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W-graph ideals

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ABSTRACT

We introduce a concept of a W-graph ideal in a Coxeter group. The main goal of this paper is to describe how to construct a W-graph from a given W-graph ideal. The principal application of this idea is in type A, where it provides an algorithm for the construction of W-graphs for Specht modules.

1. Introduction

Let (W, S) be a Coxeter system and $\mathcal{H}(W)$ its Hecke algebra over $\mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in the indeterminate q. There are certain representations of $\mathcal{H}(W)$ whose structure can be encoded by combinatorial objects called W-graphs, introduced by Kazhdan and Lusztig in [10]. A W-graph provides a compact way of providing all the information needed to construct the representation. Moreover, from the work of Gyoja [6], it is known that if W is a finite Weyl group then all irreducible $\mathcal{H}(W)$-modules can be realised as modules carried by W-graphs. However, the problem of explicitly describing these W-graphs is not completely solved.

In [10] Kazhdan and Lusztig constructed a special basis for $\mathcal{H}(W)$, using a family of polynomials in q with integer coefficients. These polynomials, now known as the *Kazhdan–Lusztig polynomials*, are parametrised by pairs of elements of W, and are defined by a recursive procedure. The Kazhdan–Lusztig basis gives the regular representation of $\mathcal{H}(W)$ a W-graph structure. Moreover, Kazhdan and Lusztig showed that W-graphs may be split into cells, which are themselves W-graphs, thus potentially providing a means of decomposing the regular representation. In type A the cells in the regular W-graph yield irreducible representations; however, constructing W-graphs for the

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irreducible representations has to date been computationally challenging because of the large number of Kazhdan–Lusztig polynomials that must be calculated.

In [3] Deodhar gave a generalisation of the Kazhdan–Lusztig construction, using *parabolic Kazhdan–Lusztig polynomials* relative to a standard parabolic subgroup W' to give W-graph structures to $\mathcal{H}(W)$-modules induced from certain one-dimensional $\mathcal{H}(W')$-modules. This raises the question whether W-graphs for other classes of representations may be constructed similarly, and to do so is one of the main objectives of our project. We introduce the concept of a *W-graph ideal* in $(W, \leqslant_L)$ (where $\leqslant_L$ is the partial order such that $u \leqslant_L v$ if and only if $l(vu^{-1}) = l(v) - l(u)$) and give a Kazhdan–Lusztig like algorithm to produce, for any such ideal $\mathscr{I}$, a W-graph with vertices indexed by the elements of $\mathscr{I}$.

Our main focus is on $\mathcal{H}(W_n)$, the Hecke algebra of type A_{n-1}. Of course in this case the Weyl group, W_n, is isomorphic to the symmetric group of degree n, and its representation theory (and that of $\mathcal{H}(W_n)$) is deeply connected with the combinatorics of tableaux. The irreducibles are parametrised by partitions of n, and for each partition the corresponding *Specht module* has basis in one-to-one correspondence with the standard tableaux of that shape. Kazhdan and Lusztig showed in [10] that for each cell of the Kazhdan–Lusztig W-graph for the left regular representation of $\mathcal{H}(W_n)$, the Robinson-Schensted algorithm provides a one-to-one correspondence between the elements of W_n in the cell and pairs of standard tableaux with a fixed first term. In [4] Dipper and James gave a combinatorial construction of Specht modules. Attempts have been made to find direct combinatorial constructions of the W-graphs carried by the cells, but only partial results have been obtained.

The unpublished draft paper [7] presented a Kazhdan–Lusztig like algorithm for computing W-graphs for Specht modules, but the algorithm's correctness was not proved. The PhD thesis [14] contains a proof that the algorithm is indeed correct, and, moreover, can be generalised to include the construction of W-graphs for modules associated with skew partitions, as well as Specht modules. The details of this will be published in another paper. The key fact is that the set of standard tableaux corresponding to a (skew) partition of n is in one-to-one correspondence with an ideal $\mathscr{I}$ in $(W, \leqslant_L)$, and it is shown that $\mathscr{I}$ is a W-graph ideal.

The present paper is organised as follows. The next three sections present basic definitions and facts concerning Coxeter groups, Hecke algebras and W-graphs. The notion of a W-graph ideal is introduced in Section 5, and in Section 6 we present an illustrative example, constructing a W-graph basis for a specific Specht module. In Section 7 we prove in general that a W-graph can be constructed from a W-graph ideal by a recursive procedure similar to the original Kazhdan–Lusztig construction, and then in Section 8 we relate our results to the constructions of Kazhdan–Lusztig and Deohdar. Finally, in Section 9, we give an alternative construction of a W-graph induced from the W-graph associated with a W-graph ideal.

2. Coxeter groups

Let (W, S) be a Coxeter system. Thus W is a group generated by a set S subject to defining relations of the form

$$\left(ss'\right)^{m(s,s')} = 1 \quad \text{for all } s, s' \in S$$

where $m(s, s') = m(s', s)$ is a positive integer or ∞ and $m(s, s') = 1$ if and only if $s = s'$. (A relation $(ss')^\infty = 1$ is regarded as vacuously true.) Elements of S are called *simple reflections*, and the cardinality of S is called the *rank* of the system. It turns out that in all cases $m(s, s')$ equals the order of ss' in W.

Let l be the length function defined on W; that is, if $w \in W$ then $l(w)$ is the minimal k such that $w = s_1 s_2 \cdots s_k$ for some elements $s_1, s_2, \ldots, s_k \in S$. If $w = s_1 s_2 \cdots s_k$ and $l(w) = k$, then $s_1 s_2 \cdots s_k$ is said to be a *reduced expression* for w. If W is finite then there is a unique longest element in W; we shall denote it by w_S.

Define $T = \{w^{-1} s w \mid s \in S,\ w \in W\}$ (the set of reflections in W). The following partial orders are defined on W.

Definition 2.1 *(Bruhat order).* The Bruhat order $\leqslant$ is the transitive closure of the relation $\xrightarrow{T}$ given by $u \xrightarrow{T} w$ if $l(u) \leqslant l(w)$ and $w = tu$ for some $t \in T \cup \{1\}$.

Definition 2.2 *(Weak order).* The left weak order $\leqslant_L$ is the transitive closure of the relation $\xrightarrow{S}$ given by $u \xrightarrow{S} w$ if $l(u) \leqslant l(w)$ and $w = su$ for some $s \in S \cup \{1\}$. If $u \leqslant_L w$, we say that u is a *suffix* of w.

Observe that $u \leqslant_L w$ implies $u \leqslant w$. It is well known that if W is finite then $u \leqslant_L w_S$ for all $u \in W$, where w_S is the maximal length element of W.

We shall employ the customary conventions that $w \geqslant u$ means the same thing as $u \leqslant w$ and that $u < w$ means $u \leqslant w$ and $u \neq w$, and so forth.

The following property of the Bruhat order is standard (see [9, Section 7.4]).

Lemma 2.3. *Let $s \in S$ and $u, w \in W$ satisfy $u < su$ and $w < sw$. Then $u \leqslant w$ if and only if $u \leqslant sw$, and $u \leqslant sw$ if and only if $su \leqslant sw$.*

Let J be an arbitrary subset of S and W_J the subgroup of W generated by J; such subgroups are called *standard parabolic subgroups* of W. It can be shown that (W_J, J) is a Coxeter system. The length function on W_J relative to the generating set J coincides with the restriction of the length function on W (see [9, Section 5.5]), and the Bruhat and weak orders on W_J are the restrictions of the corresponding orders on W (see [9, Section 5.10]). Each left coset of W_J in W contains a unique element of $D_J = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in J\}$, and $l(du) = l(d) + l(u)$ for all $u \in W_J$ and $d \in D_J$. The set D_J is called the set of distinguished (or minimal) left coset representatives in W for the subgroup W_J (see [9, Section 1.10]). If W_J is finite then we denote the longest element of W_J by w_J. If W is finite then we let d_J be the unique element in $D_J \cap w_S W_J$; then $D_J = \{w \in W \mid w \leqslant_L d_J\}$ (see [5, Lemma 2.2.1]).

Lemma 2.4. *(See [3, Lemma 2.1(iii)].) Let $J \subseteq S$. For each $s \in S$ and each $w \in D_J$, exactly one of the following occurs*:

(i) *$l(sw) < l(w)$ and $sw \in D_J$;*
(ii) *$l(sw) > l(w)$ and $sw \in D_J$;*
(iii) *$l(sw) > l(w)$ and $sw \notin D_J$, and $w^{-1}sw \in J$.*

We shall find it convenient to make use of the following definition.

Definition 2.5. If $X \subseteq W$, let $\operatorname{Pos}(X) = \{s \in S \mid l(xs) > l(x) \text{ for all } x \in X\}$.

Thus $\operatorname{Pos}(X)$ is the largest subset J of S such that $X \subseteq D_J$.

3. Hecke algebras

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q, and let $\mathcal{A}^+ = \mathbb{Z}[q]$. Let (W, S) be a Coxeter system. Then the corresponding Hecke algebra, denoted $\mathcal{H}(W)$, is the associative algebra over $\mathcal{A}$ generated by the elements $\{T_s \mid s \in S\}$ subject to the following defining relations:

$$T_s^2 = 1 + (q - q^{-1})T_s \quad \text{for all } s \in S,$$
$$T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots \quad \text{for all } s, s' \in S,$$

where in the second of these there are $m(s, s')$ factors on each side, $m(s, s')$ being the order of ss' in W. We remark that the traditional definition has $T_s^2 = q + (q - 1)T_s$ in place of the first relation above; our version is obtained by replacing q by q^2 and dividing the generators by q.

It is well known that $\mathcal{H}(W)$ is $\mathcal{A}$-free with an $\mathcal{A}$-basis $(T_w \mid w \in W)$ and multiplication satisfying

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (q - q^{-1})T_w & \text{if } l(sw) < l(w) \end{cases}$$

for all $s \in S$ and $w \in W$.

If $J \subseteq S$ then $\mathcal{H}(W_J)$, the Hecke algebra associated with the Coxeter system (W_J, J), is isomorphic to the subalgebra of $\mathcal{H}(W)$ generated by $\{T_s \mid s \in J\}$. We shall identify $\mathcal{H}(W_J)$ with this subalgebra.

4. *W*-graphs

Let $\mathcal{H} = \mathcal{H}(W)$ be the Hecke algebra associated with the Coxeter system (W, S). Let $a \mapsto \bar{a}$ be the involutory automorphism of $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ defined by $\bar{q} \mapsto q^{-1}$. This extends to an involution on $\mathcal{H}$ satisfying

$$\overline{T_s} = T_s^{-1} = T_s - \left(q - q^{-1}\right) \quad \text{for all } s \in S.$$

A W-graph is a triple (V, μ, τ) consisting of a set V, a function $\mu : V \times V \to \mathbb{Z}$ and a function τ from V to the power set of S, subject to the requirement that the free $\mathcal{A}$-module with basis V admits an $\mathcal{H}$-module structure satisfying

$$T_s v = \begin{cases} -q^{-1}v & \text{if } s \in \tau(v), \\ qv + \sum_{\{u \in V \mid s \in \tau(u)\}} \mu(u, v)u & \text{if } s \notin \tau(v), \end{cases} \tag{4.1}$$

for all $s \in S$ and $v \in V$.

The set V is called the vertex set of the W-graph, and there is a directed edge from a vertex v to u if and only if $\mu(u, v) \neq 0$. We may regard the integer $\mu(u, v)$ as the weight of the edge from v to u, and the set $\tau(v)$ as the colour of the vertex v.

Since the $\mathcal{H}$-module $\mathcal{A}V$ is $\mathcal{A}$-free it admits a unique $\mathcal{A}$-semilinear involution $\alpha \mapsto \bar{\alpha}$ such that $\bar{v} = v$ for all elements v of the basis V. It follows from (4.1) that for all $s \in S$ and $v \in V$,

$$\begin{aligned} \overline{T_s v} &= \begin{cases} -qv & \text{if } s \in \tau(v), \\ q^{-1}v + \sum_{\{u \in V \mid s \in \tau(u)\}} \mu(u, v)u & \text{if } s \notin \tau(v) \end{cases} \\ &= \left(T_s - \left(q - q^{-1}\right)\right)v \\ &= \overline{T_s}v, \end{aligned}$$

and hence $\overline{h\alpha} = \bar{h}\bar{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathcal{A}V$.

5. *W*-graph ideals

Let (W, S) be a Coxeter system and $\mathcal{H}$ the associated Hecke algebra. Let $\mathscr{I}$ be an ideal in the poset $(W, \leqslant_L)$; that is, $\mathscr{I}$ is a subset of W such that every $u \in W$ that is a suffix of an element of $\mathscr{I}$ is itself in $\mathscr{I}$. This condition implies that $\operatorname{Pos}(\mathscr{I}) = S \setminus \mathscr{I} = \{s \in S \mid s \notin \mathscr{I}\}$ (see Definition 2.5). Let J be a subset of $\operatorname{Pos}(\mathscr{I})$, so that $\mathscr{I} \subseteq D_J$. For each $w \in \mathscr{I}$ we define the following subsets of S:

$$\begin{aligned} \mathrm{SA}(w) &= \{s \in S \mid sw > w \text{ and } sw \in \mathscr{I}\}, \\ \mathrm{SD}(w) &= \{s \in S \mid sw < w\}, \\ \mathrm{WA}_J(w) &= \{s \in S \mid sw > w \text{ and } sw \in D_J \setminus \mathscr{I}\}, \\ \mathrm{WD}_J(w) &= \{s \in S \mid sw > w \text{ and } sw \notin D_J\}. \end{aligned}$$

Since $\mathscr{I} \subseteq D_J$ it is clear that, for each $w \in \mathscr{I}$, each $s \in S$ appears in exactly one of the four sets defined above. We call the elements of these sets the strong ascents, strong descents, weak ascents and weak descents of w relative to $\mathscr{I}$ and J. In contexts where the set J is fixed we frequently omit reference to J, writing $\mathrm{WA}(w)$ and $\mathrm{WD}(w)$ rather than $\mathrm{WA}_J(w)$ and $\mathrm{WD}_J(w)$. We also define the sets of descents and ascents of w by $\mathrm{D}_J(w) = \mathrm{SD}(w) \cup \mathrm{WD}_J(w)$ and $\mathrm{A}_J(w) = \mathrm{SA}(w) \cup \mathrm{WA}_J(w)$.

Remark. It follows from Lemma 2.4 that

$$\mathrm{WA}_J(w) = \{s \in S \mid sw \notin \mathscr{I} \text{ and } w^{-1}sw \notin J\},$$
$$\mathrm{WD}_J(w) = \{s \in S \mid sw \notin \mathscr{I} \text{ and } w^{-1}sw \in J\},$$

since $sw \notin \mathscr{I}$ implies that $sw > w$ (given that $\mathscr{I}$ is an ideal in $(W, \leqslant_L)$). Note also that $J = \mathrm{WD}_J(1)$.

Definition 5.1. With the above notation, the set $\mathscr{I}$ is said to be a *W-graph ideal* with respect to J if the following hypotheses are satisfied.

(i) There exists an $\mathcal{A}$-free $\mathcal{H}$-module $\mathscr{S} = \mathscr{S}(\mathscr{I}, J)$ possessing an $\mathcal{A}$-basis $B = (b_w \mid w \in \mathscr{I})$ on which the generators T_s act by

$$T_s b_w = \begin{cases} b_{sw} & \text{if } s \in \mathrm{SA}(w), \\ b_{sw} + (q - q^{-1}) b_w & \text{if } s \in \mathrm{SD}(w), \\ -q^{-1} b_w & \text{if } s \in \mathrm{WD}_J(w), \\ q b_w - \sum_{\substack{y \in \mathscr{I} \\ y < sw}} r^s_{y,w} b_y & \text{if } s \in \mathrm{WA}_J(w), \end{cases}$$

for some polynomials $r^s_{y,w} \in q\mathcal{A}^+$.

(ii) The module $\mathscr{S}$ admits an $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$ satisfying $\overline{b_1} = b_1$ and $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}$.

We shall show in Section 7 below that if $\mathscr{I}$ is a W-graph ideal with respect to J then the associated module $\mathscr{S}(\mathscr{I}, J)$ is isomorphic to a W-graph module. Moreover, the W-graph can be constructed by an algorithm that depends only on $\mathscr{I}$ and J. Hence $\mathscr{S}(\mathscr{I}, J)$ is determined up to isomorphism by $\mathscr{I}$ and J.

Remark. As we shall see in Section 8 below, it is quite possible for an ideal $\mathscr{I}$ to be a W-graph ideal with respect to two different subsets J of $\mathrm{Pos}(\mathscr{I})$, corresponding to two W-graph modules that are not isomorphic. So the set J is an important part of the definition of a W-graph ideal.

Definition 5.2. If $\Lambda \subseteq W$ and $\mathscr{I} = \{u \in W \mid u \leqslant_L w \text{ for some } w \in \Lambda\}$ is a W-graph ideal then we call Λ a *W-graph determining set*, and we call $w \in W$ a *W-graph determining element* if $\{w\}$ is a W-graph determining set.

The simplest example of a W-graph determining element is w_S, the maximal length element of a finite Coxeter group W, with J the empty subset of S. The W-graph we obtain is the Kazhdan–Lusztig W-graph corresponding to the regular representation of W. More generally, if J is an arbitrary subset of S then d_J, the minimal length element of the left coset $w_S W_J$, is a W-graph determining element with respect to J and also with respect to $\emptyset$. In both cases $\mathscr{I} = D_J$, and we recover Deodhar's parabolic analogues of the Kazhdan–Lusztig construction. See Section 8 below for the details.

6. An example

The general algorithm for constructing W-graphs from W-graph ideals is deferred to the next section. In the current section we present a motivational example.

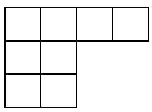
Let W_n be the Coxeter group of type A_{n-1}, which we identify with the symmetric group on $[1, n]$, the set of integers from 1 to n, by identifying the simple reflections $s_1, s_2, \ldots, s_{n-1}$ with the transpositions $(1, 2), (3, 4), \ldots, (n-1, n)$ (respectively). We use a left-operator convention for permutations, writing wi for the action of $w \in W_n$ on $i \in [1, n]$. It is well known that if $t = (i, j) \in W_n$ is an arbitrary transposition, with $i < j$, and $w \in W_n$ is an arbitrary permutation, then $wt < w$ if and only if $wi > wj$ and $tw < w$ if and only if $w^{-1}i > w^{-1}j$; moreover, $l(w)$ is the number of pairs $(i, j) \in [1, n] \times [1, n]$ such that $i < j$ and $wi > wj$.

Since our example will involve Young diagrams and tableaux, we need to start by recalling some basic definitions and establishing our notation.

A sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called a partition of n if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_k$. The λ_i are called the *parts* of λ. We define $P(n)$ to be the set of all partitions of n. For each $\lambda = (\lambda_1, \ldots, \lambda_k) \in P(n)$ we define

$$[\lambda] = \left\{(i, j) \;\middle|\; 1 \leqslant j \leqslant \lambda_i \text{ and } 1 \leqslant i \leqslant k\right\},$$

and refer to this as the Young diagram of λ. Pictorially $[\lambda]$ is represented by a left-justified array of boxes with λ_i boxes in the i-th row; the pair $(i, j) \in [\lambda]$ corresponds to the j-th box in the i-th row. Thus the Young diagram of $\lambda = (4, 2, 2)$ looks like this:



If λ is a partition of n then a λ-*tableau* is a bijection $t : [\lambda] \to [1, n]$. In other words, t is a one-to-one correspondence between the boxes of the Young diagram $[\lambda]$ and the integers from 1 to n. Of course t can be conveniently described by writing the number $t(i, j)$ in the box (i, j), for all $(i, j) \in [\lambda]$. For each $i \in [1, n]$ we define $\mathrm{row}_t(i)$ and $\mathrm{col}_t(i)$ to be the row index and column index of i in t (so that $t^{-1}(i) = (\mathrm{row}_t(i), \mathrm{col}_t(i))$). We define $\mathrm{Tab}(\lambda)$ to be the set of all λ-tableaux, and we let t^λ be the specific λ-tableau given by

$$\mathrm{t}_\lambda(i, j) = j + \sum_{h=1}^{i-1} \lambda_h$$

for all $(i, j) \in [\lambda]$. That is, the numbers $1, 2, \ldots, \lambda_1$ fill the first row of $[\lambda]$ in order from left to right, then the numbers $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ similarly fill the second row, and so on.

We define t_λ to be the λ-tableau that is the transpose of the λ'-tableau $\mathrm{t}^{\lambda'}$, where λ' is the partition dual to λ. Thus t_λ is the unique standard λ-tableau whose columns consist of sequences of consecutive numbers, while t^λ is the unique standard λ-tableau whose rows consist of sequences of consecutive numbers. We shall find it convenient to define $\mathrm{box}_\lambda(i) = \mathrm{t}_\lambda^{-1}(i)$; thus $\mathrm{box}_\lambda(i)$ is the box of $[\lambda]$ such that i is in $\mathrm{box}_\lambda(i)$ in t_λ. We say that $\mathrm{box}_\lambda(i)$ is "earlier" than $\mathrm{box}_\lambda(j)$ if $i < j$.

It is clear that for any fixed $\lambda \in P(n)$ the group W_n acts on the set of all λ-tableaux, via $(wt)(i, j) = w(t(i, j))$ for all $(i, j) \in [\lambda]$, for all λ-tableaux t and all $w \in W_n$. Moreover, the map from W_n to $\mathrm{Tab}(\lambda)$ defined by $w \mapsto w\mathrm{t}_\lambda$ for all $w \in W_n$ is bijective. We use this bijection to transfer the partial orders defined in Definitions 2.1 and 2.2 from W_n to $\mathrm{Tab}(\lambda)$. Thus if t_1, t_2 are arbitrary λ-tableaux and we write $t_1 = w_1\mathrm{t}_\lambda$ and $t_2 = w_2\mathrm{t}_\lambda$ with $w_1, w_2 \in W_n$, then by definition $t_1 \leqslant t_2$ if and only if $w_1 \leqslant w_2$, and $t_1 \leqslant_L t_2$ if and only if $w_1 \leqslant_L w_2$. Similarly, if $t = w\mathrm{t}_\lambda$ is an arbitrary λ-tableau, where $w \in W_n$, then we define $l(t) = l(w)$.

For later reference, we note the following trivial result.

Lemma 6.1. *Let $w \in W_n$ and let $t = w\mathrm{t}_\lambda$ be the corresponding λ-tableau. If $i \in [1, n-1]$ then $l(s_i t) > l(t)$ if and only if either* $\mathrm{col}_t(i) < \mathrm{col}_t(i+1)$ *or* $\mathrm{col}_t(i) = \mathrm{col}_t(i+1)$ *and* $\mathrm{row}_t(i) < \mathrm{row}_t(i+1)$.

Proof. Observe that $w^{-1}i = w^{-1}(t(\mathrm{row}_t(i), \mathrm{col}_t(i))) = \mathrm{t}_\lambda(\mathrm{row}_t(i), \mathrm{col}_t(i))$, and similarly $w^{-1}(i+1) = \mathrm{t}_\lambda(\mathrm{row}_t(i+1), \mathrm{col}_t(i+1))$. Since $\mathrm{t}_\lambda(j,k) < \mathrm{t}_\lambda(j',k')$ if and only if either $k < k'$ or $k = k'$ and $j < j'$, the condition that $\mathrm{col}_t(i) < \mathrm{col}_t(i+1)$ or $\mathrm{col}_t(i) = \mathrm{col}_t(i+1)$ and $\mathrm{row}_t(i) < \mathrm{row}_t(i+1)$ is equivalent to $w^{-1}i < w^{-1}(i+1)$. Since this in turn is equivalent to $l(s_i w) > l(w)$, the result follows. $\square$

A λ-tableau t, where $\lambda \in P(n)$, is said to be *column standard* if its entries increase down the columns, that is, if $t(i,j) < t(i+1,j)$ whenever $(i,j) \in [\lambda]$ and $(i+1,j) \in [\lambda]$. Similarly, t is said to be *row standard* if its entries increase along the rows, that is, if $t(i,j) < t(i,j+1)$ whenever $(i,j) \in [\lambda]$ and $(i,j+1) \in [\lambda]$. A *standard tableau* is a tableau that is both column standard and row standard. We write $\mathrm{CSTD}(\lambda)$, $\mathrm{RSTD}(\lambda)$ and $\mathrm{STD}(\lambda)$ for the sets of all column standard tableaux, row standard tableaux and standard tableaux for λ.

Given $\lambda \in P(n)$ we define J_λ to be the subset of S consisting of those simple reflections $s_i = (i, i+1)$ such that i and $i+1$ lie in the same column of t_λ, and we define W_λ to be the standard parabolic subgroup of W_n generated by J_λ. Thus, by our convention, W_λ is the column stabiliser of t_λ rather than the row stabiliser of t^λ. Moreover, the set of minimal left coset representatives for W_λ in W_n is the set

$$D_\lambda = \left\{d \in W_n \,\middle|\, di < d(i+1) \text{ whenever } s_i \in J_\lambda\right\}$$

since the condition $di < d(i+1)$ is equivalent to $l(ds_i) > l(d)$. It follows that $\{d\mathrm{t}_\lambda \mid d \in D_\lambda\}$ is precisely the set of column standard λ-tableaux.

Now suppose that $t \in \mathrm{STD}(\lambda)$ and $t \neq \mathrm{t}^\lambda$. Choose i to be the least integer whose position in t is not the same as its position in t^λ, and let $j = t(\mathrm{row}_{\mathrm{t}^\lambda}(i), \mathrm{col}_{\mathrm{t}^\lambda}(i))$, the number whose position in t is the position of i in t^λ. If $h = \mathrm{row}_t(j)$ then the number $j-1$ cannot appear to the left of j in the h-th row of t, or in any earlier row, since these positions are occupied by the numbers from 1 to $i-1$. Hence, since t is standard, it follows that $\mathrm{row}_t(j-1) > \mathrm{row}_t(j)$ and $\mathrm{col}_t(j-1) < \mathrm{col}_t(j)$. In particular, since $j-1$ and j are not in the same row of t or the same column of t, the tableau obtained from t by swapping the positions of $j-1$ and j is still standard. That is, $s_{j-1}t \in \mathrm{STD}(\lambda)$. But by Lemma 6.1 above we see that $l(s_{j-1}t) > l(t)$, and therefore $t <_L s_{j-1}t$. So t is not maximal in the ordering $<_L$, and it follows that t^λ is the unique maximal standard λ-tableau relative to $<_L$.

Similarly, if $t \in \mathrm{STD}(\lambda)$ and $s_j t < t$ for some $j \in [1, n-1]$, then t has $j+1$ in an earlier box than j, and since t is standard we see that $\mathrm{row}_t(j+1) > \mathrm{row}_t(j)$ and $\mathrm{col}_t(j+1) < \mathrm{col}_t(j)$. Thus $s_j t \in \mathrm{STD}(\lambda)$. So if t' is any λ-tableau such that $t' <_L t$ then t' is standard. Hence we obtain the following result (see [4, Lemma 1.5]).

Lemma 6.2. *Let $\lambda \in P(n)$ and define $v_\lambda \in W_n$ by the requirement that* $\mathrm{t}^\lambda = v_\lambda \mathrm{t}_\lambda$. *Then* $\mathrm{STD}(\lambda) = \{w\mathrm{t}_\lambda \mid w \leqslant_L v_\lambda\} = \{t \in \mathrm{Tab}(\lambda) \mid t \leqslant_L \mathrm{t}^\lambda\}$.

For $t \in \mathrm{STD}(\lambda)$, define

$$\mathrm{SA}(t) = \left\{i \in [1, n-1] \,\middle|\, \mathrm{col}_t(i) < \mathrm{col}_t(i+1) \text{ and } \mathrm{row}_t(i) \neq \mathrm{row}_t(i+1)\right\},$$

$$\mathrm{SD}(t) = \left\{i \in [1, n-1] \,\middle|\, \mathrm{col}_t(i) > \mathrm{col}_t(i+1)\right\},$$

$$\mathrm{WA}(t) = \left\{i \in [1, n-1] \,\middle|\, \mathrm{row}_t(i) = \mathrm{row}_t(i+1)\right\},$$

$$\mathrm{WD}(t) = \left\{i \in [1, n-1] \,\middle|\, \mathrm{col}_t(i) = \mathrm{col}_t(i+1)\right\}.$$

Observe that if $\mathscr{I}$ is the left ideal of $(W_n, \leqslant_L)$ generated by v_λ and if $J = J_\lambda$, then for each $w \in \mathscr{I}$ the sets $\mathrm{SA}(w), \mathrm{SD}(w), \mathrm{WA}_J(w)$ and $\mathrm{WD}_J(w)$ as defined in Section 5 coincide with the sets $\mathrm{SA}(w\mathrm{t}_\lambda), \mathrm{SD}(w\mathrm{t}_\lambda), \mathrm{WA}(w\mathrm{t}_\lambda)$ and $\mathrm{WD}(w\mathrm{t}_\lambda)$ as defined above.

Let $\mathcal{H}_n = \mathcal{H}(W_n)$ be the Hecke algebra of W_n. Thus $\mathcal{H}_n$ is generated by elements $T_1, T_2, \ldots, T_{n-1}$ satisfying $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for all $i \in [1, n-2]$ and $T_i T_j = T_j T_i$ for all $i, j \in [1, n-1]$ with $|i-j| > 1$, as well as $T_i^2 = 1 + (q - q^{-1})T_i$ for all $i \in [1, n-1]$. Let $\lambda \in P(n)$ and let S^λ be the Specht module for $\mathcal{H}_n$ corresponding to λ. It follows from results proved in [12, Chapter 3] that S^λ has an $\mathcal{A}$-basis $(b_t \mid t \in \mathrm{STD}(\lambda))$ such that for all $i \in [1, n-1]$ and $t \in \mathrm{STD}(\lambda)$,

$$T_i b_t = \begin{cases} b_{s_i t} & \text{if } i \in \mathrm{SA}(t), \\ b_{s_i t} + (q - q^{-1}) b_t & \text{if } i \in \mathrm{SD}(t), \\ -q^{-1} b_t & \text{if } i \in \mathrm{WD}(t), \\ q b_t - \sum_{s<t} r_{s,t}^{(i)} b_s & \text{if } i \in \mathrm{WA}(t), \end{cases} \tag{6.1}$$

where the $r_{s,t}^{(i)}$ in the last equation are in $\mathcal{A}$, but are not easy to describe explicitly.

The basis $(b_t \mid t \in \mathrm{STD}(\lambda))$ is known as the *standard basis* of S^λ. Note that our hypotheses and conventions are slightly different from those used in [12], and hence our formulas above are also slightly different from those in [12]. More explanation of (6.1) will be given below.

Let $\mathcal{F}$ be the field of fractions of $\mathcal{A}$, and write $\mathcal{F}S^\lambda$ for the $\mathcal{F}$-module obtained from S^λ by extension of scalars. In this context we can obtain the simpler *seminormal form* of the representation: $\mathcal{F}S^\lambda$ has an $\mathcal{F}$-basis $(b'_t \mid t \in \mathrm{STD}(\lambda))$ such that for all $i \in [1, n-1]$ and $t \in \mathrm{STD}(\lambda)$,

$$T_i b'_t = \begin{cases} -q^{-1} b'_t & \text{if } i \in \mathrm{WD}(t), \\ q b'_t & \text{if } i \in \mathrm{WA}(t), \\ p_1(d;q) b'_t + p_2(d;q) b'_{s_i t} & \text{otherwise}, \end{cases}$$

where $d = (x_1 - y_1) - (x_2 - y_2)$ if the row and column indices of i and $i+1$ in t are, respectively, x_1 and y_1 and x_2 and y_2, and

$$p_1(d;q) = \left(q^2 - 1\right) / \left(q - q^{2d+1}\right),$$
$$p_2(d;q) = \left(1 - q^{2d+2}\right) / \left(q - q^{2d+1}\right).$$

A proof of the validity of these formulas can be found in the paper of Ariki and Koike [1, Theorem 3.7]. Note that we are using a variant of $\mathcal{H}_n$ in which the eigenvalues of the generators T_i are q and q^{-1}, whereas Ariki and Koike use the traditional q and -1; hence to convert the formulas Ariki and Koike give to the ones that are appropriate for our context it was necessary to replace q by q^2 and T_i by qT_i.

The seminormal form suffers the drawback that it gives matrix coefficients that are not integral. The standard basis and the W-graph basis $(c_t \mid t \in \mathrm{STD}(\lambda))$ give integral representations but no (currently known) simple formulae for all the matrix coefficients. All three bases are related by triangular basis changes, with $c_{t_\lambda} = b_{t_\lambda} = b'_{t_\lambda}$. (This vector spans the one-dimensional subspace of S^λ consisting of those v such that $T_w v = (-q)^{-l(w)} v$ for all $w \in W_\lambda$.)

Using the seminormal form of the representation it can easily be shown that $\mathcal{F}S^\lambda$ admits a semilinear involution $v \mapsto \overline{v}$ satisfying $\overline{hv} = \overline{h}\,\overline{v}$ for all $h \in \mathcal{H}_n$ and all $v \in S^\lambda$. Indeed, if $v \in S^\lambda$ then $v = \sum_{t \in \mathrm{STD}(\lambda)} a_t b'_t$ for some coefficients $a_t \in \mathcal{F}$, and we define $\overline{v} = \sum_{t \in \mathrm{STD}(\lambda)} \overline{a_t} b'_t$. Then for all $i \in [1, n-1]$ and $t \in \mathrm{STD}(\lambda)$,

$$\overline{T_i} b'_t = T_i b'_t + \left(q^{-1} - q\right) b'_t = \overline{T_i b'_t}$$

since $p_1(d;q) + (q^{-1} - q) = p_1(d;q^{-1})$ and $p_2(d;q) = p_2(d;q^{-1})$. It follows by linearity that $\overline{T_i v} = \overline{T_i}\,\overline{v}$ for all $i \in [1, n-1]$ and all $v \in \mathcal{F}S^\lambda$, and this gives the desired result since the T_i generate $\mathcal{H}_n$.

Now for our example. We take $n = 7$ and let $\lambda = (3,3,1)$ which is a partition of 7 giving a Specht module of dimension 21. The 21 standard tableaux $t_1, t_2, \ldots, t_{21}$ are listed in order below.

$$\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5&7\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5&7\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5&7\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4&7\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4&7\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&7\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6&7\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6&7\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6&7\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6&7\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6&7\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&7\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&7\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&7\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&7\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&7\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}$$

Note that we have chosen a total ordering of $\mathrm{STD}(\lambda)$ that is consistent with the partial ordering $\leqslant$, in the sense that if $i \leqslant j$ then $t_i \leqslant t_j$. Let $b_1, b_2, \ldots, b_{21}$ be the standard basis elements corresponding (respectively) to $t_1, t_2, \ldots, t_{21}$. We shall construct a new basis $c_1, c_2, \ldots, c_{21}$ such that for all j,

$$c_j = b_j - q\sum_{i<j} f_{i,j}c_i$$

for certain $f_{i,j} \in \mathbb{Z}[q]$, to be defined recursively. In terms of this new basis the action of the algebra will be as follows:

$$T_k c_j = \begin{cases} -q^{-1}c_j & \text{if } k \in \mathrm{D}(t_j), \\ qc_j + \sum_{i\in\mathcal{R}(k,j)} \mu_{i,j}c_i & \text{if } k \in \mathrm{WA}(t_j), \\ qc_j + c_h + \sum_{i\in\mathcal{R}(k,j)} \mu_{i,j}c_i & \text{if } k \in \mathrm{SA}(t_j), \end{cases} \tag{6.2}$$

where h is defined by $s_k t_j = t_h$, the set $\mathcal{R}(k, j)$ consists of all $i < j$ such that k is a descent of t_i, and $\mu_{i,j}$ is the constant term of $f_{i,j}$.

These conditions easily yield formulas for the c_j, as listed below. To start the process, $c_1 = b_1$ is given. Now to find c_2, we first find a strong descent r of t_2; in this case, the only choice is $r = 3$. By the third formula above we must have $T_3c_1 = qc_1 + c_2$, and thus $c_2 = T_3b_1 - qc_1 = b_2 - qc_1$. In general, to find c_h given that the earlier c_j's have already been found, first find $k \in \mathrm{SD}(t_h)$, and let $t_j = s_k t_h$. Then

$$\begin{aligned} c_h &= T_k c_j - qc_j - \sum_{i\in\mathcal{R}(k,j)} \mu_{i,j}c_i \\ &= T_k\left(b_j - q\sum_{i<j} f_{i,j}c_i\right) - qc_j - \sum_{i\in\mathcal{R}(k,j)} \mu_{i,j}c_i \\ &= b_h - qc_j - q\sum_{i<j} f_{i,j}T_k c_i - \sum_{i\in\mathcal{R}(k,j)} \mu_{i,j}c_i, \end{aligned}$$

which can be expressed in the form $b_h - q\sum_{\ell<h} f_{\ell,h}c_\ell$ by using the formulas for evaluating $T_k c_i$. The crucial point is that the coefficient of each c_ℓ in $-qf_{i,j}T_k c_i$ will be a polynomial divisible by q unless $i \in \mathcal{R}(k, j)$, in which case $-qf_{i,j}T_k c_i = f_{i,j}c_i$, and the constant term $\mu_{i,j}c_i$ is cancelled by one of the terms in the second sum. In this way all the terms in the second sum also disappear.

For example, having found c_2, to find c_3 we first observe that 2 is a descent of t_3 and $s_2t_3 = t_2$, giving

$$c_3 = b_3 - qc_2 - q\sum_{i<2} f_{i,2}T_2c_i - \sum_{i\in\mathcal{R}(2,2)} \mu_{i,2}c_i.$$

Since $f_{1,2} = \mu_{1,2} = 1$ and $2 \in \mathrm{D}(t_1)$ we find that $-qf_{1,2}T_2c_1 = c_1 = \mu_{1,2}c_1$, leaving $c_3 = b_3 - qc_2$. After similarly calculating that $c_4 = b_4 - qc_2$, the calculation for c_5 proceeds as follows. Since $r = 2$ is a descent of t_5 with $s_2t_5 = t_4$,

$$\begin{aligned} c_5 &= b_5 - qc_4 - qT_2c_2 - \sum_{i \in \mathcal{R}(2,4)} \mu_{i,4}c_i \\ &= b_5 - qc_4 - q(qc_2 + c_3 + c_1) - 0, \end{aligned}$$

since the fact that $2 \notin \mathrm{D}(t_2)$ means that $\mathcal{R}(2,4)$ is empty, and $T_2c_2 = qc_2 + c_3 + c_1$ (since $2 \in D(t_1)$ and $\mu_{1,2} = 1$, and $s_2t_2 = t_3$). As a further example, the calculations involved in deriving the formula for c_{21} are given below.

$$\begin{aligned}
c_1 &= b_1, \\
c_2 &= b_2 - qc_1, \\
c_3 &= b_3 - qc_2, \\
c_4 &= b_4 - qc_2, \\
c_5 &= b_5 - qc_4 - qc_3 - q^2c_2 - qc_1, \\
c_6 &= b_6 - qc_1, \\
c_7 &= b_7 - qc_6 - qc_2 - q^2c_1, \\
c_8 &= b_8 - qc_7 - qc_3 - q^2c_2, \\
c_9 &= b_9 - qc_7 - q^2c_6 - qc_4 - q^2c_2 - qc_1, \\
c_{10} &= b_{10} - qc_9 - qc_8 - q^2c_7 - qc_5 - q^2c_4 - q^2c_3 - q^3c_2 - q^2c_1, \\
c_{11} &= b_{11} - qc_{10} - q^2c_9 - q^2c_5 - qc_4 - q^3c_1, \\
c_{12} &= b_{12} - qc_7 - qc_4 - q^2c_2, \\
c_{13} &= b_{13} - qc_{12} - qc_8 - q^2c_7 - qc_6 - qc_5 - q^2c_4 - q^2c_3 - q^3c_2 - q^2c_1, \\
c_{14} &= b_{14} - qc_{12} - qc_9 - q^2c_7 - q^2c_4 - q^3c_2 - q^2c_1, \\
c_{15} &= b_{15} - qc_{14} - qc_{13} - q^2c_{12} - qc_{10} - q^2c_9 - q^2c_8 - q^3c_7 - q^2c_6 \\
&\quad - q^2c_5 - q^3c_4 - q^3c_3 - q^4c_2 - q^3c_1, \\
c_{16} &= b_{16} - qc_{15} - qc_{14} - q^2c_{13} - qc_{12} - qc_{11} - q^2c_{10} - q^3c_9 - qc_8 \\
&\quad - q^2c_7 - q^3c_6 - q^3c_5 - q^2c_4 - q^4c_1, \\
c_{17} &= b_{17} - qc_{12} - q^2c_7, \\
c_{18} &= b_{18} - qc_{17} - qc_{13} - q^2c_{12} - q^2c_8 - q^3c_7 - q^2c_6, \\
c_{19} &= b_{19} - qc_{17} - qc_{14} - q^2c_{12} - q^2c_9 - q^3c_7 - qc_4, \\
c_{20} &= b_{20} - qc_{19} - qc_{18} - q^2c_{17} - qc_{15} - q^2c_{14} - q^2c_{13} - q^3c_{12} - q^2c_{10} \\
&\quad - q^3c_9 - q^3c_8 - q^4c_7 - q^3c_6 - qc_5 - q^2c_4 - q^2c_1,
\end{aligned}$$

$$\begin{aligned}c_{21} = b_{21} - qc_{20} - q^2c_{19} - q^2c_{18} - qc_{17} - qc_{16} - q^2c_{15} - q^3c_{14} - q^3c_{13}\\ - q^2c_{12} - q^2c_{11} - q^3c_{10} - q^4c_9 - q^2c_8 - q^3c_7 - q^4c_6 - q^2c_5\\ - \left(q^3+q\right)c_4 - qc_3 - q^2c_2 - q^3c_1.\end{aligned}$$

Here are the calculations for c_{21}. We have $s_3t_{21} = t_{20}$; so

$$\begin{aligned}c_{21} &= b_{21} - qc_{20} - \sum_{i<20} f_{i,20}T_3c_i - \sum_{i\in\mathcal{R}(3,20)} \mu_{i,20}c_i\\ &= b_{21} - qc_{20} - qT_3c_{19} - qT_3c_{18} - q^2T_3c_{17} - qT_3c_{15} - q^2T_3c_{14}\\ &\quad - q^2T_3c_{13} - q^3T_3c_{12} - q^2T_3c_{10} - q^3T_3c_9 - q^3T_3c_8 - q^4T_3c_7\\ &\quad - q^3T_3c_6 - qT_3c_5 - q^2T_3c_4 - q^2T_3c_1 - \sum_{i\in\mathcal{R}(3,20)} \mu_{i,20}c_i.\end{aligned}$$

Now 3 is a descent of t_{17}, t_{12}, t_8, t_7 and t_4; so

$$-q^2T_3c_{17} - q^3T_3c_{12} - q^3T_3c_8 - q^4T_3c_7 - q^2T_3c_4 = qc_{17} + q^2c_{12} + q^2c_8 + q^3c_7 + qc_4 \tag{6.3}$$

and we see also that the sum $\sum_{i\in\mathcal{R}(3,20)} \mu_{i,20}c_i$ has no nonzero terms. Turning to the other terms in the expression for c_{21}, the coefficient of q in the formula for c_{19} tells us that $\mu_{17,19} = \mu_{14,19} = \mu_{4,19} = 1$, and thus

$$-qT_3c_{19} = -q(qc_{19} + c_{17} + c_4) \tag{6.4}$$

since s_3t_{19} does not exist, and 3 is in $D(t_{17})$ and $D(t_4)$ but not $D(t_{14})$. Similarly

$$\begin{aligned}-qT_3c_{18} &= -q(qc_{18} + c_{17}),\\ -qT_3c_{15} &= -q(qc_{15} + c_{16}),\\ -q^2T_3c_{14} &= -q^2(qc_{14} + c_{12}),\\ -q^2T_3c_{13} &= -q^2(qc_{13} + c_{12} + c_8),\\ -q^2T_3c_{10} &= -q^2(qc_{10} + c_{11} + c_8),\\ -q^3T_3c_9 &= -q^3(qc_9 + c_7 + c_4),\\ -q^3T_3c_6 &= -q^3(qc_6 + c_7),\\ -qT_3c_5 &= -q(qc_5 + c_4 + c_3),\\ -q^2T_3c_1 &= -q^2(qc_1 + c_2),\end{aligned}$$

and we leave it to the reader to check that when these formulas together with (6.3) and (6.4) above are substituted into our expression for c_{21} the answer is as given previously.

The above example is meant to illustrate a procedure that will work for all Specht modules. Although it is clear enough that the procedure will produce a basis $(c_t \mid t \in \mathrm{STD}(\lambda))$ such that the formulas in Eq. (6.2) hold, it is not clear that the these formulas define a representation of $\mathcal{H}$. The proof that they do relies on Proposition 6.3 below, which is proved in [15]. The algorithm has been implemented using the computational algebra system MAGMA [2], and in particular has been used in the case $\lambda = (5, 5, 3, 3)$ to confirm the result of McLarnan and Warrington [13] that in this case 5 occurs as an edge-weight in the W-graph.[1]

We now briefly indicate how to adapt the discussion of the standard basis of S^λ given in [12] to yield the formulas in (6.1) above. It follows from Corollary 3.4, Corollary 3.21 and Proposition 3.22 of [12] that the Specht module S^λ (defined immediately after Corollary 3.21) has a basis $(m_t \mid t \in \mathrm{STD}(\lambda))$ such that

$$T_i m_t = \begin{cases} m_{s_i t} & \text{if } i \in \mathrm{SD}(t), \\ q m_{s_i t} + (q-1) m_t & \text{if } i \in \mathrm{SA}(t), \\ q m_t & \text{if } i \in \mathrm{WA}(t), \\ -m_t + \sum_{s<t} a_{s,t}^{(i)} m_s & \text{if } i \in \mathrm{WD}(t), \end{cases}$$

where the elements $a_{s,t}^{(i)}$ are polynomials in q. Note that [12] employs the traditional definition of $\mathcal{H}_n$, so that to make the above formulas compatible with our definitions we should replace q by q^2 and T_i by qT_i. After this we use the automorphism of $\mathcal{H}_n$ given by $T_i \to -T_i^{-1} = -\overline{T_i}$ to define a new action, obtaining a module that we call the dual Specht module. This gives

$$-\overline{T_i} m_t = \begin{cases} q^{-1} m_{s_i t} & \text{if } i \in \mathrm{SD}(t), \\ q m_{s_i t} + (q - q^{-1}) m_t & \text{if } i \in \mathrm{SA}(t), \\ q m_t & \text{if } i \in \mathrm{WA}(t), \\ -q^{-1} m_t + q^{-1} \sum_{s<t} a_{s,t}^{(i)} m_s & \text{if } i \in \mathrm{WD}(t), \end{cases}$$

where now the $a_{s,t}^{(i)}$ are polynomials in q^2. We apply these formulas for the module corresponding to the partition λ' dual to λ. This dualises again, swapping ascents and descents, and giving a module that has a basis $(m_t \mid t \in \mathrm{STD}(\lambda))$ (not the same as the m_t's we started with) satisfying

$$\overline{T_i} m_t = \begin{cases} -q^{-1} m_{s_i t} & \text{if } i \in \mathrm{SA}(t), \\ -q m_{s_i t} - (q - q^{-1}) m_t & \text{if } i \in \mathrm{SD}(t), \\ -q m_t & \text{if } i \in \mathrm{WD}(t), \\ q^{-1} m_t - q^{-1} \sum_{s<t} a_{s,t}^{(i)} m_s & \text{if } i \in \mathrm{WA}(t). \end{cases}$$

We now define $b_t = (-q)^{l(t)} \overline{m_t}$. Applying the involution $v \mapsto \bar{v}$ to both sides of the above formulas and multiplying through by $(-q)^{l(t)}$ yields (6.1) above.

The following proposition was proved in the second author's PhD thesis [14, Theorem 6.3.4]. A shorter proof is presented in [15], a recently submitted followup to the present paper.

Proposition 6.3. *The elements $r_{st}^{(i)}$ appearing in* (6.1) *are polynomials in q with zero constant term.*

In fact Theorem 6.3.4 of [14] was stronger, saying that the $r_{st}^{(i)}$ are divisible by q^2. But the weaker version is sufficient for our present needs.

Given that Proposition 6.3 is true, it follows that v_λ satisfies all the hypotheses in Definition 5.1, and is a W-graph determining element relative to J_λ. According to the theory presented in the next section, it follows that S^λ has a W-graph basis $(c_t \mid t \in \mathrm{STD}(\lambda))$ which can be computed by means of the algorithm illustrated above.

[1] The magma files used can be obtained from `http://www.maths.usyd.edu.au/u/bobh/magma/`, or from `http://magma.maths.usyd.edu.au/magma/extra/`. It is planned to include them in the next release of MAGMA.

7. Constructing the W-graph from a W-graph ideal

We return now to the situation described in Section 5 above, and let $\mathscr{I}$ be a W-graph ideal with respect to $J \subseteq S$. By Definition 5.1 there is an $\mathcal{H}$-module $\mathscr{S}$ possessing an $\mathcal{A}$-basis $B = (b_w \mid w \in \mathscr{I})$ on which the generators of $\mathcal{H}$ act via the formulas in Definition 5.1. Moreover, there is an $\mathcal{A}$-semilinear involution $v \mapsto \overline{v}$ on $\mathscr{S}$ satisfying $\overline{b_1} = b_1$ and $\overline{hv} = \overline{h}\overline{v}$ for all $h \in \mathcal{H}$ and $v \in \mathscr{S}$.

Lemma 7.1. *For each $w \in \mathscr{I}$ there exist coefficients $r_{y,w} \in \mathcal{A}$, defined for $y \in \mathscr{I}$ and $y < w$, such that $\overline{b_w} - b_w = \sum r_{y,w} b_y$ (summation over $\{y \in \mathscr{I} \mid y < w\}$).*

Proof. This is obvious when $w = 1$ since $\overline{b_1} - b_1 = 0$. Proceeding inductively, suppose that $w \in \mathscr{I}$ and $w \neq 1$, and choose $s \in S$ such that $w = su$ for some $u < w$. Then $u \in \mathscr{I}$, and by the inductive hypothesis there exist $r_{z,u} \in \mathcal{A}$ with $\overline{b_u} - b_u = \sum_{\{z \in \mathscr{I} \mid z < u\}} r_{z,u} b_z$. Moreover, $s \in \mathrm{SA}(u)$, and so $T_s b_u = b_{su} = b_w$. Thus

$$\begin{aligned}\overline{b_w} - b_w &= \overline{T_s}\,\overline{b_u} - T_s b_u \\ &= (\overline{T_s} - T_s) b_u + \overline{T_s}(\overline{b_u} - b_u) \\ &= (q^{-1} - q) b_u + \sum_{\substack{z<u \\ z \in \mathscr{I}}} r_{z,u}\big(T_s - (q - q^{-1})\big) b_z.\end{aligned}$$

Clearly $(q^{-1} - q) b_z$ is in the $\mathscr{A}$-module spanned by $\{y \in \mathscr{I} \mid y < w\}$ whenever $z \leqslant u$, and so it will suffice to show that $T_s b_z$ is in this module whenever $z \in \mathscr{I}$ and $z < u$. The formulas in Definition 5.1 describe how to express $T_s b_z$ as an $\mathcal{A}$-linear combination of elements b_x for $x \in \mathscr{I}$, and our task is simply to check that every x that occurs satisfies $x < w$.

The result is immediate if s is a weak descent of z, since in this case the only x that occurs is $x = z$, and $z < u < w$. If s is a strong descent of z then $x = z$ or $x = sz$, and in this case $sz < z$. So again $x \leqslant z < u < w$, as required.

If s is a strong ascent of z then the only x that occurs is $x = sz$. Since $z < u$ it follows from Lemma 2.3 that $sz < su = w$, giving the required result.

Finally, if s is a weak ascent of z then $T_s b_z$ is a linear combination of b_z and $\{b_x \mid x \in \mathscr{I}$ and $x < sz\}$. So either $x = z < w$ or else $x < sz < su = w$ by Lemma 2.3. □

Our aim is to construct a W-graph basis for $\mathscr{S}$. To do this we mimic the proof of Proposition 2 in Lusztig [11].

Lemma 7.2. *The module $\mathscr{S}$ has a unique $\mathcal{A}$-basis $C = (c_w \mid w \in \mathscr{I})$ such that for all $w \in \mathscr{I}$ we have $\overline{c_w} = c_w$ and*

$$b_w = c_w + q \sum_{y<w} q_{y,w} c_y \tag{7.1}$$

for certain polynomials $q_{y,w} \in \mathcal{A}^+$.

Proof. Clearly (7.1) holds for $w = 1$ if and only if $c_1 = b_1$, and defining $c_1 = b_1$ also ensures that $\overline{c_1} = c_1$, since $\overline{b_1} = b_1$ is given.

Now suppose that $w \neq 1$, and assume, inductively, that for all $y < w$ there exists a unique element $c_y \in \mathscr{S}$ such that (7.1) holds and $\overline{c_y} = c_y$. Then Lemma 7.1 gives

$$b_w - \overline{b_w} = \sum_{y<w} r_{y,w} c_y$$

for some coefficients $r_{y,w} \in \mathcal{A}$, and applying the involution $v \mapsto \overline{v}$ we see that $\overline{r_{y,w}} = -r_{y,w}$ for all $y < w$, since (7.1) and linear independence of the elements b_y ensure linear independence of the c_y. So the coefficient of q^0 in $r_{y,w}$ must be zero, and for $n > 0$ the coefficient of q^{-n} must be the negative of the coefficient of q^n. Hence $r_{y,w} = qs_{y,w} - \overline{qs_{y,w}}$ for a uniquely determined $s_{y,w} \in \mathcal{A}^+$. Moreover, $q_{y,w} = s_{y,w}$ gives the unique solution to $b_w = c_w + q\sum_{y<w} q_{y,w}c_y$ with $q_{y,w} \in \mathcal{A}^+$ and $\overline{c_w} = c_w$. So there is a unique element c_w satisfying our requirements, and the induction is complete. $\square$

Throughout the remainder of this section we let the elements c_w and the polynomials $q_{y,w}$ be defined so that the conditions of Lemma 7.2 are satisfied. We also define $\mu_{y,w}$ to be the constant term of $q_{y,w}$.

Theorem 7.3. *Let $s \in S$ and $w \in \mathscr{I}$. Then*

$$T_s c_w = \begin{cases} -q^{-1}c_w & \text{if } s \in \mathrm{D}(w), \\ qc_w + \sum_{y\in\mathcal{R}(s,w)} \mu_{y,w}c_y & \text{if } s \in \mathrm{WA}(w), \\ qc_w + c_{sw} + \sum_{y\in\mathcal{R}(s,w)} \mu_{y,w}c_y & \text{if } s \in \mathrm{SA}(w), \end{cases}$$

where the set $\mathcal{R}(s,w)$ consists of all $y \in \mathscr{I}$ such that $y < w$ and $s \in \mathrm{D}(y)$.

Proof. Suppose first that $w = 1$. If $s \notin \mathscr{I}$ then either $s \in \mathrm{WD}(1)$ (if $s \in J$) or $s \in \mathrm{WA}(1)$ (if $s \notin J$), and since $c_1 = b_1$ it follows from the formulas in Definition 5.1 that

$$T_s c_1 = \begin{cases} -q^{-1}c_1 & \text{if } s \in \mathrm{WD}(1), \\ qc_1 & \text{if } s \in \mathrm{WA}(1). \end{cases}$$

Since the set $\mathcal{R}(s,1)$ is obviously empty, the formulas in the statement of the theorem hold in these two cases. If $s \in \mathscr{I}$ then clearly $s \in \mathrm{SA}(1)$ since $s1 < 1$ is impossible, and in this case Definition 5.1 gives $T_s b_1 = b_s$. So

$$b_s - \overline{b_s} = T_s c_1 - \overline{T_s c_1} = (T_s - \overline{T_s})c_1 = \bigl(q - q^{-1}\bigr)c_1.$$

Thus $q_{1,s} = 1$ and (7.1) becomes $b_s = c_s + qc_1$, giving

$$T_s c_1 = b_s = qc_1 + c_s,$$

as required.

Proceeding by induction, suppose now that $w > 1$, and consider first the case that $s \in \mathrm{SD}(w)$. Then $y = sw < w$, and $s \in \mathrm{SA}(y)$; so the inductive hypothesis gives

$$T_s c_y = qc_y + c_w + \sum_{x\in\mathcal{R}(s,y)} \mu_{x,y}c_x,$$

which can be rewritten as

$$c_w = (T_s - q)c_y - \sum_{x\in\mathcal{R}(s,y)} \mu_{x,y}c_x.$$

Since $T_s(T_s - q) = -q^{-1}(T_s - q)$ and $T_s c_x = -q^{-1}c_x$ for all $x \in \mathcal{R}(s,y)$ (by the inductive hypothesis), it follows that

$$T_s c_w = -q^{-1}c_w,$$

as required.

Now consider the case that $s \in \mathrm{WD}(w)$. Definition 5.1 gives

$$\left(T_s + q^{-1}\right)b_w = 0,$$

and so by (7.1),

$$\left(T_s + q^{-1}\right)c_w = -q \sum_{y<w} q_{y,w}\left(T_s + q^{-1}\right)c_y. \tag{7.2}$$

If $y < w$ then $(T_s + q^{-1})c_y = 0$ if $s \in \mathrm{D}(y)$, while if $s \notin \mathrm{D}(y)$ then

$$\left(T_s + q^{-1}\right)c_y = \left(q + q^{-1}\right)c_y + v_y$$

for some v_y in $\mathscr{S}_s^-$, the subspace of $\mathscr{S}$ spanned by $\{c_x \mid s \in \mathrm{D}(x)\}$. Hence

$$\left(T_s + q^{-1}\right)c_w = \left(-\sum_{y\in Y} q_{y,w}\left(q^2+1\right)c_y\right) + v$$

for some $v \in \mathscr{S}_s^-$, where $Y = \{y \mid y < w \text{ and } s \notin \mathrm{D}(y)\}$. Since the map $v \mapsto \bar{v}$ fixes $(T_s + q^{-1})c_w$ it follows that when $(T_s + q^{-1})c_w$ is expressed as a linear combination of the basis elements c_y, all the coefficients are fixed. Hence

$$\overline{\left(q^2+1\right)q_{y,w}} = \left(q^2+1\right)q_{y,w}$$

for all $y < w$ such that $s \notin \mathrm{D}(y)$. But since $(q^2 + 1)q_{y,w}$ is a polynomial in q this forces it to be a constant, and hence forces $q_{y,w} = 0$. So all the terms on the right hand side of (7.2) disappear, and

$$T_s c_w = -q^{-1}c_w$$

as required. Note that this argument has shown that for all s such that $s \in \mathrm{WD}(w)$, the right hand side of (7.1) involves only elements c_y such that $s \in \mathrm{D}(y)$.

Suppose next that $s \in \mathrm{SA}(w)$, so that $w < sw \in \mathscr{I}$. By Lemma 7.2 and Definition 5.1 we have

$$\begin{aligned} T_s c_w &= T_s b_w - q\sum_{y<w} q_{y,w}T_s c_y \\ &= b_{sw} - q\sum_{y<w} q_{y,w}T_s c_y \\ &= c_{sw} + q\sum_{y<sw} q_{y,sw}c_y - q\sum_{y<w} q_{y,w}T_s c_y. \end{aligned}$$

Applying the inductive hypothesis to evaluate the $T_s c_y$ in the second sum gives

$$\begin{aligned} (T_s - q)c_w = c_{sw} - qc_w + q\sum_{y<sw} q_{y,sw}c_y + \sum_{y\in\mathcal{R}(s,w)} q_{y,w}c_y \\ - q\sum_{y\in\mathcal{R}'(s,w)} q_{y,w}\left(qc_y + c_{sy} + \sum_{x\in\mathcal{R}(s,y)} \mu_{x,y}c_x\right) \end{aligned}$$

where c_{sy} is to be interpreted as zero if $s \in \mathrm{WA}(y)$, and we have written $\mathcal{R}'(s,w)$ for the set of all $y < w$ such that $s \notin \mathrm{D}(y)$. Now since there are no negative powers of q appearing in any of the coefficients on the right hand side, but $\overline{(T_s - q)c_w} = (T_s - q)c_w$, we deduce that all the coefficients must simply be integers, and the positive powers of q must cancel out. So

$$(T_s - q)c_w = c_{sw} + \sum_{y \in \mathcal{R}(s,w)} \mu_{y,w} c_y,$$

where $\mu_{y,w}$ is the constant term of $q_{y,w}$, as required.

As a by-product of the above calculations we have shown that

$$\begin{aligned} -qc_w + q\sum_{y<sw} q_{y,sw}c_y + \sum_{y \in \mathcal{R}(s,w)} (q_{y,w} - \mu_{y,w})c_y \\ = q \sum_{y \in \mathcal{R}'(s,w)} q_{y,w}\left(qc_y + c_{sy} + \sum_{x \in \mathcal{R}(s,y)} \mu_{x,y}c_x\right), \end{aligned} \tag{7.3}$$

whenever $w < sw \in \mathscr{I}$. We shall return to this below, and use it to obtain a recursive formula for the polynomials $q_{y,w}$.

Finally, suppose that $s \in \mathrm{WA}(w)$. By (i) of Definition 5.1 this gives

$$(T_s - q)b_w = -\sum_{y<sw} r^s_{y,w} b_y,$$

for some $r^s_{y,w} \in q\mathcal{A}^+$, so that by Lemma 7.2

$$(T_s - q)c_w + q\sum_{y<w} q_{y,w}(T_s - q)c_y = -\sum_{y<sw} r^s_{y,w}\left(c_y + q\sum_{x<y} q_{x,y}c_x\right).$$

Hence $(T_s - q)c_w$ is equal to

$$\begin{aligned} \sum_{y \in \mathcal{R}(s,w)} q_{y,w}(q^2+1)c_y - \sum_{y \in \mathcal{R}'(s,w)} qq_{y,w}\left(c_{sy} + \sum_{x \in \mathcal{R}(s,y)} \mu_{x,y}c_x\right) \\ - \sum_{y<sw} r^s_{y,w}\left(c_y + q\sum_{x<y} q_{x,y}c_x\right), \end{aligned} \tag{7.4}$$

where again c_{sy} is interpreted as 0 if $s \in \mathrm{WA}(y)$. Since $\overline{(T_s - q)c_w} = (T_s - q)c_w$ it follows again that all terms involving positive powers of q must cancel out; this includes all of $\sum_{y<sw} r^s_{y,w}(c_y + q\sum_{x<y} q_{x,y}c_x)$ since $r^s_{y,w} \in q\mathcal{A}^+$. Hence

$$(T_s - q)c_w = \sum_{y \in \mathcal{R}(s,w)} \mu_{y,w} c_y,$$

where $\mu_{y,w}$ is the constant term of $q_{y,w}$, as required. □

Returning now to (7.3), which holds whenever $w < sw \in \mathscr{I}$, we proceed to derive the promised recursive formula for the polynomials $q_{y,w}$.

Observe first that c_w does not occur on the right hand side of (7.3) or in the last sum on the left hand side; hence it follows that $q_{w,sw} = 1$. Next, examining the coefficients of c_z when

$z \in \mathcal{R}'(s,w) = \{z < w \mid s \notin \mathrm{D}(z)\}$ gives $q_{z,sw} = qq_{z,w}$ in this case. (Note that when $z \neq w$ and $s \notin \mathrm{D}(z)$ the conditions $z < w$ and $z < sw$ are equivalent, by Lemma 2.3.) Finally, suppose that $z < sw$ and $s \in \mathrm{D}(z)$. If $z \not< w$ then $z = sy$ for some $y \in \mathcal{R}'(s,w)$, and the coefficient of c_z on the right hand side of (7.3) is $qq_{sz,w}$, while on the left hand side it is $qq_{z,sw}$. Thus $q_{sz,w} = q_{z,sw}$ in this case. If $z < w$ and $s \in \mathrm{SD}(z)$ then $sz \in \mathcal{R}'(s,w)$, and we see that c_z occurs on the right hand side of (7.3) as c_{sy} when $y = sz$, and also occurs in the sums $\sum_{x \in \mathcal{R}(s,y)} \mu_{x,y} c_x$ for those $y \in \mathcal{R}'(s,w)$ such that $z < y$. Thus the coefficient of c_z on the right hand side of (7.3) is $qq_{sz,w} + q\sum_y \mu_{z,y} q_{y,w}$, where the sum is over all $y \in \mathscr{I}$ such that $z < y < w$ and $s \notin D(y)$. On the left hand side of (7.3) the coefficient of c_z is $qq_{z,sw} + (q_{z,w} - \mu_{z,w})$. Hence

$$q_{z,sw} = -q^{-1}(q_{z,w} - \mu_{z,w}) + q_{sz,w} + \sum_y \mu_{z,y} q_{y,w} \tag{7.5}$$

where the sum is over all $y \in \mathscr{I}$ such that $z < y < w$ and $s \notin D(y)$. If $z < w$ and $s \in \mathrm{WD}(z)$ then we obtain the same formula without the $q_{sz,w}$ term.

We have proved the following result.

Corollary 7.4. *Suppose that $w < sw \in \mathscr{I}$ and $y < sw$. If $y = w$ then $q_{y,sw} = 1$, and if $y \neq w$ we have the following formulas*:

(i) $q_{y,sw} = qq_{y,w}$ *if* $s \in \mathrm{A}(y)$,
(ii) $q_{y,sw} = -q^{-1}(q_{y,w} - \mu_{y,w}) + q_{sy,w} + \sum_x \mu_{y,x} q_{x,w}$ *if* $s \in \mathrm{SD}(y)$,
(iii) $q_{y,sw} = -q^{-1}(q_{y,w} - \mu_{y,w}) + \sum_x \mu_{y,x} q_{x,w}$ *if* $s \in \mathrm{WD}(y)$,

where $q_{y,w}$ and $\mu_{y,w}$ are regarded as 0 *if $y \not< w$, and in* (ii) *and* (iii) *the sums extend over all $x \in \mathscr{I}$ such that $y < x < w$ and $s \notin \mathrm{D}(x)$.*

The following result follows easily from Corollary 7.4 by induction on $l(w) - l(y)$.

Proposition 7.5. *Let $y < w \in \mathscr{I}$. Then the degree of $q_{y,w}$ is at most $l(w) - l(y) - 1$.*

Now let $\mu : C \times C \to \mathbb{Z}$ be given by

$$\mu(c_y, c_w) = \begin{cases} \mu_{y,w} & \text{if } y < w, \\ \mu_{w,y} & \text{if } w < y, \\ 0 & \text{otherwise,} \end{cases} \tag{7.6}$$

and let τ from C to the power set of S be given by $\tau(c_w) = \mathrm{D}(w)$ for all $y \in \mathscr{I}$.

Theorem 7.6. *The triple (C, μ, τ) is a W-graph.*

Proof. In view of Theorem 7.3 it suffices to show that for all $w \in \mathscr{I}$ and $s \in S$, if $s \in \mathrm{WA}(w)$ then the set $\{y \in \mathscr{I} \mid s \in \mathrm{D}(y) \text{ and } \mu(c_y, c_w) \neq 0\}$ contains no elements $y > w$, while if $s \in \mathrm{SA}(w)$ then the only such element is sw, and $\mu(c_{sw}, c_w) = 1$.

Accordingly, suppose that $w < y \in \mathscr{I}$ with $\mu_{w,y} \neq 0$, and suppose that $s \in \mathrm{D}(y) \cap \mathrm{A}(w)$. As noted in the proof of Theorem 7.3, if $s \in \mathrm{WD}(y)$ then $q_{z,y} = 0$ for all $z < y$ with $s \in \mathrm{A}(y)$; in particular, $q_{w,y} = 0$, contradicting $\mu_{w,y} \neq 0$. Hence $s \in \mathrm{SD}(y)$. Now define $x = sy$, so that $x < sx \in \mathscr{I}$, and observe by Corollary 7.4(i) that $q_{w,sx} = qq_{w,x}$ if $w \neq x$. Since this contradicts $\mu_{w,y} \neq 0$ we conclude that $w = x$, and $q_{w,sx} = 1$, by Corollary 7.4. So $y = sw$ and $\mu_{w,sw} = 1$, as required. □

Proposition 7.7. *Let the bases $B = (b_w \mid w \in \mathscr{I})$ and $C = (c_w \mid w \in \mathscr{I})$ be as in Lemma 7.2 above. Then there exist polynomials $p_{y,w} \in \mathcal{A}^+$ such that $c_w = b_w - q\sum_{y<w} p_{y,w} b_y$ for all $w \in \mathscr{I}$, and the constant term of $p_{y,w}$ is $\mu_{y,w}$.*

Proof. It follows readily from Eq. (7.1) that the required polynomials $p_{y,w}$ are given recursively by

$$p_{y,w} = q_{y,w} - \sum_{y<x<w} q p_{y,x} q_{x,w} \quad \text{if } y < w, \tag{7.7}$$

whence the constant term of $p_{y,w}$ equals that of $q_{y,w}$. □

For our final theoretical result of this section, we show that if $\mathscr{I}$ is a W-graph ideal that is generated by a single (W-graph determining) element, then in part (i) of Definition 5.1, in the case $s \in \mathrm{WA}_J(w)$, the sum $\sum_{y\in\mathscr{I},\, y<sw} r^s_{y,w} b_y$ can be replaced by the simpler $\sum_{y\in\mathscr{I},\, y<w} r^s_{y,w} b_y$.

Lemma 7.8. *Suppose that $x, y, v, w \in W$ and $s \in S$ satisfy*

(1) $xy = vw$ *and* $l(xy) = l(x) + l(y) = l(v) + l(w)$,
(2) $sw > w$ *and* $vs > v$,
(3) $y \leqslant sw$.

Then $y \leqslant w$.

Proof. Assume that x, y, v, w and s satisfy the stated hypotheses. If $sy > y$ then the desired conclusion follows immediately from the hypotheses $y \leqslant sw$ and $sw > w$, by Lemma 2.3. So we may assume that $sy < y$. With this extra hypothesis, we use induction on $l(w)$ to prove the result.

If $l(w) = 0$ then the hypothesis (3) becomes $y \leqslant s$, and since $sy < y$ it follows that $y = s$. So $l(x) + l(y) = l(v) + l(w)$ becomes $l(x) = l(v) - 1$, and $xy = vw$ becomes $xs = v$, which together contradict the hypothesis $vs > v$. So the result is vacuously true in this case.

Now suppose that $l(w) > 0$ and that the result holds in all cases corresponding to shorter w. Choose $r \in S$ such that $w' = wr < w$. Note that since

$$l(w') + 1 = l(w) < l(sw) = l(sw'r) \leqslant l(sw') + 1$$

and also

$$l(sw') \leqslant l(w') + 1 = l(w) < l(sw) = l(sw'r)$$

it follows that $w' < sw'$ and $sw' < sw'r$.

Suppose first that $yr > y$. By hypothesis (1),

$$l(xyr) = l(vwr) = l(vw') \leqslant l(v) + l(w') = l(v) + l(w) - 1 = l(xy) - 1,$$

and so $xyr = x'y$ for some x' with $l(x') = l(x) - 1$, by the Exchange Condition. Moreover, since $sw' < sw'r = sw$ (proved above) and $y < yr$, it follows from Lemma 2.3 and the hypothesis $y < sw$ that $y \leqslant sw'$. So now we have

(1′) $x'y = vw'$ and $l(x'y) = l(x') + l(y) = l(v) + l(w')$,
(2′) $sw' > w'$ and $vs > v$,
(3′) $y \leqslant sw'$,

and since $l(w') < l(w)$ the inductive hypothesis gives $y \leqslant w'$. But $w' < w$; so $y \leqslant w$ in this case.

It remains to consider the case $yr < y$. Put $y' = yr$, and observe that

$$l(xy') \leqslant l(x) + l(y') = l(x) + l(y) - 1 = l(xy) - 1 \leqslant l(xyr) = l(xy'),$$

so that $l(xy') = l(x) + l(y')$. The same argument gives $l(vw') = l(v) + l(w')$. And since $sw' < sw'r = sw$ and $y' < y'r = y$, it follows from the hypothesis $y < sw$ and Lemma 2.3 that $y' < sw'$. So now we have

(1″) $x'y = vw'$ and $l(xy') = l(x) + l(y') = l(v) + l(w')$,
(2″) $sw' > w'$ and $vs > v$,
(3″) $y' \leqslant sw'$,

and since $l(w') < l(w)$ the inductive hypothesis gives $y' \leqslant w'$. Since $y' < y'r = y$ and $w' < w'r = w$, this yields $y < w$, by Lemma 2.3. □

Proposition 7.9. *Suppose that $u \in W$ and $\mathscr{I} = \{w \in W \mid w \leqslant_L u\}$ is a W-graph ideal with respect to J. With all the notation as in Definition* 5.1, *if $w \in \mathscr{I}$ and $s \in \mathrm{WA}_J(w)$, then every $y \in \mathscr{I}$ with $y < sw$ satisfies $y \leqslant w$.*

Proof. Suppose that $w \in \mathscr{I}$ and $s \in \mathrm{WA}_J(w)$, and that $y \in \mathscr{I}$ with $y < sw$. Since w and y are in $\mathscr{I}$ they are both suffixes of u, and so there exist $x, v \in W$ with $u = xy = vw$ and $l(u) = l(x) + l(y) = l(v) + l(w)$. If $v' = vs < v$ then $u = (v's)w = v'(sw)$, showing that $sw \leqslant_L u$ since

$$l(u) = l(v) + l(w) = l\bigl(v'\bigr) + 1 + l(w) = l\bigl(v'\bigr) + l(sw).$$

Since this contradicts the assumption that $s \in \mathrm{WA}_J(w)$, it follows that $vs > v$, and hence all the hypotheses of Lemma 7.8 are satisfied. So $y \leqslant w$, as required. □

The recursive nature of Corollary 7.4 makes it relatively straightforward to implement calculation of the polynomials $q_{y,w}$ (and hence the W-graph edge-weights $\mu_{y,w}$) using a computational algebra program. We outline one possible way to do this.

Assume that the elements of $\mathscr{I}$ are listed as $w_1, w_2, \ldots, w_d$, where $i \leqslant j$ implies that $l(w_i) \leqslant l(w_j)$, and let $S = \{s_1, s_2, \ldots, s_n\}$. The input to the process is an array `tab` such that $\texttt{tab[i,j]} = \texttt{j}$ if s_i is a weak ascent of w_j and $\texttt{tab[i,j]} = -\texttt{j}$ if s_i is a weak descent of w_j, while $\texttt{tab[i,j]} = \texttt{k}$ if s_i is a strong ascent or strong descent of w_j and $s_i w_j = w_k$. It is convenient to precompute another array `descents` such that

$$\texttt{descents[j]} = \bigl\{\texttt{i} \bigm| \texttt{tab[i,j]} < \texttt{j}\bigr\}.$$

We can now define a function `Q` such that `Q(j,k)` returns the polynomial $q_{y,z}$ if $y = w_j < z = w_k$, and returns 0 otherwise.

If $\texttt{j} \geqslant \texttt{k}$ then `Q(j,k)` immediately returns 0. Otherwise the set `descents[k]` is searched for an `s` with $\texttt{tab[s,k]} = \texttt{m} > \texttt{0}$; note that since $\texttt{m} < \texttt{k}$ the value of `Q(j,m)` can be used in the calculation of `Q(j,k)`. By (i) of Corollary 7.4, `Q(j,k)` can be set equal to $q * \texttt{Q(j,m)}$ if `s` is not in `descents[j]`. If `s` is in `descents[j]` then (ii) of Corollary 7.4 is applicable if $\texttt{tab[s,j]} > \texttt{0}$, while (iii) is applicable if $\texttt{tab[s,j]} < \texttt{0}$. Interpreting `Q(tab[s,j],m)` as zero in this latter case, the formula for `Q(j,k)` becomes

$$\texttt{Q(j,k)} = \bigl(\bigl(\texttt{mu(j,m)} - \texttt{Q(j,m)}\bigr)/q\bigr) + \texttt{Q}\bigl(\texttt{tab[s,j],m}\bigr) + \texttt{Sum}$$

where `mu(j,m)` is the constant term of `Q(j,m)`, and `Sum` denotes the sum of the values $\texttt{mu(j,i)} *$ `Q(i,m)` for `i` in the range $\texttt{j} < \texttt{i} < \texttt{m}$.

8. The Kazhdan–Lusztig and Deodhar constructions

Since every $u \in W$ occurs as a suffix of the longest element w_S, the ideal of $(W, \leqslant_L)$ generated by w_S is the whole of W. We seek to show that w_S is a W-graph determining element, or, equivalently, that W is a W-graph ideal. We are forced to let $J = \emptyset$ so that the requirement $W \subseteq D_J$

is satisfied, and this means that the sets $\mathrm{WA}_J(w)$ and $\mathrm{WD}_J(w)$ are empty for all $w \in W$. Hence to show that w_S is a W-graph determining element we need to produce an $\mathcal{H}$-module with an $\mathcal{A}$-basis $(b_w \mid w \in W)$ such that for all $s \in S$ and $w \in W$,

$$T_s b_w = \begin{cases} b_{sw} & \text{if } sw > w, \\ b_{sw} + (q - q^{-1}) b_w & \text{if } sw < w. \end{cases}$$

The module must also admit an $\mathcal{A}$-semilinear involution such that $\overline{T_w b_1} = \overline{T_w} b_1$ for all $w \in W$. Since these conditions are obviously satisfied if we put $b_w = T_w$ for all $w \in W$, the required module is the left regular module $\mathcal{H}$. Thus our construction in Section 7 will produce a W-graph basis of $\mathcal{H}$, and combining Propositions 7.5 and 7.7 yields the following result.

Proposition 8.1. *The Hecke algebra $\mathcal{H}$ has a W-graph basis $(c_w \mid w \in W)$ such that $\overline{c_w} = c_w$ and $c_w = T_w - \sum_{y<w} q p_{y,w} T_y$ for all $w \in W$, where $p_{y,w}$ is a polynomial of degree at most $l(w) - l(y) - 1$ and the W-graph edge-weight $\mu_{y,w}$ is the constant term of $p_{y,w}$.*

Converting the traditional version of $\mathcal{H}$ as used in [10] to our version requires replacing q by q^2, after which the T_w of [10] becomes $q^{l(w)} T_w$ in our context. So the formula in [10, Theorem 1.1], when converted to our context, becomes $C_w = \sum_{y \leqslant w} (-1)^{l(w)-l(y)} q^{l(w)-2l(y)} P^*_{y,w} (q^{l(y)} T_y)$, where $P^*_{y,w}$ is obtained from the Kazhdan–Lusztig polynomial $P_{y,w}$ by replacing q by q^{-2}. Since $P_{w,w} = 1$ the coefficient of T_w on the right hand side of this expression is 1, and since $P_{y,w}$ is a polynomial of degree at most $\frac{1}{2}(l(w) - l(y) - 1)$ when $y < w$ we see that the coefficient of T_y, namely $(-q)^{l(w)-l(y)} P^*_{y,w}$, is a polynomial in q with zero constant term. Since also $\overline{C_w} = C_w$, the uniqueness part of Lemma 7.2 guarantees that $C_w = c_w$, from which we can deduce a simple relationship between our polynomials $p_{y,w}$ and the Kazhdan–Lusztig polynomials.

Proposition 8.2. *The polynomials $p_{y,w}$ appearing in Proposition* 8.1 *are related to the Kazhdan–Lusztig polynomials $P_{y,w}$ via*

$$p_{y,w} = (-q)^{l(w)-l(y)-1} P^*_{y,w}, \tag{8.1}$$

where $P^*_{y,w}$ is obtained from $P_{y,w}$ by replacing q by q^{-2}. In particular, the coefficient of $q^{\frac{1}{2}(l(w)-l(y)-1)}$ in $P_{y,w}$ is $(-1)^{l(w)-l(y)-1} \mu_{y,w}$.

Note that Kazhdan and Lusztig show that $\mu_{y,w} \neq 0$ only if $l(w) - l(y) - 1$ is even.

Turning now to Deodhar's construction, let J be an arbitrary subset of S and let d_J be the longest element of D_J (which is the shortest element of $w_S W_J$). An element $u \in W$ is a suffix of d_J if and only if $u \in D_J$, and so the ideal $\mathscr{I}$ of $(W, \leqslant_L)$ generated by d_J coincides with D_J. Clearly $\mathrm{Pos}(\mathscr{I}) = J$. We shall show that $\mathscr{I} = D_J$ is a W-graph ideal with respect to J, and also that it is a W-graph ideal with respect to $\emptyset$. We consider the latter case first.

Since $D_\emptyset = W$, it follows from the definitions in Section 5 that if $w \in \mathscr{I}$ then $\mathrm{SA}(w) = \{s \in S \mid sw > w \text{ and } sw \in D_J\}$ and $\mathrm{SD}(w) = \{s \in S \mid sw < w\}$, while $\mathrm{WD}_\emptyset(w) = \{s \in S \mid sw \notin D_\emptyset\} = \emptyset$ and

$$\mathrm{WA}_\emptyset(w) = \{s \in S \mid sw \in D_\emptyset \setminus D_J\} = \{s \in S \mid sw = wt \text{ for some } t \in J\}$$

by Lemma 2.4. We proceed to construct an $\mathcal{H}$-module $\mathscr{S}$ satisfying the requirements of Definition 5.1. (Our module $\mathscr{S}$ is essentially the module M^J in [3], in the case $u = q$, the only differences being due to our non-traditional definition of $\mathcal{H}$.)

Let $\mathcal{H}_J$ be the Hecke algebra associated with the Coxeter system (W_J, J), and recall that $\mathcal{H}_J$ can be identified with the subalgebra of $\mathcal{H}$ spanned by $\{T_u \mid u \in W_J\}$. There is an $\mathcal{A}$-algebra homomorphism $\psi : \mathcal{H}_J \to \mathcal{A}$ such that $\psi(T_u) = q^{l(u)}$ for all $u \in W_J$, and this can be used to give $\mathcal{A}$ the structure of an $\mathcal{H}_J$-module, which we denote by $\mathcal{A}_\psi$. Since $\mathcal{H}$ is obviously an $(\mathcal{H}, \mathcal{H}_J)$-bimodule, the tensor product $\mathscr{S}_\psi = \mathcal{H} \otimes_{\mathcal{H}_J} \mathcal{A}_\psi$ is a (left) $\mathcal{H}$-module, and it is straightforward to show that it is $\mathcal{A}$-free with basis $B = (b_w \mid w \in D_J)$ defined by $b_w = T_w \otimes 1$ for all $w \in D_J$.

Let $w \in D_J$ and $s \in S$. If $s \in \mathrm{SA}(w)$ then $l(sw) > l(w)$, and so

$$T_s b_w = T_s(T_w \otimes 1) = (T_s T_w) \otimes 1 = T_{sw} \otimes 1 = b_{sw}$$

since $sw \in D_J$. If $s \in \mathrm{SD}(w)$ then $l(sw) < l(w)$, and so

$$T_s b_w = (T_s T_w) \otimes 1 = \big(T_{sw} + \big(q - q^{-1}\big)T_w\big) \otimes 1 = b_{sw} + \big(q - q^{-1}\big)b_w$$

since again $sw \in D_J$. There are no weak descents, and if $s \in \mathrm{WA}_\emptyset(w)$ then there is a $t \in J$ with $sw = wt$, and we find that

$$T_s b_w = (T_s T_w) \otimes 1 = (T_w T_t) \otimes 1 = T_w \otimes \psi(T_t) = q b_w.$$

So the action of the generators $\{T_s \mid s \in S\}$ on the basis B is in accordance with the requirements of Definition 5.1(i) (with all the polynomials $r^s_{y,w}$ being zero), and it only remains to check that $\mathscr{S}_\psi$ admits an $\mathcal{A}$-semilinear involution satisfying the requirements of Definition 5.1(ii). We include a proof here for the sake of completeness, although the result is proved in [3].

We show that the unique $\mathcal{A}$-semilinear map $\mathscr{S}_\psi \to \mathscr{S}_\psi$ satisfying $\overline{b_w} = \overline{T_w} \otimes 1$ for all $w \in D_J$ has the required properties. Note first that $\psi(\overline{T_u}) = \psi(T_u)^{-1} = \overline{\psi(T_u)}$ for all $u \in W_J$. Now if $x \in W$ is arbitrary then we may write $x = wu$ for some $w \in D_J$ and some $u \in W_J$, and we find that

$$\begin{aligned}\overline{T_x \otimes 1} &= \overline{T_w T_u \otimes 1} = \overline{T_w \otimes \psi(T_u)} = \overline{\psi(T_u)(T_w \otimes 1)} = \overline{\psi(T_u)}\,(\overline{T_w \otimes 1}) \\ &= \psi(\overline{T_u})(\overline{T_w} \otimes 1) = \overline{T_w} \otimes \psi(\overline{T_u}) = \overline{T_w}\,\overline{T_u} \otimes 1 = \overline{T_w T_u} \otimes 1 = \overline{T_x} \otimes 1.\end{aligned}$$

Hence $\overline{k \otimes 1} = \bar{k} \otimes 1$ for all $k \in \mathcal{H}$, and so

$$\overline{h(k \otimes 1)} = \overline{(hk) \otimes 1} = \overline{hk} \otimes 1 = (\bar{h}\bar{k}) \otimes 1 = \bar{h}(\overline{k \otimes 1})$$

for all $h, k \in \mathcal{H}$. So $\overline{h\alpha} = \bar{h}\bar{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}_\psi$, as required.

Since the requirements of Definition 5.1 have all been met, the construction in Section 7 above produces a W-graph basis in the module $\mathscr{S}_\psi$. This basis corresponds to the basis of M^J in Proposition 3.2(iii) of [3] (in the case $u = q$). Deodhar's polynomials $P^J_{y,w}$ and our polynomials are related by the obvious modification of (8.1) above.

Proposition 8.3. *The $\mathcal{H}$-module $\mathscr{S}_\psi$ has a W-graph basis $(c_w \mid w \in D_J)$ such that $\overline{c_w} = c_w$ and $c_w = b_w - \sum_{y<w} q p^J_{y,w} b_y$ for all $w \in W$, where $p^J_{y,w}$ is a polynomial of degree at most $l(w) - l(y) - 1$ and the W-graph edge-weight $\mu_{y,w}$ is the constant term of $p^J_{y,w}$. The polynomials $p^J_{y,w}$ are related to Deodhar's polynomials $P^J_{y,w}$ via*

$$p^J_{y,w} = (-q)^{l(w)-l(y)-1} P^*_{y,w}, \tag{8.2}$$

where $P^*_{y,w}$ is obtained from $P^J_{y,w}$ by replacing q by q^{-2}.

The proof that $\mathscr{I} = D_J$ is a W-graph ideal with respect to J is very similar to the proof just given. We find that

$$\mathrm{SA}(w) = \{s \in S \mid sw > w \text{ and } sw \in D_J\},$$
$$\mathrm{SD}(w) = \{s \in S \mid sw < w\},$$
$$\mathrm{WA}_J(w) = \{s \in S \mid sw \in D_J \setminus D_J\} = \emptyset,$$

and

$$\begin{aligned}\mathrm{WD}_J(w) &= \{s \in S \mid sw \notin D_J\} \\ &= \{s \in S \mid sw = wt \text{ for some } t \in J\}.\end{aligned}$$

Thus the weak ascents of the previous case are now weak descents, and vice versa. The corresponding $\mathcal{H}$-module is $\mathscr{S}_\phi = \mathcal{H} \otimes_{\mathcal{H}_J} \mathcal{A}_\phi$, where $\mathcal{A}_\phi$ is $\mathcal{A}$ made into an $\mathcal{H}_J$-module via the homomorphism $\phi : \mathcal{H}_J \to \mathcal{A}$ that satisfies $\phi(T_u) = (-q)^{-l(u)}$ for all $u \in W_J$. This corresponds to M^J in [3] in the case $u = -1$. We again define $b_w = T_w \otimes 1$ for all $w \in D_J$, and this time we find that

$$T_s b_w = \begin{cases} b_{sw} & \text{if } w \in \mathrm{SA}(w), \\ b_{sw} + (q - q^{-1}) b_w & \text{if } w \in \mathrm{SD}(w), \\ -q^{-1} b_w & \text{if } w \in \mathrm{WD}_J(w) \end{cases}$$

in accordance with the requirements of Definition 5.1. The proof that $\mathscr{S}_\phi$ admits an $\mathcal{A}$-semilinear involution with the required properties is exactly as in the previous case. Again the W-graph basis given by our construction is essentially the same as the basis of M^J in Proposition 3.2(iii) of [3] (now in the case $u = -1$).

Proposition 8.4. *The $\mathcal{H}$-module $\mathscr{S}_\phi$ has a W-graph basis $(c_w \mid w \in D_J)$ such that $\overline{c_w} = c_w$ and $c_w = b_w - \sum_{y<w} q p^J_{y,w} b_y$ for all $w \in W$, where $p^J_{y,w}$ is a polynomial of degree at most $l(w) - l(y) - 1$ and the W-graph edge-weight $\mu_{y,w}$ is the constant term of $p^J_{y,w}$. The polynomials $p^J_{y,w}$ are related to Deodhar's polynomials $P^J_{y,w}$ via*

$$p^J_{y,w} = (-q)^{l(w)-l(y)-1} P^*_{y,w}, \tag{8.3}$$

where $P^*_{y,w}$ is obtained from $P^J_{y,w}$ by replacing q by q^{-2}.

We remark that the above constructions are special cases of a more general construction to be described in the next section. If $J \subseteq S$ and $J = J_1 \cup J_2$, where no element of J_1 is conjugate in W_J to any element of J_2, then $\mathcal{H}_J$ has a one-dimensional module on whose basis element b_1 the generators T_s of $\mathcal{H}_J$ act as follows:

$$T_s b_1 = \begin{cases} -q^{-1} b_1 & \text{if } s \in J_1, \\ q b_1 & \text{if } s \in J_2. \end{cases}$$

Thus the subset of W_J consisting of the identity element alone is a W_J-graph ideal with respect to J_1, with $\mathrm{D}(1) = \mathrm{WD}(1) = J_1$ and $A(1) = \mathrm{WA}(1) = J_2$. By Theorem 9.2 below it follows that D_J is a W-graph ideal with respect to J_1. Deodhar's two constructions correspond to the cases $J_1 = \emptyset$ and $J_1 = J$.

9. Induced W-graph ideals

Let $K \subseteq S$, and let $\mathcal{H}_K$ be the Hecke algebra associated with the Coxeter system (W_K, K), identified with a subalgebra of $\mathcal{H}$ as in Section 8 above. Suppose that $\mathscr{I}_0 \subseteq W_K$ is a W_K-graph ideal with respect to $J \subseteq K$, and let $\mathscr{S}_0 = \mathscr{S}(\mathscr{I}_0, J)$ be the corresponding $\mathcal{H}_K$-module. Thus $\mathscr{S}_0$ has an $\mathcal{A}$-basis $(b_z^0 \mid z \in \mathscr{I}_0)$ such that for all $t \in K$ and $z \in \mathscr{I}_0$,

$$T_t b_z^0 = \begin{cases} b_{tz}^0 & \text{if } t \in \mathrm{SA}(K,z), \\ b_{tz}^0 + (q - q^{-1}) b_z^0 & \text{if } t \in \mathrm{SD}(K,z), \\ -q^{-1} b_z^0 & \text{if } t \in \mathrm{WD}_J(K,z), \\ q b_z^0 - \sum_{\substack{y \in \mathscr{I}_0 \\ y < tz}} r_{y,z}^t b_y^0 & \text{if } t \in \mathrm{WA}_J(K,z), \end{cases} \tag{9.1}$$

for some $r_{y,z}^t \in q\mathcal{A}^+$, where the descent and ascent sets are given by

$$\mathrm{SA}(K,z) = \{t \in K \mid tz > z \text{ and } tz \in \mathscr{I}_0\},$$

$$\mathrm{SD}(K,z) = \{t \in K \mid tz < z\},$$

$$\mathrm{WA}_J(K,z) = \{t \in K \mid tz \notin \mathscr{I}_0 \text{ and } z^{-1}tz \notin J\},$$

$$\mathrm{WD}_J(K,z) = \{t \in K \mid tz \notin \mathscr{I}_0 \text{ and } z^{-1}tz \in J\}.$$

Furthermore, $\mathscr{S}_0$ admits an $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$ satisfying $\overline{b_1^0} = b_1^0$ and $\overline{h\alpha} = \bar{h}\overline{\alpha}$ for all $h \in \mathcal{H}_K$ and $\alpha \in \mathscr{S}_0$.

We shall show that $\mathscr{I} = D_K \mathscr{I}_0 = \{dz \mid d \in D_K \text{ and } z \in \mathscr{I}_0\}$ is a W-graph ideal with respect to J. The corresponding $\mathcal{H}$-module $\mathscr{S}(\mathscr{I}, J)$ is $\mathscr{S} = \mathcal{H} \otimes_{\mathcal{H}_K} \mathscr{S}_0$.

Lemma 9.1. *The set $\mathscr{I}$ defined above is an ideal of $(W, \leqslant_L)$.*

Proof. In view of Definition 2.2, it suffices to show that $sw \in D_K \mathscr{I}_0$ whenever $s \in S$ and $w \in D_K \mathscr{I}_0$ satisfy $l(sw) < l(w)$.

Let $w = dz$, where $d \in D_K$ and $z \in \mathscr{I}_0$. Let $s \in S$, and suppose that $l(sw) < l(w)$. If $sd \in D_K$ then trivially $sw = (sd)z \in D_K \mathscr{I}_0$. Now suppose that $sd \notin D_K$. By Lemma 2.4 this gives $sd = dt$ for some $t \in K$, and since $z \in \mathscr{I}_0 \subseteq W_K$ we see that $tz \in W_K$. Hence, since $d \in D_K$,

$$l(tz) = l(dtz) - l(d) = l(sdz) - l(d) = l(sw) - l(d) < l(w) - l(d) = l(dz) - l(d) = l(z).$$

Since $t \in K$ and $z \in \mathscr{I}_0$, and $\mathscr{I}_0$ is an ideal of $(W_K, \leqslant_L)$, it follows that $tz \in \mathscr{I}_0$. Hence $sw = d(tz) \in D_K \mathscr{I}_0$ in this case also, as required. □

For each $w \in \mathscr{I}$ the sets of strong ascents, strong descents, weak ascents and weak descents of w relative to $\mathscr{I}$ and J are defined as in Section 5 above. Note that each $w \in W$ is uniquely expressible as dz with $d \in D_K$ and $z \in W_K$, and $w \in \mathscr{I}$ if and only if $z \in \mathscr{I}_0$. Moreover,

$$\mathscr{S} = \bigoplus_{d \in D_K} T_d \mathcal{H}_K \otimes_{\mathcal{H}_K} \mathscr{S}_0 = \bigoplus_{d \in D_K} T_d \otimes \mathscr{S}_0$$

and it follows that $\mathscr{S}$ is $\mathcal{A}$-free with $\mathcal{A}$-basis $(T_d \otimes b_z^0 \mid d \in D_K \text{ and } z \in \mathscr{I}_0)$. We define $b_w = T_d \otimes b_z^0$ whenever $w = dz$ as above, and proceed to show that for each $s \in S$ and $w \in \mathscr{I}$ the generator T_s of $\mathcal{H}$ acts on the basis element b_w in accordance with Definition 5.1.

Let $w = dz$, where $d \in D_K$ and $z \in \mathscr{I}_0$, and let $s \in \mathrm{SA}(w)$, so that $w < sw \in \mathscr{I}$. Suppose first that $sd \notin D_K$, so that $d < sd = dt$ for some $t \in K$, by Lemma 2.4. Then $tz \in W_K$, and since $d(tz) = sw \in D_K\mathscr{I}_0$, it follows that tz must be in $\mathscr{I}_0$. Moreover, since $l(w) < l(sw)$,

$$l(tz) = l\big(d(tz)\big) - l(d) = l(sw) - l(d) > l(w) - l(d) = l(dz) - l(d) = l(z),$$

and therefore $t \in \mathrm{SA}(K, z)$. By (9.1) above it follows that

$$T_s b_w = T_s T_d \otimes b_z^0 = T_d T_t \otimes b_z^0 = T_d \otimes T_t b_z^0 = T_d \otimes b_{tz}^0 = b_{dtz} = b_{sw}$$

in accordance with Definition 5.1. It remains to show that this same equation holds if $sd \in D_K$, and in this case we find that

$$b_{sw} = b_{(sd)z} = T_{sd} \otimes b_z^0 = T_s T_d \otimes b_z^0 = T_s b_w,$$

as required.

Suppose now that $s \in \mathrm{SD}(w)$, where $w = dz$ as above, so that $sw < w$. Suppose first that $sd \notin D_K$, so that $d < sd = dt$ for some $t \in K$, by Lemma 2.4. Then $tz \in W_K$, and since $l(w) < l(sw)$ it follows that

$$l(tz) = l\big(d(tz)\big) - l(d) = l(sw) - l(d) < l(w) - l(d) = l(dz) - l(d) = l(z),$$

whence $t \in \mathrm{SD}(K, z)$. By (9.1),

$$\begin{aligned} T_s b_w &= T_s T_d \otimes b_z^0 = T_d T_t \otimes b_z^0 = T_d \otimes T_t b_z^0 = T_d \otimes \big(b_{tz}^0 + \big(q - q^{-1}\big) b_z^0\big) \\ &= \big(T_d \otimes b_{tz}^0\big) + \big(q - q^{-1}\big)\big(T_d \otimes b_z^0\big) = b_{dtz} + \big(q - q^{-1}\big) b_{dz} = b_{sw} + \big(q - q^{-1}\big) b_w \end{aligned}$$

in accordance with Definition 5.1. It remains to show that this same equation holds if $sd \in D_K$. In this case $b_{sw} = b_{(sd)z} = T_{sd} \otimes b_z^0$, and we also find that $l(sd) = l((sd)z) - l(z) = l(sw) - l(z) < l(w) - l(z) = l(dz) - l(z) = l(d)$. So

$$\begin{aligned} T_s b_w &= T_s T_d \otimes b_z^0 = \big(T_{sd} + \big(q - q^{-1}\big) T_d\big) \otimes b_z^0 \\ &= \big(T_{sd} \otimes b_z^0\big) + \big(q - q^{-1}\big)\big(T_d \otimes b_z^0\big) = b_{sw} + \big(q - q^{-1}\big) b_w \end{aligned}$$

as required.

Next, suppose that $s \in \mathrm{WD}_J(w)$, where $w = dz$ as above, so that $sw \notin \mathscr{I}$ and $w^{-1}sw \in J$. Since $sw = (sd)z$ and $z \in \mathscr{I}_0$, the fact that $sw \notin \mathscr{I} = D_K\mathscr{I}_0$ means that $sd \notin D_K$, and so $sd = dt$ for some $t \in K$, by Lemma 2.4. Moreover, $z^{-1}tz = z^{-1}d^{-1}sdz = w^{-1}sw \in J$, so that $t \in \mathrm{WD}_J(K, z)$. By (9.1),

$$T_s b_w = T_s T_d \otimes b_z^0 = T_d T_t \otimes b_z^0 = T_d \otimes T_t b_z^0 = T_d \otimes \big(-q^{-1}\big) b_z^0 = -q^{-1} b_w$$

in accordance with Definition 5.1.

Finally, suppose that $s \in \mathrm{WA}_J(w)$, where $w = dz$ as above, so that $sw \notin \mathscr{I}$ and $w^{-1}sw \notin J$. As in the preceding case it follows that $sd \notin D_K$, and $sd = dt$ for some $t \in K$, but now $z^{-1}tz = w^{-1}sw \notin J$. So $t \in \mathrm{WA}_J(K, z)$, and by (9.1) it follows that $T_t b_z^0 = q b_z^0 - \sum_y r_{y,z}^t b_y^0$ for some polynomials $r_{y,z}^t \in q\mathcal{A}^+$ (defined whenever $y < tz$ and $y \in \mathscr{I}_0$). Hence

$$\begin{aligned} T_s b_w &= T_s T_d \otimes b_z^0 = T_d T_t \otimes b_z^0 = T_d \otimes T_t b_z^0 = T_d \otimes \left(q b_z^0 - \sum_y r_{y,z}^t b_y^0\right) \\ &= q\big(T_d \otimes b_z^0\big) - \sum_y r_{y,z}^t \big(T_d \otimes b_y^0\big) = q b_w - \sum_y r_{y,z}^t b_{dy} \end{aligned}$$

where the sums range over $y \in \mathscr{I}_0$ such that $y < tz$. Since $y \in \mathscr{I}_0$ and $y < tz$ imply that $dy \in D_K\mathscr{I}_0 = \mathscr{I}$ and $dy < dtz = sw$ (by Lemma 2.3 and an induction on $l(d)$), we conclude that in this case also the requirements of Definition 5.1 are satisfied.

To complete the proof that $\mathscr{I}$ is a W-graph ideal with respect to J it remains only to show that $\mathscr{S}$ admits a semilinear involution $\alpha \mapsto \overline{\alpha}$ such that $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}$. The proof is very similar to the corresponding proofs in Section 8 above: we set $\overline{T_d \otimes b_z^0} = \overline{T_d} \otimes \overline{b_z^0}$ for all $d \in D_K$ and $z \in \mathscr{I}_0$, using semilinearity to extend the definition to the whole of $\mathscr{S}$. We omit further details.

The discussion above enables us to state the following theorem.

Theorem 9.2. *Let $K \subseteq S$ and suppose that $\mathscr{I}_0 \subseteq W_K$ is a W_K-graph ideal with respect to $J \subseteq K$, and let $\mathscr{S}_0 = \mathscr{S}(\mathscr{I}_0, J)$ be the corresponding $\mathcal{H}_K$-module. Then $\mathscr{I} = D_K\mathscr{I}_0$ is a W-graph ideal with respect to J, the corresponding $\mathcal{H}$-module $\mathscr{S}(\mathscr{I}, J)$ being isomorphic to $\mathcal{H} \otimes_{\mathcal{H}_K} \mathscr{S}_0$.*

Remark. In the situation of Theorem 9.2, the assumption that $\mathscr{I}_0$ is a W_K-graph ideal in $(W_K, \leqslant_L)$ implies, by the construction in Section 7, that $\mathscr{S}_0$ is isomorphic to an $\mathcal{H}_K$-module arising from a W_K-graph. By [8, Theorem 5.1] it follows that the induced module $\mathscr{S}$ is isomorphic to a W-graph module. Theorem 9.2 yields an alternative construction of the induced W-graph in this special case that the W_K-graph in question comes from a W_K-graph ideal in $(W_K, \leqslant_L)$.

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