JOURNAL OF COMBINATORIAL THEORY, Series A 38, 110–112 (1985)

Note

A Counterexample to the Triangle Conjecture*

P. W. SHOR

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Communicated by the Managing Editors

Received July 15, 1983

The triangle conjecture sets a bound on the cardinality of a code formed by words of the form a^iba^j. A counterexample exceeding that bound is given. This also disproves a stronger conjecture that every code is commutatively equivalent to a prefix code. © 1985 Academic Press, Inc.

INTRODUCTION

We let $A = \{a, b\}$ be a two-letter alphabet and A^* be the free monoid generated by A. A code over A is a set of words in A^* such that any string of words in the code can be uniquely deciphered into its components. That is, if $u_1 u_2 \cdots u_j = v_1 v_2 \cdots v_k$, with u_i and v_i words in the code, then $j = k$ and $u_i = v_i$ for $i = 1, 2, \ldots, j$. Equivalently, a code is a set of words in A^* that generates a free submonoid of A^*.

Consider words of the form a^iba^j, with $i + j \leqslant m - 1$ for some fixed m. It is easy to find codes made up of m such words, for example $\{b, ab, a^2b, \ldots a^{m-1}b\}$. Since each word has a distinct beginning, the code can be deciphered by pulling words from the left of any string. Perrin and Schützenberger [3, 4] conjectured that this was the maximum possible; that is, any code consisting only of words of the form a^iba^j has at most m words, where $i + j \leqslant m - 1$.

* The author was supported during this research by a National Science Foundation Graduate Fellowship.

0097-3165/85 $3.00

The Counterexample

Here we present a set of 16 words, with $i+j \leqslant 14$, that form a code. The words are

b	a^3b	a^8b	$a^{11}b$
ba	a^3ba^2	a^8ba^2	$a^{11}ba$
ba^7	a^3ba^4	a^8ba^4	$a^{11}ba^2$
ba^{13}	a^3ba^6	a^8ba^6	
ba^{14}			

We show that in a string of words from this code, the leftmost word is uniquely determined. Suppose we have

$$a^{i_0}ba^{j_0}a^{i_1}ba^{j_1}\cdots = a^{i'_0}ba^{j'_0}a^{i'_1}ba^{j'_1}\cdots.$$

Then $i_0 = i'_0$, and $j_0 + i_1 = j'_0 + i'_1$, that is,

$$j_0 - j'_0 = i'_1 - i_1.$$

So if the leftmost word in a string is not uniquely determined, then the difference of two j-values associated with the same i-value must be equal to the difference of two i-values. We can easily check that this does not happen in the code given above. The j-values for $i=0$ are $\{0, 1, 7, 13, 14\}$, and this has difference set $\{1, 6, 7, 12, 13, 14\}$. For $i=3$ and $i=8$, we get $\{0, 2, 4, 6\}$, giving a difference set $\{2, 4, 6\}$. For $i=11$ we get a difference set of $\{1, 2\}$. All of these difference sets are disjoint from the one we get from the i-values $\{0, 3, 8, 11\}$, which is $\{3, 5, 8, 11\}$. Thus, the leftmost word is always uniquely determined. By pulling the leftmost word off a string of codewords, we obtain a shorter string of codewords. Repeating this process, we can decode the string.

By doubling m, and replacing each word a^iba^j by two words, $a^{2i}ba^{2j}$ and $a^{2i}ba^{2j+1}$, we get a larger code with the same ratio of (number of words)$/m = 16/15$. Hansel [2] has shown that this ratio cannot exceed $1+\sqrt{2}$. Thus, we have

$$16/15 \leqslant \text{supremum}\left(\frac{\text{number of words}}{\max(i+j+1)}\right) \leqslant 1 + \frac{1}{\sqrt{2}},$$

where the supremum is taken over codes with words of the form a^iba^j. It might be of interest to determine this ratio exactly.

ACKNOWLEDGMENT

I would like to thank Jeff Kahn for bringing this problem to my attention and for his help in working on it.

REFERENCES

1. C. DE FELICE, On the triangle conjecture, *Inform. Process. Lett.* **14** (1982), 197–200.
2. G. HANSEL, Baïonnettes et cardinaux, *Discrete Math.* **39** (1982), 331–335.
3. D. PERRIN AND M. P. SCHÜTZENBERGER, Un problème élémentaire de la théorie de l'information, *in* "Théorie de l'Information," Colloq. Internat. CNRS No. 276, pp. 249–260, 1977.
4. D. PERRIN AND M. P. SCHÜTZENBERGER, A conjecture on sets of differences of integer pairs, *J. Combin. Theory Ser. B* **30** (1981), 91–93.
5. J. E. PIN AND I. SIMON, A note on the triangle conjecture, *J. Combin. Theory Ser. A* **32** (1982), 106–109.

Printed in Belgium