MATHEMATICAL MODEL OF RIVER FLOW MANAGEMENT FOR THE CONTROL OF WATER QUALITY

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Abstract—A mathematical model of an industrial river is presented, taking into account man-controlled modifications of the river and adjustments of the flow rate to increase the dilution of pollution and to meet national or international water quality standards. The model is applied to the river Meuse. Different scenarios of river flow management are examined, providing the bases for a policy of construction and operation of barrages.

1. INTRODUCTION

The development of industrial activities (including navigation) along rivers has created severe problems of pollution and national or international regulations are now being implemented to control the water quality.

In many cases, the amplitude of the industrial activities is such that the new regulations cannot be satisfied, without dramatic economical problems, during certain periods of the year.

The most dramatic events occur, of course, at low water when the dilution of the pollution is not sufficient and this suggests holding part of the water—by an appropriate system of dams on tributaries—when there is plenty of it in the rainy season and to release it later in periods of low water to increase the river flow.

The policy of construction of the dams and the management of the reservoirs in relation with the objectives of water quality must take into account many geophysical, engineering, legal, economical, etc., constraints and can be best formulated with the help of a mathematical model.

The model is necessary to go beyond the simple and “static” concept of dilution—if only because increasing the flow rate may transport the pollution problems downstream—and to determine those criteria for the localization, the size and the operation of the barrages which result from the effect on the river water quality of the transit routes from the reservoirs to the river, the quantity and locations of the discharges into the river and the quality of the “transfused” water when reaching the river.

The model must take into account the modifications of the river from its “natural” configuration which the development of industrial activities and, in particular, navigation has brought about.

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For instance, in many rivers like the river Meuse which will be taken as a case study, the water level is, as much as possible, maintained constant in each of a series of reaches by successive weir-sluice complexes and the river cross-section may be assumed independent of time.

In this paper, a model of water quality in an industrial river is presented, taking into account man-controlled modifications of the river and adjustments of the flow rate according to different scenarios.

For simplicity, the model is restricted to thermal pollution (which, in most cases, is the most important factor in the determination of the river flow to satisfy water quality standards) because, although it affects all forms of pollution, it is relatively independent of them and can be treated individually, with a very good approximation, using a single partial differential dispersion equation.

The extension of the analysis to other quality criterions presents no conceptual difficulty.

2. MATHEMATICAL MODEL OF THERMAL POLLUTION

One can define thermal pollution as the difference between the actual temperature of the river and its natural temperature; a difference which is always positive for industrial rivers. River temperatures, in this context, must be understood as the mean temperatures over the river’s cross section and some appropriate period of time. Obviously an instantaneous measurement at a given point of the river is meaningless in terms of management, if only because of the natural turbulence of the water.

The natural temperature is defined as a function of time $t$ and distance $x$ along the river which results on the one hand from the heat exchanges between the river and the atmosphere (evaporation, conduction, radiation, etc.) and between the river and the ground and, on the other hand, from the cooling and heating of the river by the mixing of its water with that of the successive affluents. The natural temperature is calculated knowing the meteorological conditions over the river’s basin and neglecting all significant sources of heat pollution. It is described by a nonlinear partial differential equation which can be solved with good accuracy by standard numerical methods [1,2].

Thermal pollution can then be treated as a perturbation due to localized discharges of warm effluents.

If $T_n(t,x)$ and $\theta(t,x)$ denote respectively the natural temperature and the perturbation, the actual temperature of the river is given by

$$T(t,x) = T_n(t,x) + \theta(t,x). \quad (1)$$

The dispersion equation for $\theta$ is readily obtained from the cross-section averaged heat equation and, taking into account that $\theta$ ($\sim 5$ K) is small compared to the absolute natural river temperature ($\sim 300$ K), one can write, keeping only first order terms and neglecting longitudinal mixing as compared to longitudinal advection [3]*

$$\frac{\partial(A\theta)}{\partial t} + \frac{\partial(D\theta)}{\partial x} = -C\theta + B. \quad (2)$$

(i) $A$ denotes here the cross section of the river. If one excludes cases of very large flow rates—in flood conditions for instance—when the problem of pollution does not arise

* The paper given here in reference (Nihoul, 1979) was published before the proofs were returned and contains several misprints. The reader is advised to enquire into the corrigendum.
and cases of very low flow rates which one expects to be eliminated by operating the barrages, one may assume that the system of weirs and sluices maintains the water level reasonably constant in time. The river’s cross section $A$ may then be taken as independent of time, i.e.,

$$A = A(x).$$

(ii) $D$ is the flow rate. If the river is divided into successive sections $x_i \leq x \leq x_{i+1}$ each of which begins at a tributary, an industrial or urban affluent, a weir or a canal (pipe) diverting some of the river’s water to another basin, Eq. (3) implies that the flow rate $D$ is independent of $x$ in each section, i.e., $D$ varies only at section boundaries where tributaries, major outfalls, abductions and diversions are located.

One can thus write

$$D = D_0 + \sum_{i=1}^{N} d_i H(x - x_i),$$

where $D_0$ is the flow entering the first section and where the $d_i$’s are the inputs and outputs at the successive section boundaries $x = x_i (i = 1, 2, \ldots, N)$. $H$ denotes the Heaviside function.

One has

- $D_0 > 0$
- $d_i > 0$ at the mouth of a tributary or an urban outfall;
- $d_i = 0$ at a weir-sluice complex;
- $d_i < 0$ at the branching off of a canal or an abduction pipe;
- $d_i < 0$ at an industrial outfall because there is always some water lost or used up. (The water lost by evaporation downstream of a discharge of heated water is here regarded as a consumption and subtracted from the actual discharge at $x = x_i$.)

The flow rates $D_0$ and $d_i$ vary with time. However, in periods of low water, in which one is interested, even in natural conditions, they remain relatively constant (Fig. 1). In any case, these are the periods where water is released from the reservoirs and the flow rates of the river and its most important tributaries are then maintained constant. One may thus assume, with good approximation, that, for the periods under consideration, $D_0$ and the $d_i$’s are constant.

(iii) $C$ is the exchange coefficient with the atmosphere (in m$^2$s$^{-1}$). It can be written

$$C = \frac{EL}{\rho C_v},$$

where $E$ is the rate of heat exchange with the atmosphere per m$^2$ and per degree (W m$^{-2}$K$^{-1}$), $L$ is the width of the river, $\rho$ the specific mass of water (kg m$^{-3}$) and $C_v$ the specific heat at constant pressure (J kg$^{-1}$K$^{-1}$).

The exchange coefficient $C$ depends on the meteorological conditions and varies with time.

The annual mean of $\alpha(s^{-1})$ given by

$$\alpha = \frac{C}{A}$$

for the river Meuse, if of the order of $10^{-6}$ s$^{-1}$ with approximately 50% of the exchanges accounted for by evaporation, 30% by conduction-convection and 20% by radiation [2].
Fig. 1. Variation in 1978 (taken as a year of reference) of the flow rate of the Meuse River in natural conditions, at Ampsin-Neuville.

(iv) \( B \) is the rate of production (destruction) of thermal pollution by lateral inputs or outputs. These are located at the section boundaries and \( B \) can be written

\[
B = \sum_{i=1}^{N} b_i \delta(x - x_i),
\]

where \( \delta \) denotes the Dirac function.

The contribution of tributaries and canals to the natural temperature has been taken into account in the preliminary calculation of the latter so that the input of a tributary or a canal, at the point \( x = x_a \), is

\[
b_a = d_a [T_a - T_a(x_a)] - d_{na} [T_{na} - T_{n}(x_a)]
\]

where \( d_a \) and \( T_a \) are the actual flow rate and temperature of the affluent and \( d_{na} \) and \( T_{na} \) their respective values in natural conditions.

The contributions of an industrial or urban discharge, at \( x = x_e \), could be computed with a similar formula but it is more simply expressed as the total heat discharged per unit time \( R_e \) (expressed in watts) divided by the normalizing factor \( \rho C_p \), i.e.,

\[
b_e = \frac{R_e}{\rho C_p}.
\]

At a weir-sluice complex \( x = x_s \), \( b \) may be taken as zero but one may also argue that heat exchanges with the atmosphere are enhanced by the waterfall at the weir and that
the latter behaves like a localized sink of heat, in which case $b_s$ may be represented by a formula analogous to (9) with a negative discharge $R_s$

$$b_s = \frac{R_s}{\rho C_p}.$$  

\section*{3. EVOLUTION OF THERMAL POLLUTION IN A GIVEN SECTION}

Taking the remarks of the last section into account, one can write Eq. (2) in the form

$$A \frac{\partial \theta}{\partial t} + D \frac{\partial^2 \theta}{\partial x^2} = -C \theta + G,$$

where

$$G = \sum_{i=1}^{N} g_i \delta(x - x_i), \quad g_i = b_i - \theta_i d_i.$$  

At a weir $x = x_s$

$$g_s = b_s.$$  

At an industrial or urban outfall $d_e$ is small and

$$g_e \sim b_e.$$  

At a tributary, in natural conditions, $d_a = d_{na}, T_a = T_{na}, b_a = 0$ and $g_a$ reduces to

$$g_a = -\theta_a d_{na},$$

i.e., the effect of the tributary is simply the dilution of the river pollution by the inflowing water.

However when the tributary is used to supply fresh water to the main river, $d_a$ can be larger than $d_{na}$—actually to be determined according to optimum management criteria—and $T_a$ can be smaller than $T_{na}$ because water pumped near the bottom of a reservoir is usually cooler than natural temperature. In that case

$$g_a = d_a[T_a - T(x_a)] - d_{na}[T_{na} - T_{na}(x_a)].$$

At the branching off of a canal or an abduction pipe (negative tributary), $T_a = T(x_a), T_{na} = T_a(x_a)$ and

$$g_a = 0.$$  

It is convenient to solve Eq. 11 in each section separately. A given section, $n$, is defined by

$$x_n \leq x < x_{n+1}.$$  

It is characterized by

(i) a length

$$l_n = x_{n+1} - x_n,$$
(ii) a mean cross section

\[ A_n = \frac{1}{l_n} \int_{x_n}^{x_{n+1}} A(x) dx, \quad (21) \]

(iii) a flow rate

\[ D_n = D_0 + \sum_{i=1}^{n} d_i, \quad (22) \]

and

(iv) an exchange coefficient

\[ \alpha_n = \frac{E}{\rho C_p h_n}. \quad (23) \]

Since the sources and sinks are all located at the boundaries of the successive sections, the term \( G \) may be dropped from Eq. (11) and taken into account in the boundary conditions at \( x = x_i \) \((i = 1, \ldots, N)\). Thus if \( \theta_n \) denotes the thermal pollution in section \( n \) one shall write

\[ \theta_{n+1}(t, x_{n+1}) = \theta_n(t, x_n) + \sigma_{n+1}(t), \quad (24) \]

where \( \theta_n(t, x_{n+1}) \) denotes the value of \( \theta_n \) immediately upstream of the source (or sink) at \( x = x_{n+1} \) and where \( \sigma_{n+1} \) is the input (output) of thermal pollution at the source (sink), i.e.,

\[ \sigma_{n+1} = \frac{g_{n+1}}{D_{n+1}}. \quad (25) \]

Changing variables to

\[ \xi_n = \frac{D_n t}{A_n} \quad (26) \]

\[ \xi_n = \int_{x_n}^{x} \frac{A(x) dx}{A_n} \quad (27) \]

in section \( n \), one can write Eq. (11), in the form

\[ \frac{\partial \theta_n}{\partial \xi_n} + \frac{\partial \theta_n}{\partial \xi_n} = -\frac{\alpha_n A_n}{D_n} \theta_n \quad (28) \]

with

\[ \theta_n(0, \xi_n) = \theta_n^0(\xi_n) \quad \text{(initial condition)} \quad (29) \]

and

\[ \theta_n(\xi_n, 0) = \theta_{n-1}(\xi_n, l_{n-1}) + \sigma_n(\xi_n) \quad \text{(boundary condition)}. \quad (30) \]
The general solution of Eq. (28) can be written [4]

$$
\theta_n = f_n(\xi_n - \xi_n) e^{-\int_0^\xi \frac{\alpha_n A_n}{D_n} d\xi} ,
$$

where \( f_n \) is an arbitrary function of \( \xi_n - \xi_n \).

Using (29) and (30), one gets

$$
f_n(-\xi_n) = \theta_n(\xi_n) \xi_n = 0 ,
$$

i.e.,

$$
f_n(\xi_n - \xi_n) = \theta_n(\xi_n - \xi_n) \xi_n > \xi_n
$$

and

$$
f_n(\xi_n) e^{-\int_0^\xi \frac{\alpha_n A_n}{D_n} d\xi} = \theta_{n-1}(\xi_n, t_{n-1}) + \sigma_n(\xi_n) \xi_n = 0 ,
$$

i.e.,

$$
f_n(\xi_n - \xi_n) = \left\{ \theta_{n-1}(\xi_n - \xi_n, t_{n-1}) + \sigma_n(\xi_n - \xi_n) \right\} e^{-\int_{\xi_n - \xi_n}^\xi \frac{\alpha_n A_n}{D_n} d\xi} \xi_n < \xi_n .
$$

Now, one may take the origin of time such that the condition \( \xi_n < \xi_n \) is fulfilled in all sections, i.e.,

$$
t > \text{Sup} \frac{l_n A_n}{D_n}
$$

The influence of the initial conditions are then swept out of each section and the solution of Eq. (28) can be written simply

$$
\theta_n(\xi_n, \xi_n) = \left\{ \theta_{n-1}(\xi_n - \xi_n, t_{n-1}) + \sigma_n(\xi_n - \xi_n) \right\} e^{-\int_{\xi_n - \xi_n}^{\xi_n} \frac{\alpha_n A_n}{D_n} d\xi} .
$$

4. THE PEAKS OF THERMAL POLLUTION

It is readily seen from Eq. (37) that the largest pollution in a given section always occurs at the origin \( x = x_i \) of that section (\( \xi_i = 0 \)), i.e., even if, at a certain time, there is a maximum of pollution somewhere in the section, one can always find an earlier time when that peak was at the origin \( x_i \) with a larger amplitude. Among the points \( x_i \), only the points where discharges are located (\( i = e, \sigma_e > 0 \)) need to be considered, because, repeating the same argument, any peak at \( x = x_e \) or \( x = x_\sigma \) corresponds to a larger peak earlier at some discharge point upstream.

In a management program of a river with given regulation constraints on the maximum river temperature or the maximum rise of temperature due to any discharge, one can thus restrict attention to a discrete series of points \( x = x_e \) where

$$
\theta_e(x_e, t) = \theta_{e-1}(x_e, t) + \sigma_e(x_e, t)
$$
and where $\theta_{e-1}(x_r,t)$ is computed using the recurrence formula (37). In practice, the number of test points may be further reduced by a preliminary simulation identifying those discharges which create no pollution problem even in the worst flow conditions or those which are outside the range of a possible intervention (because, for instance, they are located upstream of all affluents on which it is technically feasible to build a dam) and which will have to comply with the regulations by a modification of their refrigeration processes.

The exchange coefficient $\alpha_n$ depends on the meteorological conditions and thus fluctuates in time. If one denotes by $\bar{\alpha}_n$ an appropriate “mean value” and by $\tilde{\alpha}_n$ the fluctuation around the mean, the exchange with the atmosphere can be written

$$e^{-\int_{\xi_n-\xi}^{\xi_n} \frac{\alpha_n A_n}{D_n} d\xi} = I_n^\alpha e^{-\frac{\bar{\alpha}_n A_n \xi_n}{D_n}},$$

(39)

where

$$I_n^\alpha = e^{-\int_{\xi_n-\xi}^{\xi_n} \frac{\tilde{\alpha}_n A_n}{D_n} d\xi},$$

(40)

is the “intermittency factor of autoepuration.”

Similarly, it is convenient to distinguish the variations with time of the functions $\sigma_i$ which are predictable (because, for instance, they are intentional and follow a known policy) and those which are unpredictable and result from small accidental fluctuations in the operation of a plant or the flow rate of a tributary.

Let

$$\sigma_n(t) = S_n \phi_n(t) I_n^\sigma,$$

(41)

where $S_n$ is an appropriate constant characterizing the magnitude of the input at $x = x_n$, $\phi_n(t)$ a known function of time, and $I_n^\sigma$ is the “intermittency factor of pollution.”

The intermittency factors cannot be foretold and cannot be included, as such, in a forecasting mathematical model.

One may argue however that they need not really be included because, on the one hand, they remain always fairly close to 1; on the other hand, the water quality regulations which will be discussed in more details in the next section, set limits on the river pollution which may not be exceeded more than some fraction of the time (say 90%) and which are not imperative at all time.

A forecasting model can then be formulated with all the intermittency factors taken equal to 1 and applied with the assumption that the legal constraints must be satisfied for all time.

The same mathematical model can be used simultaneously in hindcasting exercises to provide the basis of a statistical analysis assessing the probability of the regulations being violated some given fraction of the time and determining the best values of $\tilde{\alpha}_n$ and $S_n$ to use in forecasting.

The constant $\bar{\alpha}_n$ and $S_n$ being thus determined, one can write Eq. (37) in the simpler form

$$\theta_n(\xi_n, \xi_n) = \left\{ \theta_{n-1}(\xi_n - \xi, \xi_n - \xi) + S_n \phi_n(\xi_n - \xi_n) \right\} e^{-\frac{\bar{\alpha}_n A_n \xi_n}{D_n}}.$$

(42)
5. LEGISLATION AND ENGINEERING CONSTRAINTS ON MANAGEMENT

The system of state variables \( \theta_e(x_e, t) \) describes the thermal pollution of the river. The state variables are functions of time, of the discharge policy, of natural or man-made characteristics of the river and its affluents (widths and lengths of reaches) and of a series of control parameters such as the flow rates \( d_a \) and temperature \( T_a \) of the tributaries.

Optimal control of the river must be sought, subject to constraints on the state variables and on the control parameters.

**Water quality constraints**

As mentioned before, in most cases, national or international regulations set upper limits for both the maximum river temperature and for the maximum temperature excess at any heat discharge, i.e.,

\[
T_a(x_e, t) + \theta_e(x_e, t) \leq T_{\text{max}} \quad \forall t \quad \forall x_e
\]

\[
\sigma_e(x_e, t) \leq \Delta T_{\text{max}} \quad \forall t \quad \forall x_e.
\]

In some situations, there may be also a lower bound on the affluent temperature \( \theta_a + T_{na} \),

\[
T_{\text{min}} \geq T_{na} + \theta_a,
\]

to avoid a "thermal shock" on the ecological populations where cold dam water enters the river.

**Technical constraints**

There are geological, geographical, and hydrological restrictions on the construction of dams, on their filling capacity and on the maximum flow that can go through the valley of a given affluent.

The corresponding management constraints have the form

\[
d_a \leq d_{a,\text{max}} \quad a = a_1, a_2, \ldots
\]

\[
\sum_{a \in A} d_a dt \leq \sum_{a \in A} d_{na} dt \quad A = A_1, A_2, \ldots
\]

Eq. (46) contains the case of a tributary on which one cannot build a dam and which cannot be connected to existing dams so that its flow rate cannot exceed its natural value \( (d_{a,\text{max}} = d_{na}) \).

In Eq. (47) the sum bears on all affluents which can be interconnected to one or several dams and the integral is made over the period of time which one allows for refilling the dams (typically one year, in normal conditions).

Additional constraints arise from technical limits on the size of the dams and from the management policy of the dams and reservoirs. For instance if a reservoir is also used for recreational purposes, it can never be completely emptied.

In each case, a maximum "effective storage capacity" can be defined which sets a limit on some integral function of \( d_a - d_{na} \) and results in a constraint which may be more severe than Eq. (47).
6. EFFECT OF THE FLOW RATE ON RIVER POLLUTION

It is generally believed that increasing the flow rate, improves water quality in the river.

This argument, based on the concept of dilution of the pollution, must be re-examined in the light of the results of the mathematical model.

It is illuminating to consider a few simple cases.

One shall assume that the criterion (43) is the most severe and must be satisfied at all points of the river. The argument can easily be repeated when (44) applies or when they are both of application.

(i) The system described in Figure 2 consists of a single section \( x_1 \leq x < x_2 \) with constant heat inputs at \( x = x_1 \) and \( x = x_2 \). The river upstream of \( x_1 \) is not polluted and the temperature there is the natural temperature.

From Eqs. (42) and (25), one gets

\[
\theta_o(x_1) = 0 \\
\theta_1(x_1) = S_1 = \frac{g_1}{D_o} \\
\theta_1(x_2) = \frac{g_1}{D_o} e^{-\frac{-\alpha_t A_1 l_1}{D_o}} \\
\theta_2(x_2) = \frac{g_1}{D_o} e^{-\frac{-\alpha_t A_1 l_1}{D_o}} + \frac{g_2}{D_o} .
\]

\( \theta_1(x_2) \), the first term in the right-hand side of Eq. (51), represents the "incident pollution" at \( x = x_2 \).

It is readily seen that, for constant discharges, the incident pollution at \( x = x_2 \) is small for small and large values of \( D_o \) and has a maximum for

\[
D_o = \frac{\alpha_t A_1 l_1}{\bar{\alpha}_t}.
\]

For larger flow rates, the dilution at the point of discharge decreases the pollution load downstream. For lower flow rates, the transit time of the water masses between the point of discharge and the point of observation increases, the auto-epuration is more efficient and the pollution load again decreases at the point of observation.

Thus, increasing the flow rate is not necessarily the universal remedy for pollution as
it decreases the contribution of any individual discharge but may increase the effect at the location of that discharge of other discharges situated upstream.

Eq. (43) requires

\[
\frac{g_1}{D_0} \leq \theta_{\max}
\]

(53)

\[
\frac{g_1}{D_0} e^{-\frac{\bar{\alpha}_1 A_1 l_1}{D_0}} + \frac{g_2}{D_0} \leq \theta_{\max},
\]

(54)

where \(\theta_{\max}\) stands, in short, for \(T_{\max} - T_n\).

The criterion (43) must be satisfied at \(x = x_1\) and \(x = x_2\). At low water, this may create severe economical difficulties and one thinks naturally to increase the river flow \(D_o\) to ensure a better dilution. In that case, of course, \(g_1\) and \(g_2\) increase with the river capacity of evacuating the heat.

It is interesting to see how much the heat inputs \(g_1\) and \(g_2\) may be augmented by increasing \(D_o\).

Assuming that each source will discharge as much heat as possible, one finds

\[
g_1 = \theta_{\max} D_o
\]

(55)

\[
g_2 = \theta_{\max} \left( 1 - e^{-\frac{\bar{\alpha}_1 A_1 l_1}{D_0}} \right) D_o.
\]

(56)

Hence, while \(g_1\) may increase without bounds, \(g_2\) is limited and it is readily seen that

\[
g_2 \sim \theta_{\max} \bar{\alpha}_1 A_1 l_1 \quad \text{for} \quad D_o \gg \bar{\alpha}_1 A_1 l_1.
\]

(57)

Increasing \(D_o\) turns thus essentially to the advantage of the first source of pollution.

Taking typical values

\[
\bar{\alpha}_1 \sim 10^{-6} \text{ s}^{-1}, \quad A_1 \sim 10^3 \text{ m}^2, \quad l_1 \sim 10^4 \text{ m},
\]
one finds

$$\frac{g_2}{\alpha_1 A_1 \theta_{\text{Max}}} \sim 0.85 \text{ for } D_o \sim 30 \text{ m}^3\text{s}^{-1}$$

$$\sim 0.92 \text{ for } D_o \sim 60 \text{ m}^3\text{s}^{-1},$$

i.e., doubling the flow rate brings only 7% improvement at the second source.

This situation is the consequence of the hypothesis that the flow rate remains the same throughout the section (when increased flow rates are considered, the water is assumed to be released from the reservoirs upstream of $x_1$). The second example which follows will show a different, more equitable and economical organization.

(ii) The system described in Figure 5 consists of a single section of $x_1 \leq x < x_2$ with constant heat inputs at $x = x_1$ and $x = x_2$. As before, the river temperature upstream of $x_1$ is assumed to be the natural temperature and one examines the effect of increasing the flow rate by releasing fresh water at a temperature equal to the natural temperature of the river.

However, it is assumed that the water can be brought to the river from the reservoirs by two different pipe-lines, one ending upstream of $x_1$ and the other one upstream of $x_2$. (In natural conditions, the flow is assumed to be zero in the pipelines.)
From Eq. (42) and (17), one gets
\[ \theta_0(x_1) = 0 \]  
\[ \theta_1(x_1) = \frac{g_1}{D_o + \Delta_1 + \beta \Delta_2} \]  
\[ \theta_1(x_2) = \frac{g_1}{D_o + \Delta_1 + \beta \Delta_2} e^{-\frac{\alpha_1 A_1 l_1}{D_o + \Delta_1 + \beta \Delta_2}} \]  
\[ \theta_2(x_2) = \theta_1(x_2) - \frac{\gamma \Delta_2 \theta_1(x_2)}{D_o + \Delta_1 + \Delta_2} + \frac{g_2}{D_o + \Delta_1 + \Delta_2} \]
\[ = \frac{g_1}{D_o + \Delta_1 + \Delta_2} e^{-\frac{\alpha_1 A_1 l_1}{D_o + \Delta_1 + \beta \Delta_2} + \frac{g_2}{D_o + \Delta_1 + \Delta_2}}. \]  

The incident pollution at \( x = x_2 \) is now
\[ \frac{g_1}{D_o + \Delta_1 + \Delta_2} e^{-\frac{\alpha_1 A_1 l_1}{D_o + \Delta_1 + \beta \Delta_2} + \frac{g_2}{D_o + \Delta_1 + \Delta_2}} \]
if \( \beta < 1 \).

The term in the right-hand side of the inequality (62) represents the incident pollution one would have obtained by releasing the total flow \( D_o + \Delta_1 + \Delta_2 \) upstream of \( x_1 \). As mentioned before, a better repartition of the water supplies to the river decreases the effect of one source on the other. One can also show that the total amount of water to be supplied is the smallest the larger the supply at \( x = x_2 \) compatible with the respect of condition (43) at \( x_1 \) and \( x_2 \).

If \( g_1 \) and \( g_2 \) are fixed at their maximum value, condition (43) requires
\[ \frac{g_1}{D_o + \Delta_1 + \beta \Delta_2} \leq \theta_{\text{Max}} \]  
(the equality corresponds to \( \beta = 0 \))
\[ \frac{g_1}{D_o + \Delta_1 + \Delta_2} e^{-\frac{\alpha_1 A_1 l_1}{D_o + \Delta_1 + \beta \Delta_2} + \frac{g_2}{D_o + \Delta_1 + \Delta_2}} = \theta_{\text{Max}}. \]  

Introducing nondimensional variables and constants
\[ y = \frac{D_o + \Delta_1 + \Delta_2}{\alpha_1 A_1 l_1} \]
\[ x = \frac{\gamma \Delta_2}{\alpha_1 A_1 l_1} \]
\[ r_1 = \frac{g_1}{\alpha_1 A_1 l_1 \theta_{\text{Max}}} \]
\[ r_2 = \frac{g_2}{\alpha_1 A_1 l_1 \theta_{\text{Max}}}. \]
one can write Eq. (64) in the form
\[ y = r_2 + r_1 e^{\frac{1}{y - x}}. \]  

(69)

Differentiating, one finds
\[ \frac{dy}{dx} = -\frac{r_1 e^{\frac{1}{y - x}}}{\left(1 - \frac{r_1 e^{\frac{1}{y - x}}}{(y - x)^2}\right)(y - x)^2}. \]  

(70)

Now the function
\[ e^{\frac{1}{y - x}} \]

is always less than 1 (its maximum value is $e^{-1}$). Condition (63) gives then
\[ \frac{r_1}{y - x} \leq 1, \]

(71)

\[ \frac{r_1 e^{\frac{1}{y - x}}}{(y - x)^2} < 1, \]

(72)

and
\[ \frac{dy}{dx} < 0. \]  

(73)

Hence the larger $x$ (the larger $\gamma$), the smaller $y$. The most economical situation (the smallest total supply) corresponds to $\gamma = 1$.

When one can do otherwise, one should thus avoid to discharge the water supply required by different sources of pollution all in one block upstream of the first source. Of course, other factors must be taken into account like the possibility and cost of connecting the dams to the river, but the point deserves consideration.

Furthermore, it has been assumed above that the first source requests a certain flow to evacuate a given heat power and then sticks to the agreed policy even if more water becomes available by the requirements of another source, downstream. If this is not the case, maintaining an unnecessary high flow rate at a given source of pollution may induce trespassing against authorizations and create an even more severe problem downstream.

This is particularly important when the first source is across the border and only tied down to the international standards, at the border.
7. APPLICATION TO THE RIVER MEUSE

The mathematical model has been applied to the Meuse River. The river is shown in Figure 6. The main tributaries are indicated by wavy arrows, the main inputs and outputs by straight arrows. Three tributaries (Houille, Hoyoux and Ourthe) have been underlined to indicate possible candidates for carrying the water from the reservoirs to the river. The double arrow at Namur, the mouth of the Sambre river, emphasizes that this heavily industrialized river (and the urban discharges of the town of Namur) constitute more a source of pollution than a supply of fresh water. The total input at Namur is computed by a separate submodel of the river Sambre.

About 1 m³s⁻¹ of river water is pumped at Tailfer to provide drinking water. The (positive) contribution of urban discharges along the river is approximately of the same order but mainly located at the major town of Liège where industrial activities consume about 1.5 m³s⁻¹ of water.

The most important diversion is the Albert Canal with a flow of about 22 m³s⁻¹. About 3 m³s⁻¹ are returned from the canal to the river as a result of sluice operations and pumping by the iron and steel industry at Chertal.

Table 1. Summarizes the most important heat discharges

<table>
<thead>
<tr>
<th>Place</th>
<th>Nature</th>
<th>R (watts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chooz</td>
<td>Power-station</td>
<td>6.4 \times 10^8</td>
</tr>
<tr>
<td>Tihange</td>
<td>Power-station</td>
<td>3.3 \times 10^8</td>
</tr>
<tr>
<td>Awirs</td>
<td>Power-station</td>
<td>5.1 \times 10^7</td>
</tr>
<tr>
<td>Seraing</td>
<td>Steel and iron industry</td>
<td>3 \times 10^6</td>
</tr>
<tr>
<td>Fragnee</td>
<td>Power-station</td>
<td>1.1 \times 10^6</td>
</tr>
<tr>
<td>Bressoux</td>
<td>Power-station</td>
<td>1.4 \times 10^6</td>
</tr>
<tr>
<td>Monsin</td>
<td>Power-station</td>
<td>0.4 \times 10^6</td>
</tr>
<tr>
<td>Chertal</td>
<td>Steel and iron industry</td>
<td>8 \times 10^5</td>
</tr>
</tbody>
</table>
The flow rate at Ampsin Neuville is used as a reference and the model is applied to a period when, in natural condition, the flow rate at Ampsin Neuville varies from 20 to 100 m$^3$s$^{-1}$. The effect of maintaining the flow rate at the levels of 30, 40, 50, and 60 m$^3$s$^{-1}$, by supplying fresh water from the reservoirs is studied according to three different scenari:

(i) Total supply of water in the region of the Upper Meuse (region of Chooz);
(ii) Total supply of water at Huy, i.e., directly upstream of Tihange;
(iii) The same supply of water at Huy with an additional supply by the river Ourthe to maintain the ratio of the Ourthe's and Meuse's flow rates—and thus the dilution effect—the same as in natural conditions.

For each scenario, three cases are considered for which the water is supplied at a temperature respectively equal, two degrees lower and four degrees lower than the natural temperature of the Meuse. It is assumed however, that the Ourthe's temperature remains always the same, i.e., two degrees lower than the Meuse’s natural temperature.

The model is applied to the determination of the peaks of pollution $\theta_e(x_e,t)$ and, in particular, the maximum thermal pollution (which may occur at different times and places for different scenari) and the thermal pollution at the Dutch border.

The complete results are given in Smitz et al. [5]. They are summarized in Figs. 7 to 12.

The data at the Dutch border confirm the conclusions of the simple analysis of the previous section: Increasing the flow rate decreases the efficiency of autoepuration, and if the supply of water is not large enough to compensate by the effect of dilution at the pollution peaks, the thermal pollution actually increases at the Dutch border with increasing river flow.

Fig. 7. Maximum thermal pollution in the river, $\theta_{\text{max}}$, for scenario 1 and three different temperatures of the water supplied from the reservoirs.

- $\theta_{\text{max}}$ in natural flow conditions ($D = D_n$)
- $\theta_{\text{max}}$ with a supply $d$ at $T = T_n - 2$
- $\theta_{\text{max}}$ with a supply $d$ at $T = T_n - 4$

$D_n$ is the natural flow rate at Ampsin Neuville. The upper curves (here indistinguishable) correspond to a water supply adjusted to maintain the flow rate at 30 m$^3$s$^{-1}$ (upper scale). The lower curves correspond to a water supply adjusted to maintain the flow rate at 60 m$^3$s$^{-1}$ (lower scale).
Fig. 8. Thermal pollution at the Dutch border, $\theta_n$, for scenario 1 and three different temperatures of the water supplied from the reservoirs.

- - - - - - $\theta_n$ in natural flow conditions ($D = D_n$)
- - - - - - $\theta_n$ with a supply $d$ at $T = T_n$
- - - - - - $\theta_n$ with a supply $d$ at $T = T_n - 2$
- - - - - - $\theta_n$ with a supply $d$ at $T = T_n - 4$

$D_n$ is the natural flow rate at Ampsin Neuville. The upper curves correspond to a water supply adjusted to maintain the flow rate at 60 m$^3$s$^{-1}$ (upper scale). The lower curves (here indistinguishable) correspond to a water supply adjusted to maintain the flow rate at 30 m$^3$s$^{-1}$ (lower scale).

Fig. 9. Maximum thermal pollution in the river, $\theta_{\text{Max}}$, for scenario 2 and three different temperatures of the water supplied from the reservoirs.

- - - - - - $\theta_{\text{Max}}$ in natural flow conditions ($D = D_n$)
- - - - - - $\theta_{\text{Max}}$ with a supply $d$ at $T = T_n$
- - - - - - $\theta_{\text{Max}}$ with a supply $d$ at $T = T_n - 2$
- - - - - - $\theta_{\text{Max}}$ with a supply $d$ at $T = T_n - 4$

$D_n$ is the natural flow rate at Ampsin Neuville. The upper curves correspond to a water supply adjusted to maintain the flow rate at 60 m$^3$s$^{-1}$ (upper scale). The lower curves correspond to a water supply adjusted to maintain the flow rate at 30 m$^3$s$^{-1}$ (lower scale).
Fig. 10. Thermal pollution at the Dutch border, $\theta_n$, for scenario 2 and three different temperatures of the water supplied from the reservoirs.

- - - - - - $\theta_n$ in natural flow conditions ($D = D_n$)
- - - - - - $\theta_n$ with a supply $d$ at $T = T_n$
- - - - - - $\theta_n$ with a supply $d$ at $T = T_n - 2$

$D_n$ is the natural flow rate at Ampsin-Neuville. The water supply is adjusted to maintain the flow rate at 60 m$^3$s$^{-1}$. (The curves corresponding to an adjustment at 30 m$^3$s$^{-1}$ are almost the same as for scenario 1 and are not reproduced here.)

Fig. 11. Maximum thermal pollution in the river, $\theta_{\text{Max}}$, for scenario 3 and three different temperatures of the water supplied from the reservoirs.

- - - - - - $\theta_{\text{Max}}$ in natural flow conditions ($D = D_n$)
- - - - - - $\theta_{\text{Max}}$ with a supply $d$ at $T = T_n$ and an additional supply $d'$ by the river Ourthe at $T = T_n - 2$
- - - - - - $\theta_{\text{Max}}$ with a supply $d$ at $T = T_n - 2$ and an additional supply $d'$ by the river Ourthe at $T = T_n - 2$

$D_n$ is the natural flow rate at Ampsin-Neuville. The upper curves correspond to a water supply adjusted to maintain the flow rate at 30 m$^3$s$^{-1}$ (upper scale). The lower curves correspond to a water supply adjusted to maintain the flow rate at 60 m$^3$s$^{-1}$ (lower scale).
Fig. 12. Thermal pollution at the Dutch border, $\theta_d$, for scenario 3 and three different temperatures of the water supplied from the reservoirs

- - - - - $\theta_d$ in natural flow conditions ($D = D_n$)
- - - - - $\theta_d$ with a supply $d$ at $T = T_n$ and an additional supply $d'$ by the river Ourthe at $T = T_n - 2$
- - - - - $\theta_d$ with a supply $d$ at $T = T_n - 2$ and an additional supply $d'$ by the river Ourthe at $T = T_n - 2$

$D_n$ is the natural flow rate at Ampsin Neuville. The water supply is adjusted to maintain the flow rate at 60 m$^3$s$^{-1}$. (The curves corresponding to an adjustment at 30 m$^3$s$^{-1}$ are almost the same as for scenario 1 and are not reproduced here.)

The results of the model are presently being exploited to determine the management policy of the River Meuse: number, locations, sizes of the dams, connections to the river, release operations program, etc.

REFERENCES