1)\overbrace{p\cdots p}^{k-s-1}(p-1)$ tuples in $Y^j \in B_s$, $1 \le s < k$ and $p-1$ tuples in $Y^j \in B_k$ for $s = k$), the subcases $B_s(a)$, $B_s(b_i)$, and $B_s(c_i)$ $(1 \le i \le \ell - k)$ admit the same estimates as in subcases $A(a)$, $A(b_i)$, and $A(c_i)$, resp., except for subcase $B_s(b_{\ell-k})$, where $\gamma^i_{\ell-1,0} = \cdots = \gamma^j_{k,0} = 0$ (note $y^j_{k-1,0} = \cdots = y^j_{k-s,0} = 0$ but $y^j_{k-s-1,0} \neq 0$) and so there is 1 tuple in $\Gamma^j \ni$

$$H_j \approx p^{(\ell-k+1)+s}.$$

Similarly, for case C_s $(1 \leq s \leq k)$: $y^j_{-1,0} \neq 0$, $y^j_{k-1,0} = \cdots = y^j_{k-s,0} = p - 1$ but $y^j_{k-s-1,0} < p-1$ (there are $(p-1)\overbrace{p \cdots p}^{k-s-1}(p-1)$ tuples in $Y^j \in C_s$, $1 \leq s < k$ and $p-1$ tuples in $Y^j \in C_k$ for $s = k$), the subcases $C_s(a)$, $C_s(b_i)$, and $C_s(c_i)$ $(1 \leq i \leq \ell - k)$ admit the same estimates as in subcases $B_s(a)$, $B_s(b_i)$, and $B_s(c_i)$, resp., except for subcase $C_s(c_{\ell-k})$, where $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0} = p-1$ (note $y^j_{k-1,0} = \cdots = y^j_{k-s,0} = p-1$ but $y^j_{k-s-1,0} < p-1$) and so there is 1 tuple in $\Gamma^j \ni$

$$H_j \approx p^{(\ell-k+1)+s}.$$

Case D. $y^j_{-1,0} = y^j_{0,0} = \cdots = y^j_{k-1,0} = 0$ (there is 1 tuple in $Y^j \in D$). The only subcase with $H_j \neq 0$ is $D(b_{\ell-k})$, where $\gamma^j_{\ell-1,0} = \cdots = \gamma^j_{k,0}$ and so there is 1 tuple in $\Gamma^j \ni$

$$H_j = p^{\ell} \approx p^{(\ell-k+1)+k}.$$

Case E. $y^j_{-1,0} = 0$ but $y^j_{n,0} \neq 0$ for some $0 \leq n \leq k-1$. Then

$$H_j = 0.$$

Now we are able to do the sum $I(x) := \sum_{\alpha=(\alpha_k,\ldots,\alpha_{\ell-1})} d(x,\alpha)$. For each $x = x_{-1}\beta^{-1} + \cdots + x_{k-1}\beta^{k-1} := (x_{-1},\ldots,x_{k-1})$, consider the mapping $\Theta_x : GF(q)^{\ell-k} \mapsto \prod_{j=0}^{c-1}(Y^j, GF(p)^{\ell-k})$ given by $(\alpha_k,\ldots,\alpha_{\ell-1}) \mapsto \prod_{j=0}^{c-1}(Y^j,\Gamma^j)$, where $Y^j = (y^j_{-1,0},\ldots,y^j_{k-1,0})$, $\Gamma^j = (\gamma^j_{k,0},\ldots,\gamma^j_{\ell-1,0})$ are determined by the last equation in (10); clearly if x is given, then Y^j are given, $0 \leq j \leq c-1$.

Using essentially the same argument as in the proof of Lemma 8 of [17], we can show that the mapping Θ_x is a bijection for given x (and Y^j's). Hence, putting $\prod_{j=0}^{c-1}\Gamma^j = (\Gamma^0,\Gamma^1,\ldots,\Gamma^{c-1})$,

$$I(x) = \sum_{(\Gamma^0,\Gamma^1,\ldots,\Gamma^{c-1})} \prod_{j=0}^{c-1} H(Y^j,\Gamma^j) = \sum_{\prod_{j=0}^{c-1}\Gamma^j} \prod_{j=0}^{c-1} H_j.$$

Denote by $\mathscr{T}$ the set of all cases in Y^j with $H_j \neq 0$ and $\mathscr{S}$ the set of all cases in Γ^j. Write $\mathscr{T} = \{A, B_s, C_s, D, E, 1 \leq s \leq k\}$ and $\mathscr{S} = \{(a),(b_i),(c_i), 1 \leq i \leq \ell - k\}$. Let $I(\tau \times \sigma)$ denote the sum of all terms $\prod_{j=0}^{c-1} H(Y^j,\Gamma^j)$

indexed by $\prod_{j=0}^{c-1}\Gamma^j \in \sigma := \prod_{j=0}^{c-1}\sigma_j \in \overbrace{\mathscr{S}\times\cdots\times\mathscr{S}}^{c} := \mathscr{S}^c$ for given $\prod_{j=0}^{c-1}Y^j \in \tau := \prod_{j=0}^{c-1}\tau_j \in \overbrace{\mathscr{T}\times\cdots\times\mathscr{T}}^{c}$, where $\prod_{j=0}^{c-1}\Gamma^j \in \sigma$ means for each j, $\Gamma^j \in \sigma_j$. Then given τ, there are $(2\ell - 2k + 1)^c$ cases in $\sigma \in \mathscr{S}^c$ for $I(\tau \times \sigma)$ (because there are $2\ell - 2k + 1$ cases in $\mathscr{S}$).

We have

$$I(x) = \sum_{\sigma\in\mathscr{S}^c} I(\tau\times\sigma) = \sum_{\sigma=\prod_{j=0}^{c-1}\sigma_j}\ \sum_{\prod_{j=0}^{c-1}\Gamma^j\in\sigma}\ \prod_{j=0}^{c-1} H_j.$$

Claim 1. $I(\tau\times\sigma) \le q^{\ell+k+1}p^{\|\Theta(x)\|}$ (with obvious convention $p^{-\infty} = 0$ if $\|\Theta(x)\| = -\infty$); here the mapping $\Theta : x \mapsto \prod_{j=0}^{c-1}Y^j$, $Y^j = (y^j_{-1,0}, \dots, y^j_{k-1,0})$ is defined by the last equation in (9) and the "norm" $\|\cdot\|$ is given by $\|\Theta(x)\| = \sum_{j=0}^{c-1}\|Y^j\|$ with

$$\|Y^j\| = \begin{cases} 0 & \text{if } Y^j \in A \\ s & \text{if } Y^j \in B_s \text{ or } C_s,\ 1 \le s \le k \\ k & \text{if } Y^j \in D \\ -\infty & \text{if } Y^j \in E. \end{cases}$$

Note that the RHS of the above inequality is independent of σ.

If Claim 1 is true, we will obtain the first inequality of the Main Lemma (ii):

$$\begin{aligned} I(x) &\le (2\ell - 2k + 1)^c q^{\ell-k+1} p^{\|\Theta(x)\|} \\ &\le C(\ell - k)^c q^{\ell-k} p^{\|\Theta(x)\|}. \end{aligned}$$

So we need to verify Claim 1. Indeed, by definition

$$\begin{aligned} I(\tau\times\sigma) &\le N\left(\prod_{j=0}^{c-1}\Gamma^j \in \sigma\right) \sup_{\prod_{j=0}^{c-1}\Gamma^j\in\sigma}\ \prod_{j=0}^{c-1} H(Y^j, \Gamma^j) \\ &= \prod_{j=0}^{c-1} N\big(\Gamma^j \in \sigma_j\big) \prod_{j=0}^{c-1} \sup_{\Gamma^j\in\sigma_j} H(Y^j, \Gamma^j) \\ &= \prod_{j=0}^{c-1} N\big(\Gamma^j \in \sigma_j\big) \sup_{\Gamma^j\in\sigma_j} H_j, \end{aligned}$$

where $N(\Gamma^j \in \sigma_j)$ denotes the number of tuples in $\Gamma^j \in \sigma_j$ and the supremum $\sup_{\Gamma^j\in\sigma_j}$ is taken over all $\Gamma^j \in \sigma_j$ for given $Y_j \in \tau_j$ (in subcase $\tau_j(\sigma_j)$); the notation for the "product case" is then self-evident. It is enough to show that

$$J := N\big(\Gamma^j \in \sigma_j\big) \sup_{\Gamma^j \in \sigma_j} H(Y^j, \Gamma^j) \le p^{\ell - k + 1 + \|Y^j\|}.$$

But this can easily be verified by our analysis of those subcases discussed earlier. For instance,

(i) Let $\tau_j = A$, $\sigma_j = b_1$. Then from $A(b_1)$

$$N\big(\Gamma^j \in b_1\big) = \overbrace{p \cdots p}^{\ell - k - 2}(p - 1)$$

$$\sup_{(Y^j, \Gamma^j) \in A(b_1)} H(Y^j, \Gamma^j) \le p^2$$

so that

$$J \le p^{\ell - k + 1}$$

($\|Y^j\| = 0$ if $Y^j \in A$).

(ii) Let $\tau_j = B_s$ $(1 \le s \le k)$, $\sigma_j = b_{\ell - k}$. Then from $B_s(b_{\ell - k})$

$$N\big(\Gamma^j \in b_{\ell - k}\big) = 1$$

$$\sup_{B_s(b_{\ell - k})} H(Y^j, \Gamma^j) \le p^{(\ell - k + 1) + s},$$

so that

$$J \le p^{\ell - k + 1 + s}$$

($\|Y^j\| = s$ if $Y^j \in B_s$).

(iii) Let $\tau_j = B_s$, $\sigma_j = c_i$ $(1 \le s \le k,\ 1 \le i \le \ell - k)$. Then from $B_s(c_i)$, $1 \le i < \ell - k$, we have

$$N\big(\Gamma^j \in c_i\big) = \overbrace{p \cdots p}^{\ell - k - i - 1}(p - 1)$$

$$\sup_{B_s(c_i)} H(Y^j, \Gamma^j) \le p^{i + 1},$$

so that

$$J \le p^{\ell - k + 1} \le p^{\ell - k + 1 + s}$$

($\|Y^j\| = s$ if $Y^j \in B_s$), and from $B_s(c_{\ell - k})$ we have

$$N\big(\Gamma^j \in c_{\ell - k}\big) = 1$$

$$\sup_{B_s(c_{\ell - k})} H(Y^j, \Gamma^j) \le p^{\ell - k + 1},$$

so that

$$J \le p^{\ell - k + 1 + s}.$$

(iv) Let $\tau_j = D$, $\sigma_j = b_{\ell-k}$. Then from $D(b_{\ell-k})$

$$N\big(\Gamma^j \in b_{\ell-k}\big) = 1$$
$$\sup_{D(b_{\ell-k})} H(Y^j, \Gamma^j) \le p^{(\ell-k+1)+k}$$

($\|Y^j\| = k$ if $Y^j \in D$).

The other cases can be checked similarly.

Finally we prove the second inequality of the Main Lemma (ii). Noting from [17, Lemma 8] that the mapping $\Theta : x \mapsto \prod_{j=0}^{c-1} Y^j$ (as defined in Claim 1) is a bijection, we have

$$Q := \sum_{\substack{x=(x_{-1}, x_0, \dots, x_{k-1}) \\ \in GF(q)^{k+1}}} p^{\|\Theta(x)\|} = \sum_{\prod_{j=0}^{c-1} Y^j} p^{\sum_{i=0}^{c-1} \|Y^j\|}.$$

Similar to the estimation of $I(x)$, let $Q(\tau)$ denote the sum of all terms indexed by $\prod_{j=0}^{c-1} Y^j \in \tau = \prod_{j=0}^{c-1} \tau_j$, $\tau_j \in \mathscr{T}$. Then there are $(2k+2)^c$ cases in $\tau \in \overbrace{\mathscr{T} \times \cdots \times \mathscr{T}}^{c}$ such that $p^{\sum_{i=0}^{c-1}\|Y^j\|} \ne 0$ (because there are $2k+2$ cases, namely, A, D, B_s, C_s in $\mathscr{T}$ such that $\|Y^j\| \ne -\infty$). We compute

$$\begin{aligned} Q &= \sum_{\tau = \prod_{j=0}^{c-1} \tau_j} Q(\tau) = \sum_{\tau} \sum_{\prod_{j=0}^{c-1} Y^j \in \tau} p^{\sum_{j=0}^{c-1} \|Y^j\|} \\ &= \sum_{\tau} N\left(\prod_{j=0}^{c-1} Y^j \in \tau \right) p^{\sum_{j=0}^{c-1} \|Y^j\|} \\ &= \sum_{\tau} \prod_{j=0}^{c-1} N\big(Y^j \in \tau_j\big) \prod_{j=0}^{c-1} p^{\|Y^j\|} = \sum_{\tau} \prod_{j=0}^{c-1} N\big(Y^j \in \tau_j\big) p^{\|Y^j\|}, \end{aligned}$$

where $N(Y^j \in \tau_j)$ denotes the number of tuples in $Y^j \in \tau_j$ (the notation $N(\prod_{j=0}^{c-1} Y^j \in \tau)$ is then self-evident), and we note that $\|Y^j\|$ depends on the case $\tau_j \in \mathscr{T}$ only.

Claim 2. $N(Y^j \in \tau_j)p^{\|Y^j\|} \leq p^{k+1}$.

If Claim 2 is true, we will obtain the second inequality of the Main Lemma (ii):

$$Q \leq (2k+2)^c \prod_{j=0}^{c-1} p^{k+1} \leq Ck^c q^k.$$

It remains to verify Claim 2. But this also easily follows from our analysis of the cases in $\mathscr{T}$. For instance, (i) let $\tau_j = A$. Then from Case A we learn that

$$N(Y^j \in A) = (p-1)\overbrace{p \cdots p}^{k-1}(p-2),$$

$$\|Y^j\| = 0 \qquad \text{for } Y^j \in A,$$

so that

$$N(Y^j \in A)p^{\|Y^j\|} \leq p^{k+1}.$$

(ii) Let $\tau_j = B_s$ $(1 \leq s \leq k)$. Then from Case B_s, we learn that if $1 \leq s < k$,

$$N\big(Y^j \in B_s\big) = (p-1)\overbrace{p \cdots p}^{k-s-1}(p-1),$$

$$\|Y^j\| = s \qquad \text{for } Y^j \in B_s$$

so that

$$N\big(Y^j \in B_s\big)p^{\|Y^j\|} \leq p^{k+1};$$

if $s = k$,

$$N\big(Y^j \in B_k\big) = p - 1,$$

$$\|Y^j\| = k \qquad \text{for } Y^j \in B_k,$$

so that

$$N\big(Y^j \in B_k\big)p^{\|Y^j\|} \leq p^{k+1}.$$

(iii) The same is valid for $\tau_j = C_s$ $(1 \leq s \leq k)$.

(iv) Let $\tau_j = D$, then from Case D we learn that

$$N(Y^j \in D) = 1,$$

$$\|Y^j\| = k \qquad \text{for } Y^j \in D$$

so that

$$N(Y^j \in D)\,p^{\|Y^j\|} = p^k \le p^{k+1}.$$

This proves Claim 2 and hence concludes the proof of the Main Lemma.

Remark. A little more careful inspection of the proof will verify that the estimates in the Main Lemma (ii) are the best possible, in the sense that the inverse inequalities hold up to a constant multiple.

It is natural (compare the classical case in [10]) to conjecture that the estimate in the main theorem is sharp in the sense that σ^* is *not* bounded from $H^{1/2}$ to $L^{1/2}$. Nevertheless, it would be interesting to verify that σ^* maps $H^{1/2}$ into weak-$L^{1/2}$ ($= L^{1/2,\infty}$) using the estimates obtained in the proof of the main theorem.

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