Interpolation orbits and optimal Sobolev’s embeddings

Amiran Gogatishvili a,1,2, Vladimir I. Ovchinnikov b,*1,3

a Mathematical Institute of the Academy of Science of the Czech Republic, Zitna 25, CZ-115 67 Prague 1, Czech Republic
b Mathematical Department, Voronezh State University, Universitetskaya pl., 1, Voronezh 394006, Russia

Received 4 April 2006; accepted 29 August 2007

Communicated by G. Pisier

Abstract

We consider Sobolev’s embeddings for spaces based on rearrangement invariant spaces (not necessarily with the Fatou property) on domains with a sufficiently smooth boundary in \( \mathbb{R}^n \). We show that each optimal embedding \( W^m E \subset G \), where \( m < n \), can be obtained by the real interpolation of the well-known endpoint embeddings. We also give an orbital description of the optimal range space in Sobolev’s embedding.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Rearrangement invariant spaces; Interpolation spaces; Interpolation orbits; Sobolev spaces; Lorentz spaces; Embedding theorems; The Hardy type operators

0. Introduction

The construction of intermediate embeddings with the help of a pair of endpoint embeddings is one of the most natural applications of interpolation of linear operators. Indeed, if we have \( X_0 \subset Y_0 \) and \( X_1 \subset Y_1 \), then we deduce that \( \mathcal{F}(X_0, X_1) \subset \mathcal{F}(Y_0, Y_1) \) for each interpolation construction (interpolation functor) \( \mathcal{F} \). The situation described above exactly takes place for embeddings of full Sobolev’s spaces of smooth functions to rearrangement invariant spaces. Here we
have two sharp endpoint embeddings, \( W^m A_{m/n}(\Omega) \subset L_\infty(\Omega) \) and \( W^m L_1(\Omega) \subset A_{1-m/n}(\Omega) \), where the integer smoothness \( m \) is less than the dimension \( n \) of the underlying domain \( \Omega \) (we suppose that \( \Omega \) has a sufficiently smooth boundary). By definition, \( W^m E \) consists of functions such that all partial derivatives up to the degree \( m \) belong to the rearrangement invariant space \( E \), and \( A_\alpha \) denotes the Lorentz space. It is natural that we are interested in sharp or optimal embeddings, and our estimation of an embedding from the point of view of optimality depends on the class of spaces, where we choose the spaces for comparison. Recall that the embedding \( W^m E \subset G \) is said to be optimal with respect to some class of spaces if it is impossible to improve the embedding in the given class of spaces, i.e., to enlarge the space \( E \) or to decrease the space \( G \). The most natural class of spaces in our case is the class of all rearrangement invariant spaces, however some conditions on the norms are often presumed.

From the first sight, the embeddings \( \mathcal{F}(X_0, X_1) \subset \mathcal{F}(Y_0, Y_1) \) scarcely look optimal, even if the original endpoint embeddings are the best possible.

The main result of the present paper is that, if we apply a real method interpolation construction \( \mathcal{F} \) to the endpoint Sobolev’s embedding, we always obtain an optimal embedding with respect to the class of all interpolation rearrangement invariant spaces (the spaces which are interpolation spaces between \( L_1 \) and \( L_\infty \)). Moreover, each optimal embedding of the form \( W^m E \subset G \) is obtained by the real interpolation between the classical endpoint embeddings. The approach we used here goes up to the description of embeddings \( W^m E \subset G \) in terms of the Hardy operator, considered in [13] and [6]. There is also some improvement in comparison with [13], because following [6] we deal with interpolation rearrangement invariant spaces which may have not the Fatou property that is presumed in [13]. The class of rearrangement invariant spaces without the Fatou property is very rich, e.g., see the recent paper [8], where a number of constructions of such spaces are presented. Perhaps some may regard these spaces as exotic. That is why we present simple examples of optimal embeddings for spaces which are intermediate between the classical Marcinkiewicz space \( M_\theta \) and its separable part \( M_\theta' \). Thus, we obtain some new optimal embeddings which are “invisible” from the point of view of embedding inequalities.

The essence of our approach consists of reduction to the study of linear operators, taking the couple \( \{L_1, A_{m/n}\} \) to the couple \( \{A_{1-m/n}, L_\infty\} \). Thus, we come to an orbital description of the optimal range spaces for Sobolev’s embeddings. As an application, we consider embeddings \( W^m L_{\rho,p} \subset G \), where \( L_{\rho,p} \) are the Lorentz spaces with arbitrary quasi-concave function parameter \( \rho \), such that \( L_{\rho,p} \) is an intermediate space between \( L_1 \) and \( A_{m/n} \). We give a simple explicit description of the optimal range spaces, similar to the limiting optimal embedding obtained in [6].

The paper is organized as follows. In Section 1, we recall rearrangement invariant spaces, orbital equivalence of intermediate spaces, and orbital equivalence of Banach couples. The embeddings of Sobolev’s spaces are considered in Section 2. The formulation of our main results is given here. We present optimal embeddings, implied by the orbital description of optimal range spaces, as well as the optimal embeddings for some non-separable spaces intermediate between the Marcinkiewicz space and its separable part. Interpolation orbits are used in Section 3 for the analysis of the Hardy operator. The final Section 4 contains the proof of the main theorem.

1. Rearrangement invariant spaces. Orbital equivalence

We shall consider spaces of functions defined on bounded domains \( \Omega \) in \( \mathbb{R}^n \) with the Lebesgue measure. Assume \( \text{mes}(\Omega) = 1 \).
If $f$ is a measurable function on $\Omega$, then $f^*(t)$ denotes a non-increasing function on $[0, 1]$ which is equimeasurable with $|f|$ on $\Omega$.

Recall that a space $E$ of measurable functions is called rearrangement invariant (r.i. space) if

$$g^* \leq f^* \quad \text{and} \quad f \in E$$

imply that $g \in E$ and $\|g\|_E \leq \|f\|_E$. These spaces are very natural in analysis of local structure of functions independent of the position of the argument. The $L_p$ spaces, the Orlicz spaces, and the Lorentz spaces $L_{p,q}$ (e.g., see [1]) are the well-known examples of r.i. spaces. In this paper we consider only those r.i. spaces which are interpolation spaces between $L_1$ and $L_\infty$.

These interpolation r.i. spaces are remarkable due to the possibility of defining the analog of each interpolation r.i. space of functions on an arbitrary measure space. The relation between r.i. spaces defined on different measure spaces is established with the help of orbital equivalence.

Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be two Banach couples. Recall that $x \in X_0 + X_1$ and $y \in Y_0 + Y_1$ are called orbital equivalent relative to $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ if there exist linear bounded operators $T : \{X_0, X_1\} \to \{Y_0, Y_1\}$ and $S : \{Y_0, Y_1\} \to \{X_0, X_1\}$, such that $Tx = y$ and $Sy = x$.

Spaces $X \subset X_0 + X_1$ and $Y \subset Y_0 + Y_1$ are called orbital equivalent relative to Banach couples $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$, if each $x \in X$ is orbital equivalent to some $y_x \in Y$, and each $y \in Y$ is orbital equivalent to some $x_y \in X$.

Couples $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ are called orbital equivalent if $X_0 + X_1$ is orbital equivalent to $Y_0 + Y_1$.

We readily have that the interpolation spaces of orbital equivalent couples are in a one-to-one correspondence (corresponding spaces are orbital equivalent).

Note that the couple $\{L_1(\Omega), L_\infty(\Omega)\}$, where $\Omega$ is a bounded open set in $\mathbb{R}^n$, is orbital equivalent to the couple $\{L_1(0, 1), L_\infty(0, 1)\}$ (see [4]). The space consisting of $f \in L_1(0, 1)$ that are orbital equivalents to elements of $E(\Omega)$ is exactly the Luxemburg space $\widetilde{E}$ representing $E(\Omega)$. It will cause no confusion if we use the same letter $E$ for the Luxemburg space.

If $F$ is a sequence space and $w_k$ is a positive sequence (a weight), then by $F(w_k)$ we denote the space of sequences $\xi_k$ such that $\xi_k w_k \in F$, with the natural norm $\|\xi\|_{F(w_k)} = \|\xi_k w_k\|_F$.

The couple $\{L_1(0, 1), L_\infty(0, 1)\)$ is orbital equivalent to the couple $\{l_1(2^{-k}), l_\infty\}$, where $l_\infty$ and $l_1$ are standard sequence spaces (we mean $k \in \mathbb{N}$). Let us denote by $E_d$ the interpolation space between $l_1(2^{-k})$ and $l_\infty$, which is orbital equivalent to the interpolation space $E$ between $L_1$ and $L_\infty$.

The space $E_d$ can be described explicitly. We have

$$E_d = \left\{ \{\xi_k\}_{k=0}^\infty : \sum_{k=0}^\infty \xi_k x(2^{-k}, 2^{-k+1}) \ (t) \in E \right\}. \quad (1)$$

Note that the space $l_0(2^{-k/p})$ is orbital equivalent to the $L_{p,q}$ relative to the couples $\{l_1(2^{-k}), l_\infty\}$ and $\{L_1, L_\infty\}$, where $L_{p,q}$ denotes the Lorentz space. In particular, $l_1(2^{-\frac{m}{2} k})$ is orbital equivalent to the space $L_{m,1}$. This implies, by the way, that $E$ is orbital equivalent to $E_d$ relative to couples $\{L_1, L_{m,1}\}$ and $\{l_1(2^{-k}), l_1(2^{-k \frac{m}{2}})\}$ as well.

We use the notation $A_\alpha$ for the space $L_{1/\alpha,1}$ if $0 < \alpha < 1$. Recall that $f \in A_\alpha(\Omega)$ if

$$\|f\|_{A_\alpha} = \int_0^1 f^*(t) \, dt^\alpha < \infty.$$
The following definitions are used for the formulations and the proofs of our main results.

**Definition 1.1.** The orbit of \( x \in X_0 + X_1 \) with respect to linear bounded operators mapping the couple \( \{X_0, X_1\} \) to a couple \( \{Y_0, Y_1\} \), denoted by

\[
\text{Orb}(x, \{X_0, X_1\} \to \{Y_0, Y_1\}),
\]

is a linear space of \( y \in Y_0 + Y_1 \) such that

\[
y = \sum_{j=1}^{\infty} T_j x_j \quad \text{(convergence in } Y_0 + Y_1),
\]

where

\[
\sum_{j=1}^{\infty} \max(\|T_j\|_{X_0 \to Y_0}, \|T_j\|_{X_1 \to Y_1})\|x_j\|_X < \infty.
\]

**Definition 1.2.** The orbit of the space \( X \subset X_0 + X_1 \) with respect to linear bounded operators mapping the couple \( \{X_0, X_1\} \) to the couple \( \{Y_0, Y_1\} \), denoted by

\[
\text{Orb}(X, \{X_0, X_1\} \to \{Y_0, Y_1\}),
\]

is a linear space of \( y \in Y_0 + Y_1 \) such that

\[
y = \sum_{j=1}^{\infty} T_j x_j \quad \text{(convergence in } Y_0 + Y_1), \quad (2)
\]

where

\[
\sum_{j=1}^{\infty} \max(\|T_j\|_{X_0 \to Y_0}, \|T_j\|_{X_1 \to Y_1})\|x_j\|_X < \infty.
\]

2. **Optimal Sobolev’s embeddings**

Recall that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). Assume that \( \Omega \) is a domain with the minimally smooth boundary such that we can extend smooth functions from \( \Omega \) to \( \mathbb{R}^n \) (see [17]).

Let \( E \) be an interpolation r.i. space on \( \Omega \). The Sobolev space \( W^m E(\Omega) \), where \( m \in \mathbb{N} \), is defined in the usual fashion with the help of the norm

\[
\|f\|_{W^m E} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_E,
\]

where \( D^\alpha \) is a mixed partial derivative of the order \( \alpha = (\alpha_1, \ldots, \alpha_n) \), and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

As usual, \( W^m L_\rho = W^m_\rho \).

We consider only the case \( m < n \). The extreme embeddings in this case are well known

\[
W^m_1 \subset A^{-\frac{m}{n}}, \quad W^m A^{\frac{m}{n}} \subset L_\infty \quad (3)
\]

(e.g., see [13]).

Let us denote by \( \mathcal{F} \) any interpolation construction (interpolation functor), and apply \( \mathcal{F} \) to the couples \( \{W^m_1, W^m A^{\frac{m}{n}}\} \) and \( \{A^{1-\frac{m}{n}}, L_\infty\} \). Thus, we obtain

\[
\mathcal{F}(W^m_1, W^m A^{\frac{m}{n}}) \subset \mathcal{F}(A^{1-\frac{m}{n}}, L_\infty).
\]
It would be natural to move $F$ inside, i.e.,

$$F(W^m_1, W^m A_{\frac{m}{n}}) = W^m F(L_1, A_{\frac{m}{n}}).$$  \hspace{1cm} (4)

Thus we come to Sobolev’s embedding

$$W^m F(L_1, A_{\frac{m}{n}}) \subset F(A_{1-\frac{m}{n}}, L_\infty).$$

With the help of the De Vore–Sherer theorem [7], Nilsson [12] showed that (4) really takes place for the $K$-method interpolation constructions. Recall that these interpolation constructions are obtained with the help of the $K$-functional.

As usual, we denote by $K(t, x, \{X_0, X_1\})$ the $K$-functional of the couple $\{X_0, X_1\}$, i.e.,

$$K(t, x, \{X_0, X_1\}) = \inf_{x = x_0 + x_1} \|x_0\|_{X_0} + t\|x\|_{X_1},$$

where the infimum is taken over all representations of $x$ as a sum of $x_0 \in X_0$ and $x_1 \in X_1$.

An interpolation functor $F$ is called the $K$-method functor if there exists a Banach lattice, denoted by $F$, of sequences $\{\xi_l\}_{l=-\infty}^{\infty}$ such that

$$F(X_0, X_1) = (X_0, X_1)^K_F.$$

By definition, $x \in (X_0, X_1)^K_F$ if $x \in X_0 + X_1$ and

$$\{K(2^l, x, \{L_1, L_\infty\})\} \in F,$$

and we have the equivalence of the norms

$$\|x\|_{F(X_0, X_1)} \asymp \|\{K(2^l, x, \{L_1, L_\infty\})\}\|_F.$$

Thus we have

$$W^m (L_1, A_{\frac{m}{n}})^K_F \subset (A_{1-\frac{m}{n}}, L_\infty)^K_F$$

for each parameter space $F$ of the $K$-method.

If we choose any r.i. spaces $E$ and $G$, such that $E \subset (L_1, A_{\frac{m}{n}})^K_F$ and $(A_{1-\frac{m}{n}}, L_\infty)^K_F \subset G$, we readily have

$$W^m E \subset G.$$  \hspace{1cm} (5)

The main result of the present paper is that we have no other pairs of interpolation r.i. spaces satisfying (5), i.e. the following theorem is true.

**Theorem 2.1.** If $1 \leq m < n$ and

$$W^m E(\Omega) \subset G(\Omega)$$

for some interpolation r.i. spaces $E$ and $G$, then there exists $F$, such that

$$E \subset (L_1, A_{\frac{m}{n}})^K_F \text{ and } (A_{1-\frac{m}{n}}, L_\infty)^K_F \subset G.$$
The embedding $W^m E(Ω) ⊂ G(Ω)$ is called optimal with respect to interpolation r.i. spaces if $G$ cannot be replaced by a smaller interpolation r.i. space and $E$ cannot be replaced by a larger one.

**Corollary 2.1.** If $W^m E(Ω) ⊂ G(Ω)$ is an optimal embedding with respect to interpolation r.i. spaces, then there exists $F$, such that $E = (L_1, A_{m/2})^K_F$ and $G = (A_{1-m/2}, L_∞)^K_F$.

Corollary 2.1 gives us opportunity to construct new optimal embeddings with the help of known ones.

**Corollary 2.2.** If $W^m E_0 ⊂ G_0$ and $W^m E_1 ⊂ G_1$ are optimal embeddings, and $F$ is an arbitrary $K$-method or $J$-method interpolation functor, then $W^m F(E_0, E_1) ⊂ F(G_0, G_1)$ is an optimal embedding.

The proof is based on the reiteration theorems for the $K$-method spaces (e.g., see [12] and [4]). In particular we obtain that the embedding $W^m (E_0 ∩ E_1) ⊂ G_0 ∩ G_1$ is optimal if $W^m E_0 ⊂ G_0$ and $W^m E_1 ⊂ G_1$ are optimal.

Theorem 2.1 can be interpreted as a description of the optimal range space in the embedding (5).

**Theorem 2.2.** For each interpolation r.i. space $E$ the smallest interpolation r.i. space $G$ in (5) is equal to $\text{Orb}(E, [L_1, A_{m/2}] → [A_{1-m/2}, L_∞])$.

As an application of Theorem 2.2, consider optimal embeddings for the Lorentz spaces $L_{ρ,p}$ as the domain space $E$, where $ρ$ is a function parameter, such that $C_ρ(t) ≤ t^{m/2}$. The latter condition means that $L_{ρ,p}$ is an intermediate space between $L_1$ and $A_{m/2}$. Thus we cover a series of special cases considered in [9].

By definition, for any quasi-concave $ρ(t)$ on $(0, ∞)$ and $1 ≤ p ≤ ∞$ we have

$$L_{ρ,p} = (L_1, L_∞)_{ρ,p},$$

where $(X_0, X_1)_{ρ,p}$ is the Janson space. Recall that (see [11]), by definition, $x ∈ (X_0, X_1)_{ρ,p}$ if

$$\left\{ \frac{K(t_k, x, \{X_0, X_1\})}{ρ(t_k)} \right\} \in l_p,$$

where $t_k$ is a two-sided sequence, constructed inductively by $t_0 = 1$, and

$$\min\left(\frac{ρ(t_{k+1})}{ρ(t_k)}, \frac{t_{k+1}ρ(t_k)}{ρ((t_{k+1})t_k)}\right) = 2.$$

In our cases, such that $X_0 ⊃ X_1$, the behavior of $ρ(t)$ for $t > 1$ is unimportant, and $\{t_k\}$ is constructed only for $k ≤ 0$.

Note that these spaces $L_{ρ,p}$ are also known as $Γ$-spaces.

The optimal range space for the embedding

$$W^m L_{ρ,p} ⊂ G$$
is equal to $\text{Orb}(L_\rho,p, \{L_1, A_{\frac{m}{n}}\} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty\})$ by Theorem 2.2. The calculation of the orbit can be easily reduced to the calculation of $\text{Orb}(L_\rho,p, \{L_1, \Lambda_{m/n}\} \rightarrow \{\Lambda_{1-m/n}, L_\infty\}(2^{-k}))$. This leads (we leave details for another paper) to

$$\text{Orb}(L_\rho,p, \{L_1, A_{\frac{m}{n}}\} \rightarrow \{\Lambda_{1-m/n}, L_\infty\}) = (\Lambda_{1-m/n}, L_\infty)_{\sigma,p},$$

where $\sigma$ is a quasi-concave function

$$\sigma(t) = \|\min(1, t/s^{1-\frac{m}{n}})\|_{L_{p',p}},$$

where $\tilde{\rho}(t) = t/\rho(t)$, and $p' = p/(p-1)$. Since, by the Holmstedt formula (see [2]), the $K$-functional of the couple $\{A_{1-\frac{m}{n}}, L_\infty\}$ can be explicitly calculated by

$$K\left(t^{1-\frac{m}{n}}, f, \{A_{1-\frac{m}{n}}, L_\infty\}\right) = \int_0^t f^*(s) \, ds^{1-\frac{m}{n}},$$

we obtain a description of the optimal range space.

It is instructive to consider the limiting case $\rho(t) = t^{\frac{m}{n}}$. Note that this case was also considered in [6], which in turn goes up to the famous Hansson’s and Brezis–Wainger’s results (see [10] and [3]), and we present it just as an illustration. The optimal range space $G$ now is equal to $(\Lambda_{1-\frac{m}{n}}, L_\infty)_{\sigma,p}$, where

$$\sigma(t) = \|\min(1, t/s^{1-\frac{m}{n}})\|_{L_{\frac{n-m}{n}},p'} \asymp t(\log_2 1/t)^{\frac{1}{p'}}$$

for $t \leq 1/2$.

In the limiting case, we readily have $(\Lambda_{1-\frac{m}{n}}, L_\infty)_{\sigma,p} = (L_1, L_\infty)_{\sigma,p}$ in view of reiteration for Janson’s spaces (see [11]). Hence, $G = L_{\sigma,p}$.

Note that for $\sigma(t)$ above, we can take $t_k = 2^{-2^{|k|}}$. That is why we see that the optimal range space in the embedding $W^m L_{\frac{m}{n},p} \subset G$ consists of $f \in L_1$, such that

$$\left\{ \frac{K(2^{-2^{|k|}}, f, \{L_1, L_\infty\})}{\sigma(2^{-2^{|k|}})} \right\} \in l_p.$$ (6)

It is well known (see [4]) that

$$K(t, f, L_1, L_\infty) = \int_0^t f^*(s) \, ds,$$

therefore the relation (6) means

$$\sum_{l=0}^{\infty} 2^{-\frac{p}{p'} l} \left(2^{2^l} \int_0^{2^{-2^l}} f^*(t) \, dt\right)^p = \sum_{l=0}^{\infty} 2^l (2^{-2^l} f^{**}(2^{-2^l}))^p < \infty,$$

if $p < \infty$. If $p = \infty$, we have
\[
\sup_{t \geq 0} 2^{-l} f^{**}(2^{-2^l}) < \infty,
\]
where as usual we denote
\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds.
\]

Now we turn to the simplest examples of embeddings which are “invisible” from the point of view of r.i. spaces with the Fatou property. As usual, we denote by \( M_\theta \), where \( 0 \leq \theta \leq 1 \), the Marcinkiewicz space, i.e.
\[
M_\theta = \left\{ f \in L_1 : \sup_{t > 0} \int_0^t f^*(s) \, ds / t^{\theta} < \infty \right\}.
\]

Let \( M_\theta^0 \) denote the closure of \( L_\infty \) in \( M_\theta \) with the same norm. If \( 0 < \theta < 1 - \frac{m}{n} \), then \( M_\theta^0 \) is an interpolation space between \( L_1 \) and \( \Lambda^{m \over n} \). Moreover, \( M_\theta^0 = (L_1, \Lambda^{m \over n})_F^K \), where \( F = c_0(2^{-k\mu}) \) and \( \mu = \theta / (1 - \frac{m}{n}) \). Likewise \( M_{\theta + \frac{m}{n}}^0 = (\Lambda^{1 - \frac{m}{n}}, L_\infty)_F^K \). Hence by Corollary 2.1 we obtain an optimal embedding
\[
W^m M_\theta^0 \subset M_{\theta + \frac{m}{n}}^0.
\]

There are many of intermediate r.i. spaces between \( M_\theta^0 \) and \( M_\theta \). In particular consider a monotone sequence \( u_k \to 0 \). Denote by \( M_{\theta, \{u_k\}}^0 \) the space of \( f \in M_\theta \) for which
\[
\int_0^{u_k} \frac{f^*(s) \, ds}{u_k^{\theta}} \to 0.
\]

It can be easily shown that
\[
M_{\theta, \{u_k\}}^0 = (L_1, \Lambda^{m \over n})_F^K \quad \text{and} \quad M_{\theta + \frac{m}{n}, \{u_k\}}^0 = (\Lambda^{1 - \frac{m}{n}}, L_\infty)_F^K,
\]
where the parameter space \( F \) is constructed with the help of the sequence \( u_k \) as follows
\[
F = \{ \xi \in l_\infty(2^{-\mu n}) : \xi_{m_k} 2^{-\mu m_k} \to 0 \},
\]
where \( m_k = \lceil -(1 - \frac{m}{n}) \log_2 u_k \rceil \).

Thus, we obtain an optimal embedding
\[
W^m M_{\theta, \{u_k\}}^0 \subset M_{\theta + \frac{m}{n}, \{u_k\}}^0.
\]

3. The dominating property of the Hardy operator \( H_n^m \)

The proof of Theorems 2.1 and 2.2 is based on the study of the Hardy operator
\[
H_n^m : f \mapsto \int_{\nu}^{1} s^{-\frac{m}{n} - 1} f(s) \, ds,
\]
considered in spaces of functions, defined on $(0, 1)$. Here we go alongside the ideas of [9,13]. However, we extend the family of spaces for which the results similar to those of [9,13] can be applied.

Our next proposition is similar to the necessity part of the main Theorem A in [13], which claims that $H_m: E \rightarrow G$ is necessary and sufficient for $W^m E \subset G$ for any r.i. spaces $E$ and $G$, having the Fatou property.

**Proposition 3.1.** If $1 \leq m < n$ and if

$$W^m E(\Omega) \subset G(\Omega)$$

for interpolation r.i. spaces $E$ and $G$, then

$$H_m: E \rightarrow G.$$

Unfortunately, we are unable to cite [13] directly because we consider interpolation r.i. spaces instead of r.i. spaces with the Fatou property. However, the proof of the necessary part of Theorem A in [13], based on the reduction to radial functions in (5), can be applied to interpolation r.i. spaces without any changes. The situation with the sufficient part is quite different.

Direct calculations imply that $H_m: \{L_1, A_m^n\} \rightarrow \{A_{1-m/n}, L_\infty\}$. (Formally, we can use (3) and Theorem A from [13].) The next proposition is a central point of the paper. It says that the Hardy operator is a dominating operator among all linear operators $T: \{L_1, A_m^n\} \rightarrow \{A_{1-m/n}, L_\infty\}$ mapping interpolation r.i. spaces. A somewhat analogous statement was obtained in [6] for a special family of interpolation r.i. spaces.

**Proposition 3.2.** If $E$ is an interpolation space between $L_1$ and $L_\infty$, then

$$\text{Orb}(E, \{L_1, A_m^n\} \rightarrow \{A_{1-m/n}, L_\infty\}) = \{h: h < H_m(f) \text{ for some } f \in E\},$$

where $h < g$ means $\int_0^t h^*(s)\, ds \leq \int_0^t g^*(s)\, ds$ for all $t > 0$.

**Proof.** The proof is divided into several lemmas and intermediate remarks. For brevity, we denote

$$K = \{h: h < H_m(f), \text{ for some } f \in E\},$$

and $J = \text{Orb}(E, \{L_1, A_m^n\} \rightarrow \{A_{1-m/n}, L_\infty\})$.

**Lemma 3.1.** $K = \text{Orb}(E_d, \{l_1(2^{-k(1+1/n)}) , l_1(2^{-k 1/n})\} \rightarrow \{L_1, L_\infty\})$.

**Proof.** Since the interpolation from the couple $\{l_1(2^{-k(1+1/n)}) , l_1(2^{-k 1/n})\}$ into the couple $\{L_1, L_\infty\}$ is described by the $K$-method (e.g., see [4] or [14]), the orbit of $E_d$ consists of $h \in L_1$, such that there exists $\xi \in E_d$ with

$$K(t, h, \{L_1, L_\infty\}) \leq K(t, \xi, \{l_1(2^{-k(1+1/n)}) , l_1(2^{-k 1/n})\}).$$
We assume that \( \xi \geq 0 \), and define
\[
f = \sum_{k=0}^{\infty} \xi_k \chi(2^{-k-1}, 2^{-k}) \quad \text{and} \quad g = \sum_{k=0}^{\infty} \xi_k 2^{-k \frac{m}{n}} \chi(0, 2^{-k}).
\]

Easy calculations show that
\[
K(t, \xi, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\}) = \sum_{k=0}^{\infty} \xi_k \min(2^{-k(1+\frac{m}{n})}, t 2^{-k \frac{m}{n}})
\]
\[
= \sum_{k=0}^{\infty} \xi_k 2^{-k \frac{m}{n}} \min(2^{-k}, t) = \int_0^t \left[ \sum_{k=0}^{\infty} \xi_k 2^{-k \frac{m}{n}} \chi(0, 2^{-k})(s) \right] ds
\]
\[
= \int_0^t g(s) ds = K(t, g, \{L_1, L_\infty\})
\]
since \( g \) is decreasing as a sum of decreasing functions.

Let us prove that \( g^* = g < H_{\frac{m}{n}}(f) \). If \( 2^{-l-1} < t < 2^{-l} \) then
\[
g(t) = \sum_{k=0}^{l} \xi_k 2^{-k \frac{m}{n}} \chi(0, 2^{-k})(t) = \sum_{k=0}^{l} \xi_k 2^{-k \frac{m}{n}}
\]
\[
= \sum_{k=0}^{l} \frac{1}{2^{-k-1}} \int_{2^{-k-1}}^{2^{-k}} f(s) ds 2^{-k \frac{m}{n}} = 2 \sum_{k=0}^{l} 2^{-k \frac{m}{n}+k} \int_{2^{-k-1}}^{2^{-k}} f(s) ds
\]
\[
\leq 2 \sum_{k=0}^{l} \int_{2^{-k-1}}^{2^{-k}} s^{\frac{m}{n}-1} f(s) ds = 2 \int_{2^{-l-1}}^{1} s^{\frac{m}{n}-1} f(s) ds
\]
\[
\leq 2 \int_{t/2}^{1} s^{\frac{m}{n}-1} f(s) ds = 2 H_{\frac{m}{n}}(f)(t/2).
\]

Thus,
\[
K(t, h, \{L_1, L_\infty\}) \leq K(t, \xi, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\})
\]
\[
\asymp K(t, g, \{L_1, L_\infty\}) \leq K(t, H_{\frac{m}{n}}(4f), \{L_1, L_\infty\}),
\]

where \( f \in E \) if \( \xi \in E_d \). Hence,
\[
\text{Orb}(E_d; \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\} \to \{L_1, L_\infty\}) \subset K.
\]
The opposite inclusion is proved similarly. If \( h \in K \) then
\[
K(t, h, \{L_1, L_\infty\}) \leq K(t, H_{\frac{m}{n}}(f), \{L_1, L_\infty\})
\]
for some \( f \in E \). By the Calderón–Mityagin theorem, we can find a linear bounded operator
\[
S: \{L_1, L_\infty\} \to \{L_1, L_\infty\},
\]
such that \( h = S(H_{\frac{m}{n}}f) \). Now we intend to show that \( H_{\frac{m}{n}}f \) is an image of some \( \xi \in E_d \) under a linear operator
\[
T : \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\} \to \{L_1, L_\infty\}.
\]

Assume \( f \geq 0 \), and denote
\[
\xi_k = \frac{1}{2^{-k-1}} \int_{2^{-k-1}}^{2^{-k}} f(s) \, ds.
\]
Since \( E \) is an interpolation space between \( L_1 \) and \( L_\infty \), we conclude that \( \xi \in E_d \). Denote by \( U \) the linear operator which takes the standard basis vectors \( e_k \) to \( 2^{-k \frac{m}{n}} \chi_{(0,2^{-k})} \). The operator \( U \) can be easily extended, and it takes the couple \( \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\} \) to \( \{L_1, L_\infty\} \) since
\[
\|2^{-k \frac{m}{n}} \chi_{(0,2^{-k})}\|_{L_1} = 2^{-k(1+\frac{m}{n})} \quad \text{and} \quad \|2^{-k \frac{m}{n}} \chi_{(0,2^{-k})}\|_{L_\infty} = 2^{-k \frac{m}{n}}.
\]
By definition, for \( 2^{-l-1} < t < 2^{-l} \),
\[
U(\xi)(t) = \sum_{k=1}^{\infty} \xi_k 2^{-k \frac{m}{n}} \chi_{(0,2^{-k})}(t) = \sum_{k=1}^{l} \frac{1}{2^{-k-1}} \int_{2^{-k-1}}^{2^{-k}} f(s) \, ds 2^{-k \frac{m}{n}}
\]
\[
\geq 2^{-l-1} \sum_{k=1}^{l} \int_{2^{-k-1}}^{2^{-k}} s^\frac{m}{n} f(s) \, ds \geq 2^{-l-1} \frac{1}{t} \int_{t}^{1} s^\frac{m}{n} f(s) \, ds = 2^{-l-1} H_{\frac{m}{n}}(f)(t).
\]
Hence, \( h = ST(\xi) \) where \( T \) is a composition of \( U \) and multiplication by a bounded function. Thus,
\[
T : \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\} \to \{L_1, L_\infty\},
\]
and we obtain
\[
\text{Orb}(E_d; \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k \frac{m}{n}})\} \to \{L_1, L_\infty\}) = K.
\]
Lemma is proved. \( \square \)
Denote by $\widetilde{E}_d$ the space of sequences $\eta$, such that

$$K \{t, \eta, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}\} \leq CK \{t, \xi, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}\}$$

for some $\xi \in E_d$. Since the interpolation in couples of weighted $l_1$ spaces is described by the $K$-method (e.g., see [4]), the space $\widetilde{E}_d$ is equal to $\text{Orb}(E_d, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\})$. (We use the notation $\text{Orb}(X, \{X_0, X_1\})$ for $\text{Orb}(X, \{X_0, X_1\} \to \{X_0, X_1\}$).

Moreover, we can easily deduce from Lemma 3.1 that

$$K = \text{Orb}(\widetilde{E}_d; \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\} \to \{L_1, L_\infty\}).$$

Let us denote by $\widetilde{\widetilde{E}}_d$ the orbit of $E_d$ relative to the “internal” couple $\{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}$, i.e.,

$$\widetilde{\widetilde{E}}_d = \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}).$$

The space $l_1(2^{-k})$ is an interpolation space between $l_1(2^{-k(1+\frac{m}{n})})$ and $l_1(2^{-k\frac{m}{n}})$, therefore $\widetilde{E}_d \subset \widetilde{\widetilde{E}}_d$.

**Lemma 3.2.** $\widetilde{E}_d = \widetilde{\widetilde{E}}_d$.

**Proof.** Consider four spaces

$$l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k}), l_1(2^{-k\frac{m}{n}}), l_\infty.$$ 

It is well known (e.g., see [4] or [1]) that the spaces $l_1(2^{-k})$ and $l_1(2^{-k\frac{m}{n}})$ are elements of the interpolation scale $(l_1(2^{-k(1+\frac{m}{n})}), l_\infty)_{\alpha, 1}$ connecting the endpoint spaces.

We want to show that

$$\text{Orb}(E_d, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}) = \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}),$$

provided that $E_d$ is an interpolation space for the couple $\{l_1(2^{-k}), l_\infty\}$. Recall that $E_d$ is such a space if $E$ is an interpolation space for the couple $\{L_1, L_\infty\}$.

For proving (7), it is sufficient to prove

$$\text{Orb}(F, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}) = \text{Orb}(F, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}) + \text{Orb}(F, \{l_1(2^{-k}, l_\infty)\})$$

for each $F \subset l_1(2^{-k})$. Indeed, for $F = E_d$, we should have

$$\text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}) = \text{Orb}(E_d, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}) + \text{Orb}(E_d, \{l_1(2^{-k}, l_\infty)\}),$$

where $\text{Orb}(E_d, \{l_1(2^{-k}), l_\infty\}) = E_d$ since $E_d$ is an interpolation space between $l_1(2^{-k})$ and $l_\infty$. Hence,

$$\text{Orb}(E_d, \{l_1(2^{-k}), l_\infty\}) = \text{Orb}(E_d, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}).$$
The identity (8) is easily reduced to
\[ \text{Orb}(\xi, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}) \]
\[ = \text{Orb}(\xi, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\}) + \text{Orb}(\xi, \{l_1(2^{-k}), l_\infty\}) \]  (9)
for all \( \xi \in l_1(2^{-k}) \). This really takes place in even a more general setting. In [5] it is proved that
\[ \text{Orb}(\xi, \{X_{\alpha_0}, X_{\alpha_1}\}) = \text{Orb}(\xi, \{X_0, X_{\alpha_1}\}) + \text{Orb}(\xi, \{X_{\alpha_0}, X_1\}) \]
for arbitrary interpolation scale \( X_\alpha \) connecting \( X_0 \) and \( X_1 \). Thus if we take \( X_0 = l_1(2^{-k(1+\frac{m}{n})}) \), \( X_1 = l_\infty \) and \( X_\alpha = (X_0, X_1)_{\alpha,1} \) and an appropriate \( \alpha_0 \) and \( \alpha_1 \), we obtain (9). Thereby, Lemma 3.2 is proved. \( \square \)

Hence,
\[ K = \text{Orb}(\tilde{E}_d, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{L_1, L_\infty\}) \]  (10)
where now the space \( \tilde{E}_d \) is an interpolation space for the couple \( \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \).

**Lemma 3.3.** \( J = \text{Orb}(\tilde{E}_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty\}) \).

**Proof.** Since \( E \) and \( E_d \) are orbital equivalent relative to \( \{L_1, A_{\frac{m}{n}}\} \) and \( \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \), then
\[ J = \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty\}) \]
We evidently have
\[ \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty\}) \]
\[ = \text{Orb}(\tilde{E}_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty\}) \],
in view of \( \tilde{E}_d = \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\}) \). Lemma is proved. \( \square \)

Thus, to complete the proof of Proposition 3.2, we have to prove that
\[ J = \text{Orb}(\tilde{E}_d, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty\}) \]
and
\[ K = \text{Orb}(\tilde{E}_d, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}})\} \rightarrow \{L_1, L_\infty\}) \]
coincide, provided \( \tilde{E}_d \) is an interpolation space between \( l_1(2^{-k}) \) and \( l_1(2^{-k\frac{m}{n}}) \).

To this end we prove coincidence of the corresponding orbits for the simplest interpolation spaces between \( l_1(2^{-k}) \) and \( l_1(2^{-k\frac{m}{n}}) \), namely for interpolation orbits of arbitrary elements of \( l_1(2^{-k}) \).
Lemma 3.4. For all \( \xi \in l_1(2^{-k}) \)

\[
\text{Orb}(\mathcal{E}, \{l_1(2^{-k}), l_1(2^{-k(1+\frac{m}{n})}) \} \rightarrow \{A_{1-\frac{m}{n}}, L_\infty \}) \\
= \text{Orb}(\mathcal{E}, \{l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k\frac{m}{n}}) \} \rightarrow \{L_1, L_\infty \}),
\]

(11)

where \( \mathcal{E} = \text{Orb}(\xi, \{l_1(2^{-k}), l_1(2^{-k\frac{m}{n}})) \}). \)

Proof. To prove (11), it is convenient to replace the triple of spaces \( l_1(2^{-k(1+\frac{m}{n})}), l_1(2^{-k}) \) and \( l_1(2^{-k\frac{m}{n}}) \) by an isomorphic triple of \( l_1(2^{-k}), l_1(2^{-k\alpha}) \) and \( l_1 \) for an appropriate \( 0 < \alpha < 1 \) as well as the triple of spaces \( L_1, A_{1-\frac{m}{n}} \) and \( L_\infty \) by the triple \( l_1(2^{-k}), l_1(2^{-k\alpha}) \) and \( l_\infty \) consisting of orbital equivalent spaces relative to the couples \( \{L_1, L_\infty\} \) and \( \{l_1(2^{-k}), l_\infty\} \).

Recall that interpolation from the couple \( \{l_1(2^{-k\alpha}), l_1\} \) to any relatively complete couple is described by the \( K \)-method. Thus we have in particular,

\[
\text{Orb}(\xi, \{l_1(2^{-k\alpha}), l_1\}) = (l_1(2^{-k\alpha}), l_1)_{\rho, \infty},
\]

and

\[
\text{Orb}(\xi, \{l_1(2^{-k}), l_1\} \rightarrow \{l_1(2^{-k\alpha}), l_\infty\}) = (l_1(2^{-k\alpha}), l_\infty)_{\rho, \infty},
\]

where \( \rho(t) = K(t, \xi, \{l_1(2^{-k\alpha}), l_1\}) \), and

\[
(X_0, X_1)_{\rho, \infty} = \{ x \in X_0 + X_1: \sup_t\frac{K(t, x, \{X_0, X_1\})}{\rho(t)} < \infty \}
\]

by definition.

We evidently have

\[
\text{Orb}(\xi, \{l_1(2^{-k\alpha}), l_1\} \rightarrow \{l_1(2^{-k}), l_\infty\}) \\
= \text{Orb}(\text{Orb}(\xi, \{l_1(2^{-k\alpha}), l_1\}), \{l_1(2^{-k\alpha}), l_1\} \rightarrow \{l_1(2^{-k}), l_\infty\}).
\]

Thus, we mean to show that

\[
\text{Orb}(\{l_1(2^{-k\alpha}), l_1\} \rightarrow \{l_1(2^{-k}), l_\infty\}) \\
= \text{Orb}(\{l_1(2^{-k\alpha}), l_1\} \rightarrow \{l_1(2^{-k}), l_\infty\}) \\
= (l_1(2^{-k\alpha}), l_\infty)_{\rho, \infty}
\]

(12)

for all quasi-concave \( \rho \).

The case of bounded \( \rho(t) \) corresponds to \( \xi \in l_\infty \) and is trivial, so we consider the case of non-degenerate \( \rho(t) \).

Denote by \( r_k \) the nodes of a linear step function equivalent to \( \tilde{\rho}(t) = t\rho(1/t) \). (For definition and properties of linear step functions see [15] or [16].) Then (see [15])

\[
(l_1(2^{-k\alpha}), l_1)_{\rho, \infty} = (l_1, l_1(2^{-k\alpha}))_{\tilde{\rho}, \infty} = l_\infty(l_1^{M_{2n}} \oplus l_1^{M_{2n+1}})(1/\tilde{\rho}(2^\alpha)),
\]

(13)
as well as
\[(l_1(2^{-k}\alpha), l_\infty)_{\tilde{\rho}, \infty} = (l_\infty, l_1(2^{-k}\alpha)) = l_\infty(l_2^{M_{2n}} \oplus l_1^{M_{2n+1}}(1/\tilde{\rho}(2^{\alpha k}))) \tag{14},\]

where \(M_m = \{k: r_m \leq 2^{k\alpha} < r_{m+1}\} \).

Thus, for the proof of (12) we have to show that each element of the space (14) is the image of an element from the space (13) under a linear operator
\[T : \{l_1(2^{-k}), l_1\} \to \{l_1(2^{-k}), l_\infty\}.\]

Let us introduce
\[J_n : l_1^{M_{2n}} \to l_\infty^{M_{2n}}.\]

Denote by \([b_n, e_n]\) the interval \(M_{2n}\) and put
\[J_n(\xi) = \left(\sum_{k=b_n}^{b_n+1} \xi_k, \sum_{k=b_n}^{b_n} \xi_k, \ldots, \sum_{k=b_n}^{e_n} \xi_k\right).\]

We have
\[\|J_n\|_{l_1^{M_{2n}} \to l_\infty^{M_{2n}}} = 1,\]

and since \(\tilde{\rho}\) is constant on the intervals \([r_{2n}, r_{2n+1}]\), we conclude
\[\|J_n\|_{l_1^{M_{2n}}(1/\tilde{\rho}(2^{\alpha k})) \to l_\infty^{M_{2n}}(1/\tilde{\rho}(2^{\alpha k}))} = 1.\]

Denote by
\[I_n : l_1^{M_{2n+1}} \to l_1^{M_{2n+1}}\]

the identity operator, and consider
\[Q = \bigoplus_{n=1}^{\infty} J_n \oplus I_n.\]

This operator \(Q\) takes \(l_1\) to \(l_\infty\) and \(l_1(2^{-k})\) to itself. Indeed, any \(I_n\) has the unit norm anywhere. The operators
\[J_n : l_1^{M_{2n}} \to l_\infty^{M_{2n}}\]

have unit norms, and norms of
\[J_n : l_1^{M_{2n}}(2^{-k}) \to l_1^{M_{2n}}(2^{-k})\]
are uniformly bounded, since they are less than the norm of the operator of partial sums mapping the space $l_1(2^{-k})$.

Evidently the module of each element of $l_{\infty}^{M_{2^n}}(1/\tilde{\rho}(2^{\alpha k}))$ is less than an element from the image of the operator $Q$ on $l_{\infty}^{M_{2^n}}(1/\tilde{\rho}(2^{\alpha k}))$. That is why each element of $l_{\infty}^{M_{2^n}}(1/\tilde{\rho}(2^{\alpha k}))$ is an image of some element of $l_1^{M_{2^n}}(1/\tilde{\rho}(2^{\alpha k}))$ under a composition of $Q$ and multiplication by a bounded sequence.

Lemma is proved. \(\Box\)

We evidently have

\[
\tilde{E}_d = \bigcup_{\xi \in \tilde{E}_d} \xi = \bigcup_{\xi \in \tilde{E}_d} \text{Orb}(\xi, \{l_1(2^{-k}), l_1(2^{-k} m/\tilde{\rho})\}),
\]

therefore

\[
\text{Orb}(\tilde{E}_d, \{l_1(2^{-k}), l_1(2^{-k} m/\tilde{\rho})\} \rightarrow \{A_{1-m/\tilde{\rho}}, L_{\infty}\})
= \text{Orb}(\tilde{E}_d, \{l_1(2^{-k} (1+m/\tilde{\rho})), l_1(2^{-k} m/\tilde{\rho})\} \rightarrow \{L_1, L_{\infty}\})
\]

by (11). Thus, $K = J$.

Proposition is proved. \(\Box\)

### 4. Proof of main theorem

**Proof of Theorem 2.1.** Recall that we have to find a parameter space $F$, such that

\[
E \subset (L_1, A_{m/\tilde{\rho}})^K_F \quad \text{and} \quad (A_{1-m/\tilde{\rho}}, L_{\infty})^K_F \subset G,
\]

provided that $H_m : E \rightarrow G$, in view of Proposition 3.1.

For an interpolation r.i. space $E$, we evidently have

\[
E \subset \text{Orb}(E, \{L_1, A_{m/\tilde{\rho}}\}).
\]

Since $\{L_1, A_{m/\tilde{\rho}}\}$ is orbital equivalent to the couple $\{l_1(2^{-k}), l_1(2^{-k} m/\tilde{\rho})\}$, we have

\[
\text{Orb}(E, \{L_1, A_{m/\tilde{\rho}}\} \rightarrow \{X_0, X_1\}) = \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k} m/\tilde{\rho})\} \rightarrow \{X_0, X_1\})
\]

for any Banach couple $\{X_0, X_1\}$.

Hence, for any relatively complete couple $\{X_0, X_1\}$

\[
\text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k} m/\tilde{\rho})\} \rightarrow \{X_0, X_1\}) = (X_0, X_1)^K_F,
\]

where

\[
F = \text{Orb}(E_d, \{l_1(2^{-k}), l_1(2^{-k} m/\tilde{\rho})\} \rightarrow \{l_{\infty}, l_{\infty}(2^{-k})\})
= \text{Orb}(E, \{L_1, A_{m/\tilde{\rho}}\} \rightarrow \{l_{\infty}, l_{\infty}(2^{-k})\})
\]

(see [4] or [14]).
Since our couples, \( \{L^1, \Lambda_n^m\} \) and \( \{\Lambda_n^m - L^\infty\} \), are relatively complete, we obtain

\[
\text{Orb}(E, \{L^1, \Lambda_n^m\}) = (L^1, \Lambda_n^m)^K_F
\]

and

\[
\text{Orb}(E, \{L^1, \Lambda_n^m\} \rightarrow \{\Lambda_n^m - L^\infty\}) = (\Lambda_n^m - L^\infty)^K_F.
\]

Thus, Proposition 3.2 actually proves that for each \( h \in (\Lambda_n^m - L^\infty)^K_F \) there exists \( f \in E \) such that \( h \prec H_n^m(f) \). Therefore, by the Calderón–Mityagin theorem, \( h \in G \) since \( H_n^m(f) \in G \) and \( G \) is an interpolation r.i. space. Hence,

\[
(\Lambda_n^m - L^\infty)^K_F \subset G,
\]

and in view of (15) theorem is proved. □

The same ideas lead to the following.

**Proof of Theorem 2.2.** Indeed, if \( G \) is the optimal range space then \( G = (\Lambda_n^m - L^\infty)^K_F \). Therefore,

\[
G = \text{Orb}(E, \{L^1, \Lambda_n^m\} \rightarrow \{\Lambda_n^m - L^\infty\})
\]

Theorem is proved. □

**References**