JOURNAL OF DIFFERENTIAL EQUATIONS 15, 269–277 (1974)

Oscillations of Functional Differential Equations with Retarded Argument

T. KUSANO

Department of Mathematics, Hiroshima University, Hiroshima, Japan

AND

H. ONOSE

Department of Mathematics, Ibaraki University, Mito, Japan

Received March 7, 1973

1. INTRODUCTION

Recently there has been an increasing interest in studying the oscillatory character of differential equations with retarded argument. Among numerous papers dealing with the subject we refer in particular to [5–11, 13–18] in which oscillation criteria for equations of arbitrary order have been established.

This paper is concerned with the oscillatory behavior of solutions of the delay differential equation

$$x^{(n)}(t) + p(t) f(x(g(t))) = q(t) \tag{$*$}$$

where the following conditions are always assumed to hold:

(a) $p, q \in C[[0, \infty), R]$, $p(t) \geqslant 0$;

(b) $g \in C^{(1)}[[0, \infty), R]$, $g(t) \leqslant t$, $\lim_{t\to\infty} g(t) = \infty$, $g'(t) \geqslant 0$;

(c) $f \in C^{(1)}[R, R]$, $\operatorname{sgn} f(y) = \operatorname{sgn} y$, $f'(y) \geqslant 0$.

It will be tacitly assumed that under the initial condition

$$x(t) = \phi(t), \quad t \leqslant t_0, \qquad \text{and} \qquad x^{(j)}(t_0) = x_j^0, \quad j = 1,\ldots, n-1,$$

Eq. $(*)$ has a solution which can be continued to $[t_0, \infty)$. A solution of $(*)$ is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

In this paper we shall prove two oscillation theorems for Eq. $(*)$. The first theorem deals with the case $q(t) \equiv 0$ and extends a result of Kamenev [1] on the oscillation of differential equations without delay. It unifies previous results of Ševelo and Vareh [13] and Kusano and Onose [6] for linear and nonlinear delay equations. The second theorem is an extension of results of Kartsatos [2, 3] on the forced oscillation of differential equations without delay. It asserts that the presence of a forcing term $q(t)$ which is either small or periodic does not affect the oscillatory character of the associated unforced delay equation.

2. Unforced Oscillation

Recently, Kamenev [1] has obtained an oscillation criterion for the ordinary differential equation $x^{(n)} + p(t)f(x) = 0$, which includes as special cases criteria of Kiguradze [4, Theorem 7] for the linear oscillation and of Ryder and Wend [12, Theorem 1] for the nonlinear oscillation.

The purpose of this section is to generalize Kamenev's theorem to the delay equation

$$x^{(n)}(t) + p(t)f(x(g(t))) = 0. \tag{A}$$

Theorem 1. *In addition to* (a)–(c) *assume that there exists a function* $\phi(y)$ *satisfying the following conditions*:

$$\phi \in C^{(1)}[(0, \infty), R], \qquad \phi(y) > 0, \qquad \phi'(y) \geqslant 0;$$

$$\int_{\epsilon}^{\infty} \frac{dy}{f(y)\,\phi(y^{1/(n-1)})} < \infty, \qquad \int_{-\epsilon}^{-\infty} \frac{dy}{f(y)\,\phi(-y^{1/(n-1)})} < \infty \text{ for any } \epsilon > 0; \tag{1}$$

$$\int^{\infty} \frac{[g(t)]^{n-1} p(t)}{\phi(g(t))}\, dt = \infty. \tag{2}$$

Then, if n is even, every solution of (A) *is oscillatory, while if n is odd, every solution is oscillatory or tends monotonically to zero as $t \to \infty$ together with its first $n - 1$ derivatives.*

Proof. Let $x(t)$ be a nonoscillatory solution of (A). We assume that $x(t) > 0$ for $t \geqslant t_0$. The case $x(t) < 0$ can be treated similarly. Because of condition (b) there is a $t_1 \geqslant t_0$ such that $x(g(t)) > 0$ for $t \geqslant t_1$. In view of Eq. (A) and condition (a) we obtain $x^{(n)}(t) \leqslant 0$ for $t \geqslant t_1$. Appealing to Kiguradze's lemma [4, Lemma 2], we conclude that there exists an integer

l, $0 \leqslant l \leqslant n-1$, which is odd if n is even and even if n is odd and such that for $t \geqslant t_1$

$$x^{(j)}(t) \geqslant 0, \quad j = 0, 1, \ldots, l, \quad (-1)^{l+j} x^{(j)}(t) \geqslant 0, \quad j = l+1, \ldots, n, \tag{3}$$

$$x^{(l)}(t) \leqslant j!(t-t_1)^{-j}\, x^{(l-j)}(t), \qquad j = 1, \ldots, l. \tag{4}$$

If one takes a $t_2 \geqslant t_1$ such that $g(t) \geqslant 2t_1$ for $t \geqslant t_2$, then it follows from (4) that

$$x^{(l)}(g(t)) \leqslant 2^{l-1}(l-1)!\,[g(t)]^{-l+1}\, x'(g(t)), \qquad t \geqslant t_2 . \tag{5}$$

Let n be even. We multiply Eq. (A) by $[g(t)]^{n-1}/\phi(g(t))f(x(g(t)))$ and integrate from t_2 to t. After manipulations including integration by parts we obtain

$$\begin{aligned}\int_{t_2}^{t} \frac{[g(s)]^{n-1}\, p(s)}{\phi(g(s))}\, ds = &- \frac{[g(s)]^{n-1}\, x^{(n-1)}(s)}{\phi(g(s))\, f(x(g(s)))}\Bigg|_{t_2}^{t} \\ &+ (n-1)\int_{t_2}^{t} \frac{[g(s)]^{n-2}\, g'(s)\, x^{(n-1)}(s)}{\phi(g(s))\, f(x(g(s)))}\, ds \\ &+ \int_{t_2}^{t} [g(s)]^{n-1}\, x^{(n-1)}(s)\, d\{[\phi(g(s))\, f(x(g(s)))]^{-1}\}.\end{aligned}$$

Since $x^{(n-1)}(t) \geqslant 0$ (by (3)) and $d\{[\phi(g(s))\, f(x(g(s)))]^{-1}\} \leqslant 0$ (by (b), (c), and (3)), we get from the above

$$\int_{t_2}^{t} \frac{[g(s)]^{n-1}\, p(s)}{\phi(g(s))}\, ds \leqslant C + (n-1)\int_{t_2}^{t} \frac{[g(s)]^{n-2}\, g'(s)\, x^{(n-1)}(s)}{\phi(g(s))\, f(x(g(s)))}\, ds,$$

where C is a constant, from which, using the nonincreasing character of $x^{(n-1)}(t)$, we obtain

$$\int_{t_2}^{t} \frac{[g(s)]^{n-1}\, p(s)}{\phi(g(s))}\, ds \leqslant C + (n-1)\int_{t_2}^{t} \frac{[g(s)]^{n-2}\, g'(s)\, x^{(n-1)}(g(s))}{\phi(g(s))\, f(x(g(s)))}\, ds. \tag{6}$$

Successive integration by parts of the last integral of (6) then leads to

$$\begin{aligned}&\int_{t_2}^{t} \frac{[g(s)]^{n-1}\, p(s)}{\phi(g(s))}\, ds \\ &\quad \leqslant C' + \frac{(n-1)\, w(t)}{\phi(g(t))\, f(x(g(t)))} \\ &\qquad - (n-1)\int_{t_2}^{t} w(s)\, d\{[\phi(g(s))\, f(x(g(s)))]^{-1}\} \\ &\qquad + (-1)^{n-l-1}\,(n-1)(n-2)\cdots l \int_{t_2}^{t} \frac{[g(s)]^{l-1}\, g'(s)\, x^{(l)}(g(s))}{\phi(g(s))\, f(x(g(s)))}\, ds,\end{aligned} \tag{7}$$

where C' is a constant and

$$w(t) = [g(t)]^{n-2} x^{(n-2)}(g(t)) - (n-2)[g(t)]^{n-3} x^{(n-3)}(g(t)) + \cdots + (-1)^{n-l}(n-2)(n-3) \cdots (l+1)[g(t)]^{l} x^{(l)}(g(t)).$$

Observing that $w(t) \leqslant 0$ by (3) and using (5), we find from (7)

$$\int_{t_2}^{t} \frac{[g(s)]^{n-1} p(s)}{\phi(g(s))} ds \leqslant C' + 2^{l-1}(n-1)! \int_{t_2}^{t} \frac{x'(g(s))\, g'(s)}{\phi(g(s))\, f(x(g(s)))} ds. \quad (8)$$

Since $x^{(n)}(t) \leqslant 0$ for $t \geqslant t_1$, by Taylor's theorem, there exists a constant $a \geqslant 1$ such that $x(t) \leqslant at^{n-1}$ for $t \geqslant t_1$, which yields

$$[x(g(t))/a]^{1/(n-1)} \leqslant g(t), \qquad t \geqslant t_2. \quad (9)$$

With the aid of (9) and the nondecreasing character of ϕ, f, g, x, we obtain

$$\begin{aligned} \int_{t_2}^{t} \frac{x'(g(s))\, g'(s)}{\phi(g(s))\, f(x(g(s)))} ds &\leqslant \int_{t_2}^{t} \frac{x'(g(s))\, g'(s)}{\phi([x(g(s))/a]^{1/(n-1)})\, f(x(g(s)))} ds \\ &= a \int_{x(g(t_2))/a}^{x(g(t))/a} \frac{dy}{f(ay)\, \phi(y^{1/(n-1)})} \\ &\leqslant a \int_{x(g(t_2))/a}^{\infty} \frac{dy}{f(y)\, \phi(y^{1/(n-1)})}, \end{aligned}$$

so that, because of (1), the second integral of (8) remains bounded above as $t \to \infty$. Hence, we conclude that

$$\int^{\infty} \frac{[g(t)]^{n-1} p(t)}{\phi(g(t))} dt < \infty, \quad (10)$$

which contradicts (2).

Now let n be odd. If the integer l in (3) associated with the nonoscillatory solution $x(t)$ under consideration is positive, then, arguing as before, we are led to the contradiction (10). If $l = 0$, then, (3) implies that $x'(t) \leqslant 0$ for $t \geqslant t_1$, so $x(t)$ decreases to a finite limit $c \geqslant 0$ as $t \to \infty$. We claim that $c = 0$. Otherwise, choose a $t_3 \geqslant t_1$ such that $f(x(g(t))) \geqslant f(c)/2 > 0$ for $t \geqslant t_3$; then from Eq. (A) we have

$$x^{(n)}(t) + (f(c)/2)\, p(t) \leqslant 0, \qquad t \geqslant t_3. \quad (11)$$

Multiplying both sides of (11) by t^{n-1} and integrating from t_3 to t give

$$t^{n-1}x^{(n-1)}(t) - (n-1)\, t^{n-2}x^{(n-2)}(t) + \cdots + (n-1)!\, x(t) + \frac{f(c)}{2} \int_{t_3}^{t} s^{n-1}p(s)\, ds = C'',$$

where C'' is a constant, from which, using (3) and letting $t \to \infty$ we obtain $\int^{\infty} t^{n-1}p(t)\,dt < \infty$. This inequality implies (10), because $\phi(t)$ is nondecreasing and $g(t) \leqslant t$. This contradiction shows that $c = 0$. Thus, if n is odd, a nonoscillatory solution $x(t)$ of (A) must approach zero as t tends to infinity. In this case, it is easily verified that the first $n - 1$ derivatives of $x(t)$ also tend monotonically to zero as $t \to \infty$.

The proof is therefore complete.

Remark. Let the retarded argument $g(t)$ satisfy instead of (b) the following condition:

(b*) $g \in C[[0, \infty), R]$ and there exists a function $g_* \in C^{(1)}[[0, \infty), R]$ with the property: $g_*(t) \leqslant g(t)$, $g_*'(t) \geqslant 0$, $\lim_{t\to\infty} g_*(t) = \infty$. Then, it can be shown as in [6] that the conclusion of Theorem 1 remains true if we replace the integral condition (2) by

$$\int^{\infty} \frac{[g_*(t)]^{n-1}\,p(t)}{\phi(g_*(t))}\,dt = \infty.$$

This remark also applies to Theorem 2 which will be proved in the next section.

Remark. When $f(y) \equiv y$, Theorem 1 improves a result of Ševelo and Vareh [13, Theorem 2], while if we take $\phi(y) \equiv 1$, Theorem 1 reduces to a result of the present authors [6, Theorem 2].

EXAMPLE 1. Consider the equation

$$x^{(4)}(t) + (15/16t^{15/4}\log(1 + t^{1/4}))\,x(g(t))\log[1 + |x(g(t))|] = 0. \qquad (12)$$

If $g(t)$ is such that $\liminf_{t\to\infty} g(t)/t = c > 0$, then conditions (1) and (2) are satisfied with $\phi(y) = [\log(1 + |y|)]^{\alpha}$, $0 < \alpha < 1$. Hence, by Theorem 1, every solution of (12) is oscillatory. On the other hand, if $g(t) \equiv t^{1/2}$, the hypotheses of Theorem 1 are not satisfied and Eq. (12) has a nonoscillatory solution $x(t) = t^{1/2}$. This shows that the oscillatory character of a delay equation is affected by the rate of growth for large t of its retarded argument.

3. Forced Oscillation

It is a question of mathematical and physical interest to know whether one can maintain the oscillatory property of Eq. (A) by adding a forcing term. In the case of ordinary differential equations, that is, when $g(t) \equiv t$, this question has recently been answered by Teufel [19] for second order equations and by Kartsatos [2, 3] for n-th order equations.

In this section we shall extend part of results of Kartsatos [2, 3] to the delay equations, presenting a theorem to the effect that the oscillation of Eq. (A) is maintained under the influence of a forcing term, provided it is periodic or sufficiently small.

THEOREM 2. *Consider the delay equation*

$$x^{(n)}(t) + p(t) f(x(g(t))) = q(t). \tag{B}$$

Assume that the hypotheses of Theorem 1 *are satisfied. Let there exist a function* $Q \in C^{(n)}[[0, \infty), R]$ *such that* $Q^{(n)}(t) = q(t)$ *for* $t \geq 0$ *and either*:

(I) $\lim_{t\to\infty} Q(t) = 0$; *or*

(II) *there exist constants* q_1, q_2 *and sequences* $\{t_m'\}$, $\{t_m''\}$ *with the following property*:

$$\lim_{m\to\infty} t_m' = \lim_{m\to\infty} t_m'' = \infty, \qquad Q(t_m') = q_1, \qquad Q(t_m'') = q_2,$$

$$q_1 \leq Q(t) \leq q_2 \qquad \textit{for} \quad t \geq 0.$$

Let (I) *hold. Then, every solution* $x(t)$ *of* (B) *is oscillatory or such that* $\lim_{t\to\infty} x(t) = 0$.

Let (II) *hold. Then, if n is even, every solution* $x(t)$ *of* (B) *is oscillatory, while if n is odd, every solution is oscillatory or such that* $\lim_{t\to\infty} [x(t) - Q(t)] = -q_1$ *or* $-q_2$.

Proof. Our proof is an adaptation of the arguments developed by Kartsatos in [2, 3].

Case (I). Assume the existence of a nonoscillatory solution $x(t)$ of (B). Let $x(t) > 0$ for $t \geq t_0$ and choose a $t_1 \geq t_0$ such that $x(g(t)) > 0$ for $t \geq t_1$. If we put $y(t) \equiv x(t) - Q(t)$, then $y(t)$ is a solution of

$$y^{(n)}(t) + p(t) f(y(g(t)) + Q(g(t))) = 0 \tag{13}$$

such that $y(g(t)) + Q(g(t)) > 0$ for $t \geq t_1$. From (13) we obtain

$$y^{(n)}(t) \leq 0 \qquad \text{for} \quad t \geq t_1. \tag{14}$$

Suppose that $x(t)$ is unbounded for all large t; then so is $y(t)$ and it follows from (14) that the first $n - 1$ derivatives of $y(t)$ are eventually of fixed sign. In particular, $y'(t) \geq 0$ for $t \geq t_1$ and $\lim_{t\to\infty} y(t) = \infty$. There exist a $t_2 \geq t_1$ and an $\epsilon > 0$ such that

$$y(g(t)) + Q(g(t)) > y(g(t)) - \epsilon > 0 \qquad \text{for} \quad t \geq t_2.$$

Put $z(t) \equiv y(t) - \epsilon$. Then, $\lim_{t\to\infty} z(t) = \infty$. On the other hand, $z(t)$ satisfies the delay equation

$$z^{(n)}(t) + p_1(t) f(z(g(t))) = 0$$

where

$$p_1(t) \equiv p(t)[f(y(g(t)) + Q(g(t)))]/f(y(g(t)) - \epsilon) \geqslant p(t), \qquad t \geqslant t_2 .$$

In view of (2) we obtain

$$\int^{\infty} \frac{[g(t)]^{n-1} p_1(t)}{\phi(g(t))} \, dt = \infty.$$

Therefore it follows from Theorem 1 that $z(t)$ is oscillatory or tends to zero as $t \to \infty$, contradicting the fact $\lim_{t\to\infty} z(t) = \infty$ obtained above. Consequently, there are no positive solutions of (B) which are unbounded for all large t.

Next suppose that $x(t)$ remains bounded as $t \to \infty$. Then $y(t)$ is also bounded and, with the use of (14), it can be verified that for $t \geqslant t_1$

$$(-1)^{n+j} y^{(j)}(t) \leqslant 0, \qquad j = 1,\ldots, n. \tag{15}$$

Let n be even. Then, (15) implies $y'(t) \geqslant 0$, $t \geqslant t_1$. If $y(t) > 0$ for all large t, then we have $\lim_{t\to\infty} y(t) > 0$, from which we can derive a contradiction by considering the function $z(t) \equiv y(t) - \epsilon$ and arguing as in the case $x(t)$ is unbounded. Hence we must have $y(t) \leqslant 0$, that is, $x(t) \leqslant Q(t)$ for all large t. Of course, this is possible only when $Q(t)$ is eventually positive, and in this case we have $\lim_{t\to\infty} x(t) = 0$. Let n be odd. Then, by (15), $y'(t) \leqslant 0$ for $t \geqslant t_1$, so that $y(t)$ decreases to a limit c as $t \to \infty$. Again it cannot happen that $c > 0$. Since $y(t) + Q(t)$ is eventually positive, the case $c < 0$ cannot occur. Thus, we conclude that $c = 0$, which implies $\lim_{t\to\infty} x(t) = 0$.

An analogous proof holds if we assume that $x(t)$ is negative for all large t, and this completes the proof of Case (I).

Case (II). Let $x(t)$ be a nonoscillatory solution of (B) such that $x(g(t)) > 0$ for $t \geqslant t_1$.

Assume that $x(t)$ is unbounded; then, as in the corresponding part of Case (I), it can be shown that $y(t) \equiv x(t) - Q(t)$ has the property: $y'(t) \geqslant 0$ for $t \geqslant t_1$, $\lim_{t\to\infty} y(t) = \infty$, and there is a $t_2 \geqslant t_1$ such that

$$y(g(t)) + Q(g(t)) \geqslant y(g(t)) + q_1 > 0 \tag{16}$$

for $t \geqslant t_2$. The function $w(t) \equiv y(t) + q_1$ is a solution of the delay equation

$$w^{(n)}(t) + p_2(t) f(w(g(t))) = 0 \tag{17}$$

where

$$p_2(t) \equiv p(t)[f(y(g(t)) + Q(g(t)))]/f(y(g(t)) + q_1) \geqslant p(t), \qquad t \geqslant t_2 .$$

Since

$$\int^{\infty} \frac{[g(t)]^{n-1}\, p_2(t)}{\phi(g(t))}\, dt = \infty,$$

according to Theorem 1, $w(t)$ has to be oscillatory or tend to zero as $t \to \infty$. This is a contradiction to $\lim_{t\to\infty} w(t) = \infty$.

Assume now that $x(t)$ is bounded. Let n be even; then $y(t) \equiv x(t) - Q(t)$ also satisfies $y'(t) \geqslant 0$ for $t \geqslant t_1$, and (16) holds for all large t. Thus, we arrive again at Eq. (17) which implies a contradiction to $\lim_{t\to\infty}[y(t) + q_1] > 0$. Let n be odd; then $y(t) \equiv x(t) - Q(t)$ satisfies $y'(t) \leqslant 0$ for $t \geqslant t_1$. Suppose that $y(\tau) + q_1 \leqslant 0$ for some $\tau \geqslant t_1$. Then, $y(t) + q_1 \leqslant 0$ for all $t \geqslant \tau$, which contradicts the eventual positivity of $y(t) + Q(t)$. Therefore, $y(t) + q_1 > 0$ for all $t \geqslant t_1$ and it follows from Theorem 1 applied to Eq. (17) that $\lim_{t\to\infty} [y(t) + q_1] = 0$, that is, $\lim_{t\to\infty}[x(t) - Q(t)] = -q_1$.

A parallel argument holds if we assume that $x(t)$ is a negative nonoscillatory solution of (B), and this completes the proof of Case (II).

Remark. If we take $g(t) \equiv t$ and $\phi(y) \equiv 1$, Case (I) of Theorem 2 becomes a result of Kartsatos [2, Theorem 3] and Case (II) reduces to a situation which is covered by a general theorem of Kartsatos [3, Theorem 1].

EXAMPLE 2. Consider the equation

$$\begin{aligned} x^{(4)}(t) + ((15 + 16t^{7/2}e^{-t})/16t^{15/4} \log(1 + t^{1/4}))\, x(g(t)) \log[1 + |\, x(g(t))|] \\ = e^{-t}. \end{aligned} \tag{18}$$

If $g(t)$ is such that $\liminf_{t\to\infty} g(t)/t = c > 0$, then conditions (1) and (2) are satisfied with $\phi(y) = [\log(1 + |\, y\,|)]^{\alpha}$, $0 < \alpha < 1$. Thus it follows from Theorem 2 (Case I) that every solution of (18) either oscillates or tends to zero as $t \to \infty$. On the other hand, if $g(t) \equiv t^{1/2}$, the associated unforced equation does not belong to a class of equations to which Theorem 1 applies, and Eq. (18) has an unbounded nonoscillatory solution $x(t) = t^{1/2}$.

ACKNOWLEDGMENT

The authors would like to thank the referee for his helpful suggestions.

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