FERRERS DIGRAPHS AND THRESHOLD GRAPHS

Olivier COGIS

C.R.I.G., Université des Sciences et Techniques du Languedoc, 34075 Montpellier Cedex, France

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We deduce a set of known characterizations of threshold graphs (Theorem 3) from a set of characterizations of Ferrers digraphs (Theorem 1) by investigating the connection between symmetric Ferrers digraphs and threshold graphs. A direct proof of Theorem 3 is easier than the one provided in here, but the purpose of this paper is to view Theorem 1 as an extension of Theorem 3 to the directed case (this extension point of view still holds on an algorithmic ground).

1. Introduction and definitions

A digraph $G$ is a couple $G = (X, U)$ such that $U \subseteq X^2$; $X$ (resp. $U$) is the set of vertices (resp. arcs) of $G$. A simple graph $G$ is a couple $G = (X, E)$ such that $E \subseteq \mathcal{P}_2(X)$, the set of subsets of $X$ of cardinality 2; $E$ is the set of edges of $G$ (thus loops and multiple edges are not allowed; from now on we say graph for simple graph).

All throughout this paper we deal with finite graphs and digraphs (that is $X$ is finite). $U$ will always denote a set of arcs of some digraph, while $E$ will always denote the set of edges of some graph. We simply note $xy \in U$ (resp. $xy \in E$) for $(x, y) \in U$ (resp. $\{x, y\} \in E$).

If $G$ is a graph, $S \subseteq X$ is an independent (resp. complete) set for $G$ when the induced subgraph $G(S)$ is edgeless (resp. a clique, that is it contains all possible edges on $S$).

We denote the complementary of $G$ (graph or digraph) by $\bar{G}$.

If $G$ is a digraph, $G^*$ denotes its dual (reverse every arc). If $G'$ is another digraph with the same set of vertices as $G$, $GG'$ is the digraph such that $xz$ is an arc of $GG'$ iff there exists a vertex $y$ such that $xy$ and $yz$ are arcs of $G$ and $G'$ respectively.

Threshold graphs were introduced by Chvátal and Hammer in [2]. A threshold graph is a graph $G = (X, E)$ for which there exists a real mapping $v : X \rightarrow \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that, for every $S \subseteq X$, $S$ is an independent set for $G$ iff $v(S) = \sum_{x \in S} v(x) \leq t$. The graph of Fig. 3 is a threshold graph: for $x = 1, 2, \ldots, 7$, let $v(x) = 1, 2, 2, 6, 6, 10, 10$ and $t = 11$.

Ferrers digraphs were defined by Riguet in [8], though from an algebraic point of view rather than from a graphic one: a Ferrers digraph is a digraph $G = (X, U)$.
such that $G\bar{G}^\rightarrow$ is a partial digraph of $G$ (that is, if $V$ is the set of arcs of $G\bar{G}^\rightarrow$, then $V \subset L$). The digraph of Fig. 1 is a Ferrers digraph (the definition not being easy to handle by itself, it is more convenient to use Theorem 1, (F8) for instance).

Many characterizations of threshold graphs are known [2, 4, 5], and the same goes for Ferrers digraphs [1, 3, 6, 7, 8]. A certain similarity (see (T8) of Theorem 2 and (F8) of Theorem 1 for instance) led us to investigate the question as could Ferrers digraph characterizations be viewed as generalisations (in the directed case) of threshold graph characterizations.

The purpose of this paper is to provide a positive answer to this question through the usual connection: a graph is but a symmetric digraph (that is if $xy$ is an arc of the digraph, then $yx$ is too) without loops.

Theorem 1 states some characterizations of Ferrers digraphs (some of which are new and were directly inspired from known results about threshold graphs). Theorem 2 investigates the connection with graphs in the special case of symmetric digraphs. Theorem 3 states known characterizations of threshold graphs (it is viewed as an obvious corollary of Theorem 1 and Theorem 2; still, it is much easier to give a direct proof of Theorem 3). Theorem 4 gives more precision about the connection between symmetric Ferrers digraphs and threshold graphs.

We now state some more definitions and notations.

If $A$ is a set, $|A|$ is the cardinality of $A$. A digraph $G = (X, U)$ (resp. a graph $G = G(X, E)$) is of order $n$ when $|X| = n$.

Given any digraph $G$, $x$ is a predecessor of $y$ and $y$ is a successor of $x$ when $xy \in U$; the set of successors (resp. predecessors) of $x$ is denoted by $\Gamma^+(x)$ (resp. $\Gamma^-(x)$). The outdegree (resp. indegree) of $x$ is denoted by $d^+(x)$ (resp. $d^-(x)$); that is, $d^+(x) = |\Gamma^+(x)|$ and $d^-(x) = |\Gamma^-(x)|$.

We denote by $d_0^+ < d_1^+ < \cdots < d_k^+$ (resp. $d_0^- < d_1^- < \cdots < d_k^-$) the sequence of outdegrees (resp. indegrees) of $G$ with the further assumption that $d_0^+ = 0$ (resp. $d_0^- = 0$) even if no vertex of $G$ has no successor (resp. predecessor). We denote by $D_i^+$, where $0 \leq i \leq k^+$ (resp. $D_i^-$, where $0 \leq i \leq k^-$) the set of vertices of $G$ whose
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outdegree (resp. indegree) is $d_i^+$ (resp. $d_i^-$), and we denote by $\delta^+(x)$ (resp. $\delta^-(x)$) the index $i$ such that $x \in D_i^+$ (resp. $x \in D_i^-$). Thus $D_0^+$ and $D_0^-$ are possibly empty but not $D_i^+$ for $1 \leq i \leq k^*$ nor $D_i^-$ for $1 \leq i \leq k^-$. An example is given in Fig. 2.

Given a graph $G$, we define neighbour. $I'$, $d$, $k$, $0 = d_0 < d_1 < \cdots < d_k$, $D_i$ for $0 \leq i \leq k$, and $\delta(x)$ in a similar way. An example is given in Fig. 3.

We say that a graph $G$ induces a 4-alternated-cycle $c$ when $c$ is a cycle of length 4 of the clique $G \cup \bar{G}$ such that its edges belong alternately, along the cycle, to $G$ and to $\bar{G}$ (thus $c$ is an elementary cycle of $G \cup \bar{G}$).

The directed corresponding notion is a little less simple. We say that a digraph $G$ induces a 4-alternated-anticircuit $c$ when $c$ is an anticircuit of length 4 of $G \cup \bar{G}$ such that its arcs belong alternately, along the anticircuit, to $G$ and to $\bar{G}$ (thus $c$ is an anticircuit of length 4 of the complete symmetric digraph $G \cup \bar{G}$, by anticircuit, we mean an even circuit from which every second arc, along the circuit, has been reversed).

Loops being allowed, a 4-alternated-anticircuit can be supported by 2, 3 or 4 vertices. All 4-alternated-anticircuits are displayed in Fig. 4 where plain arcs belong to $G$ while dotted arcs belong to $\bar{G}$, or conversely.

Fig. 2. $d_0 = 0$, $d_1 = 1$, $d_2 = 2$, $d_3 = 5$ and $k^* = 3$. $D_0^+ = \{3\}$, $D_1^+ = \{5\}$, $D_2^+ = \{1, 2\}$, $D_3^+ = \{4\}$. $d_0 = 0$, $d_1 = 2$ and $k^- = 1$. $D_0^- = \{0\}$. $D_1^- = \{1, 2, 3, 4, 5\}$. $\delta^+(1) = 2$, $\delta^-(1) = 1$, $\delta^+(3) = 1$, $\delta^-(5) = 1$.

Fig. 3. $d_0 = 0$. $d_1 = 2$, $d_2 = 3$, $d_3 = 5$ and $k = 3$. $D_0 = \{1\}$, $D_1 = \{2, 3\}$, $D_2 = \{4, 5\}$, $D_3 = \{6, 7\}$. $\delta(1) = 0$, $\delta(2) = 1$, $\ldots$, $\delta(7) = 3$. 
For $i = 2, 3, 4$, $\mathcal{G}_i$ is defined as the isomorphism class of digraphs of order $i$ that induce a 4-alternated-anticircuit of type $i$ (that is, which is supported by $i$ different vertices), and we let $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$; for $i = 2, 3$, a digraph $G \in \mathcal{G}_i$ is in $\mathcal{G}_i'$ (resp. $\mathcal{G}_i''$) when it induces a 4-alternated-anticircuit with its loop(s) belonging to $G$ (resp. to $G'$).

We define $\mathcal{F}$ as the class of graphs constructible, starting from the empty graph $(\emptyset, \emptyset)$, by applying repeatedly the following rules:

1. Add an isolated vertex $z$ to $G = (X, E) \in \mathcal{F}$ (that is, $G' = (X \cup \{z\}, E)$)
2. Add a dominating vertex $z$ to $G = (X, E) \in \mathcal{F}$ (that is, $G' = (X \cup \{z\}, E \cup \{zx \mid x \in X\}$).

The directed corresponding notion is a little more complicated to state. For example, instead of an isolated vertex, to $G = (X, U)$ we add a vertex $z$ with no loop, we add no arc from $X$ to $z$, and we add every arc from $z$ to $A \subseteq X$ where $A$ is chosen in such a way that: $A = \Gamma^+(x)$ for some $x \in X$, or there exist some $x, y \in X$ such that $\{x\} \subseteq A \subseteq \Gamma^+(y)$ and there is no $t \in X$ such that $\Gamma^+(x) \subseteq \Gamma^+(t) \subseteq \Gamma^+(y)$, or the same holds when replacing $\Gamma^+(x)$ (resp. $\Gamma^+(y)$) by $\emptyset$ (resp. $X$).

In other words, $\mathcal{F}$ is the class of digraphs constructible, starting from the empty digraph $(\emptyset, \emptyset)$, by applying the following rules:

1. Add a vertex $z$ to $G = (X, U) \in \mathcal{F}$, no loop on $z$, no arc from $X$ to $z$, every arc from $z$ to $A \subseteq X$ where $A$ is chosen so that $A' \subseteq A \subseteq A''$, where $A' = \Gamma^+(x) \cap A \cap \Gamma^+(y)$ and, for every $x \in X$, $A' \subseteq \Gamma^+(x) \subseteq A'' \Rightarrow A' = \Gamma^+(x)$
2. Add a vertex $z$ to $G = (X, U) \in \mathcal{F}$, a loop on $z$, every arc from $A \subseteq X$ to $z$, every arc from $A \subseteq X$ to $z$, where $A$ is chosen as in (1) where every $\Gamma^+$ is replaced by $\Gamma$.

At last, if $G$ is a symmetric digraph, $\varphi(G)$ denotes the graph with same vertices as $G$ and such that $xy$ is an edge of $\varphi(G)$ iff $xy$ is an arc of $G$. When dealing with $G$ and $\varphi(G)$ simultaneously we shall use $\Gamma(x)$, $d^-(x)$, $k^+$, $D_i$, etc. with no further precision; but no confusion should arise since notions regarding the digraph will be indexed by a sign $+$ or $-$, while those dealing with the graph will not.

Fig. 4.
2. Some characterizations of Ferrers digraphs

There exist many characterizations of Ferrers digraphs. Theorem 1 states those which generalise known characterizations of threshold graphs.

**Theorem 1.** For any digraph $G = (X, U)$, the following statements are equivalent.

- **(F0)** $G \bar{G}^{-} G$ is a partial digraph of $G$.
- **(F1)** There exist two mappings $v^+, v^- : X \to \mathbb{R}$ and $t \in \mathbb{R}$ such that
  
  a. $\forall x \in X : v^+(x), v^-(x) \leq t$.
  
  b. if $Y, Z \subseteq X$ and $v^+(Y), v^-(Z) \leq t$, then
      
      $(Y \times Z) \cap U = \emptyset \iff v^+(Y) + v^-(Z) \leq t$.
  
- **(F2)** There exist two mappings $v^+, v^- : X \to \mathbb{R}$ and $t \in \mathbb{R}$ such that
      
      $\forall x, y \in X : xy \in U \iff v^+(x) + v^-(y) > t$.
  
- **(F3)** a. $k^+ = k^-$.
  
      b. $\forall x, y \in X : xy \in U \iff \delta^+(x) + \delta^-(y) > k^+$.
  
- **(F4)** a. $k^+ = k^-$.
  
      b. $\forall i \in [0, k^+] : d_{i+1}^+ = d_i^+ + |D_k^+|$.
      
      c. $\forall i \in [0, k^-] : d_{i+1}^- = d_i^- + |D_k^-|$.
  
- **(F5)** There exists a linear order ($\mathcal{O}$, $\leq$) and two mappings $f_1, f_2 : X \to \mathcal{O}$ such that
      
      $\forall x, y \in X : xy \in U \iff f_1(x) \leq f_2(y)$.
  
- **(F6)** $G \in \mathcal{F}$.
  
- **(F7)** No subgraph of $G$ is isomorphic to a digraph of $\mathcal{F}$.
  
- **(F7bis)** $G$ induces no 4-alternated-anticircuit.
  
- **(F8)** The set $\{\Gamma^+(x) : x \in X\}$ is linearly ordered by inclusion.
  
- **(F8bis)** the set $\{\Gamma^-(x) : x \in X\}$ is linearly ordered by inclusion.

((F0), (F8) and (F8bis) can be found in [6, 8], (F0) and (F8) in [7], (F5) in [1] and (F5) and (F7bis) in [3])

**Proof.** We want to use the fewest arguments as possible but, on the other hand, we want to make use of the most obvious ones: we come off from this dilemma with the following suggestions.

- $(F0) \iff (F7bis) \iff (F7)$. (F0) is a restatement of (F7bis) and so is (F7).
- not $(F8bis) \iff$ not $(F7bis) \iff$ not (F8). Obvious.
- $(F3) \Rightarrow (F4)$. Obvious.
- $(F4) \Rightarrow (F3)$. If $p > q$, let $\emptyset = \bigcup_{i=p}^q A_i$ for any $A_i$'s and $0 = \bigcup_{i=p}^q a_i$ for any $a_i$'s; let $k = k^+ = k^-$ for short; it is clear that, for any $i \in [0, k]$,

$$d_i^+ = \sum_{i=k+1}^k |D_i^+|$$

and $d_i^- = \sum_{i=k+1}^k |D_i^-|$. 

it can be used to prove by simultaneous induction on $i$, that, for every $i \in [0, k]$,

(a) $\Gamma^-(D^-_i) = \bigcup_{j=0}^{k-i} D^-_j$, and

(b) $\Gamma^+(D^+_k) = \bigcup_{j=0}^{k-i} D^+_j$ (right before proving (a) (resp. (b)), for some $i$, one can prove it with $= \Rightarrow$ changed into $\geq$ (resp. $\leq$), then use the degree equality).

Note that both (b) and (c) of (F4) are required in order to prove (F3).

(F5) $\Rightarrow$ (F6). Let $|X| \geq 2$ and $m = \min\{f_1(X) \cup f_2(X)\}$.

(a) if $m \neq f_1(x)$ for all $x \in X$, then $m = f_2(z)$ for some $z \in X$ and $\Gamma^-(z) = \emptyset$; let $\Lambda' = \{x \in X; \exists y \in X - \{z\}, f_1(z) < f_1(y) \leq f_2(x)\}$; let $\Lambda'' = X - \{x \in X; \exists y \in X - \{z\}, f_2(z) < f_2(y) \leq f_1(x)\}$.

(b) if $m = f_1(z)$ for some $z \in X$, then $\Gamma^-(z) = \emptyset$; let $\Lambda' = \{x \in X; \exists y \in X - \{z\}, f_1(x) \leq f_1(y) \leq f_2(z)\}$; let $\Lambda'' = X - \{x \in X; \exists y \in X - \{z\}, f_2(z) \leq f_2(y) \leq f_1(x)\}$.

(F6) $\Rightarrow$ (F8): Obvious by induction (and keeping in mind that (F8) $\iff$ (F8bis)).

(F8) $\Rightarrow$ (F5): Let $X = \{x_1, x_2, \ldots, x_n\}$ such that $i < j \Rightarrow \Gamma^+(x_i) \supseteq \Gamma^+(x_j)$; let $f_1(x_i) = i$ and $f_2(x_i) = |\Gamma^-(x_i)|$.

(F1) $\Rightarrow$ (F2): Obvious with same $v^+$, $v^-$ and $t$.

(F2) $\Rightarrow$ (F5): There exists $t' \geq t$ such that, for every $x, y \in X$, $xy \in U$ iff $v^+(x) + v^-(y) \geq t'$; let $\emptyset \leq \emptyset = (\emptyset, \emptyset) = f_1 \leftarrow t' - v^+$ and $f_2 \leftarrow v^-$.

(F5) $\Rightarrow$ (F3): Let $\emptyset' = \emptyset \cup \{m, M\}$ with $m, M \notin \emptyset$ so that $(\emptyset', \leq)$ is an extension of $(\emptyset, \leq)$, $m$ and $M$ being min $\emptyset'$ and max $\emptyset'$ respectively, and let $\min(\emptyset) = M$ and $\max(\emptyset) = m$; for every $x \in X$, first let $h_1(x) = \min\{f_2(y); f_1(x) \leq f_2(y)\}$, and let $h_2(x) = \max\{f_2(y); f_2(y) \leq f_2(x)\}$; let $H = \{a_1, a_2, \ldots, a_k\} = h_1(X) \cap h_2(X)$ so that $a_1 < a_2 \ldots < a_k$; then $H \subseteq h_1(X) \subseteq H \cup \{M\}$, $H \subseteq h_2(X) \subseteq H \cup \{m\}$, and ‘still’ $\forall x, y \in X$, $xy \in U \iff h_1(x) \leq h_2(y)$; let $a_0 = m$ and $a_{k+1} = M$; then $k' = k^+ = k$ and, for any $i \in [0, k]$, $D^-_i = h_i^{-1}(a_i)$ and $D^+_i = h_i^{-1}(a_{i+1})$.

(F3) $\Rightarrow$ (F1): The $v^+$, $v^-$ and $t$ we define here make use of larger values than necessary, but, doing so, we avoid any discussion about whether $k$ ($= k^+ = k^-$) is odd or even; we assume that $k > 0$; then let $p = \lfloor \frac{k}{2} \rfloor$ (that is the largest integer less or equal to $\frac{1}{2}k$). $b = 2 |X| + 1$ and, for every $x \in X$:

$$
\delta^+(x) \in [0, p] \Rightarrow v^+(x) = b^p - x,
$$

$$
\delta^+(x) \in [p + 1, k] \Rightarrow v^+(x) = b^{p+2} - b^{k+1} - x,
$$

$$
\delta^-(x) \in [0, p] \Rightarrow v^-(x) = b^{p+1} - x,
$$

$$
\delta^-(x) \in [p + 1, k] \Rightarrow v^-(x) = b^{p+2} - b^{k+1} - x,
$$

now let $t = b^{p+2} - 1$; assuming that $\emptyset, \emptyset \in \{+, -\}$, one can see that:

(a) $v^\emptyset(x) + v^\emptyset(x') > t \Leftrightarrow \delta^\emptyset(x) + \delta^\emptyset(x') > k$,

(b) $v^+(\bigcup_{i=0}^{p} D^+_i) \cap v^-(\bigcup_{i=0}^{p} D^-_i) \leq t$;

if $\delta^\emptyset(x) = i > p$, then

$$
v^\emptyset(x) + v^+(\bigcup_{i=0}^{k-i} D^+_i) + v^-(\bigcup_{i=0}^{k-i} D^-_i) \leq t;
$$
as suggested in [4], where similar ideas are used, doing so in arithmetic base \( h \) might help; then (F1) follows easily (one can note the usefulness of requiring that \( v^+(Y), v^-(Z) \leq t \) in (F1)(b)).

**Comments.** (1) (F1) is rather awkward; first, the idea was to 'separate' empty rectangles (that is \( Y \times Z \) such that \( (Y \times Z) \cap U = \emptyset \)) from not empty ones by means of \( v^-, v^- \) and \( t \); unfortunately, this is not always possible (for instance with \( G = (\{1, 2, 3\}, \{21, 22, 31, 32\}) \)).

(2) The (F3) \( \Rightarrow \) (F1) and (F1) \( \Rightarrow \) (F2) proofs show that, when speaking of \( v^+, v^- \) and \( t \) in (F1) and (F2), one can demand that:

\[
\begin{align*}
(a) & \quad v^+, v^- : \rightarrow \mathbb{N}^+ \text{ and } t \in \mathbb{N}^+; \\
(b) & \quad \text{if } x \in D^+_v \text{ and } y \in D^-_v, \text{ then } v^+(x) = v^-(y). 
\end{align*}
\]

We shall use these requests when proving Theorem 2.

From Theorem 1 we can derive an algorithm to test whether a given digraph \( G = (X, U) \) is a Ferrers digraph or not.

**Algorithm.** Assume that \( |X| = n \).

*Step 0.* For each vertex \( x \in X \), evaluate \( d^+(x) \) and \( d^-(x) \); define two labelings \( \lambda \) and \( \mu \) of the vertices so that:

\[
\begin{align*}
&d^+(x_{\lambda(1)}) \geq d^+(x_{\lambda(2)}) \geq \cdots \geq d^+(x_{\lambda(n)}), \\
&d^-(x_{\mu(1)}) \geq d^-(x_{\mu(2)}) \geq \cdots \geq d^-(x_{\mu(n)}); \\
&h \leftarrow p \leftarrow 0; q \leftarrow n; \\
&\text{Step 1. If } h = 2n, \text{ then stop;} \\
&\text{Step 2. } h \leftarrow h + 1; \text{ if } q = 0, \text{ then go to Step 4;} \\
&\text{Step 3. If } d^-(x_{\mu(q)}) = p, \text{ then } [f_1(x_{\mu(q)}) \leftarrow h; q \leftarrow q - 1; \text{ go to Step 1}]; \\
&\text{Step 4. } p \leftarrow p + 1; \text{ if } d^+(x_{\lambda(p)}) = q, \text{ then } [f_1(x_{\lambda(p)}) \leftarrow h; \text{ go to Step 1}]; \\
&\text{Step 5. } p \leftarrow n - 1; \text{ if } h \leftarrow h - 1; \text{ stop};
\end{align*}
\]

**Proposition.** The algorithm takes \( O(n^2) \) steps: if it terminates in Step 1, then \( f_1 \) and \( f_2 \) are two mappings from \( X \) to \( \mathcal{V}_0 \) such that: \forall x, y \in X, xy \in U \Leftrightarrow f_1(x) \leq f_2(y) \) and thus \( G \) is a Ferrers digraph; if it terminates in Step 5, then \( G \) is not a Ferrers digraph.

**Proof.** The computation of \( d^+ \) and \( d^- \) takes \( O(n^2) \) steps while the construction of \( \lambda \) and \( \mu \) takes \( O(n \log n) \) steps, and to perform the rest of the algorithm takes only \( O(n) \) steps. Now, one can prove \( \gamma \) by induction on \( h \) that, at Step 1, one has (Fig. 5 displays a picture of a 0-1 matrix associated to \( G \) such that the element of}
row $i$ and column $j$ is 1 iff $x_{\lambda(i)\mu(i)} \in U$:

1. $h = n + p - q$ and $0 = h \leq 2n$;
2. $0 \leq q \leq n$;
3. $(p = n$ and $q > 0) \Rightarrow d^-(x_{\mu(0)}) = n$;
4. $0 \leq p \leq n$;
5. for $1 \leq i \leq p$, $p < i' \leq n$, $1 \leq j \leq q$, $q < j' \leq n$:
   (i) $x_{\lambda(i)\mu(i)} \in U$,
   (ii) $x_{\lambda(i)\mu(i)} \in U \Leftrightarrow f_1(x_{\lambda(i)}) < f_2(x_{\mu(i)})$,
   (iii) $x_{\lambda(i)\mu(i)} \notin U$.

then, only two cases can occur:

Case 1. The algorithm terminates in Step 1 with $h = 2n$, $p = n$, $q = 0$ and $G$ is a Ferrers digraph: $f_1$ and $f_2$ satisfy (F5) of Theorem 1 since, for every $x, y \in X$, $f_1(x) < f_2(y)$ iff $f_1(x) \leq f_2(y)$.

Case 2. The algorithm terminates in Step 5: then $f_1(x_{\lambda(1)}), f_1(x_{\lambda(2)}), \ldots, f_1(x_{\lambda(p)})$ and $f_2(x_{\mu(1)}), f_2(x_{\mu(2)}), \ldots, f_2(x_{\mu(q)})$ have been given a value, all different, from 1 to actual $h$, and the 5-statement above is valid with actual $h$, $p$, $q, f_1, f_2$: moreover, $d(x_{\mu(0)}) > p$ and $d^+(x_{\lambda(p+1)}) < q$: if $x_{\lambda(p+1)} \in U$, then for some $i \in [1, q]$, $x_{\lambda(i)} \in U$; then for some $i' \in [p+1, n]$, $x_{\lambda(i)} \in U$ while $x_{\lambda(i')} \in U$ (because $d(x_{\mu(0)}) > d^-(x_{\mu(q)})$); if $x_{\lambda(p+1)} \notin U$, then for some $i' \in [p+1, n]$, $x_{\lambda(i)} \in U$; then for some $i \in [1, q]$, $x_{\lambda(i)} \notin U$ while $x_{\lambda(i')} \in U$ (because $d^+(x_{\lambda(p+1)}) > d^+(x_{\lambda(i)}))$: in any case, $x_{\lambda(p+1)}$ and $x_{\mu(q)}$ belong to a 4-alternated-anticircuit induced by $G$.

3. Some properties of symmetric Ferrers digraphs and some characterizations of threshold graphs

When a Ferrers digraph $G$ is symmetric, properties (F1), \ldots, (F8bis) of Theorem 1 can be expressed in terms of $\varphi(G)$. 

Fig. 5.
Theorem 2. For \( i = 1, 2, \ldots, 7 \), this 8. the following statements hold:

(a) if \( G \) is a symmetric digraph satisfying (Fi), then \( \varphi(G) \) satisfies (Ti);
(b) if \( G \) is a graph satisfying (Ti), then there exists a symmetric digraph \( G' \in \varphi^{-1}(G) \) satisfying (Fi);

where the (Fi)'s are the properties of Theorem 1 and the (Ti)'s are the following ones, with \( G = (X, E) \) being a graph:

(T1) there exists a mapping \( v : X \to \mathbb{R} \) and a threshold \( t \in \mathbb{R} \) such that \( \forall S \subseteq X, S \) is an independent set for \( G \Leftrightarrow v(S) \leq t \);
(T2) there exists a mapping \( v : X \to \mathbb{R} \) and a threshold \( t \in \mathbb{R} \) such that

\[
\forall x, y \in X, \quad xy \in E \Leftrightarrow v(x) + v(y) > t;
\]

(T3) \( \forall x, y \in X, \ xy \in E \Leftrightarrow \delta(x) + \delta(y) > k \);
(T4) (a) \( \forall \ell \in \{0, 1, 2, \ldots, k \} \), \( k \), \( \ell = \ell_1 + \ell_2 = d_1 + |D_{k-1}| \).
(b) \( d_{k+1} = d_{k+1} + |D_{k+1}| - 1 \);
(T5) \( X = X_1 \cup X_2, \ X_1 \cap X_2 = \emptyset \), where \( X_1 \) (resp. \( X_2 \)) is an independent (resp. complete) set for \( G \) and there exist a linear order \( (\emptyset, \leq) \) and a mapping \( f : X \to \emptyset \) such that

\[
\forall x \in X_1, \forall y \in X_2, \ xy \in E \Leftrightarrow f(x) \leq f(y);
\]

(T6) \( G \in \mathcal{F} \);
(T7) no subgraph of \( G \) is isomorphic to \( 2K_2, P_4 \) or \( C_4 \), that is two parallel edges, a chain with 4 vertices and a cycle with 4 vertices respectively (these graphs are displayed in Fig. 6);
(T7bis) \( G \) induces no 4-alternated-cycle;
(T8) the relation \( R \) defined on \( X \) by

\[
\forall x, y \in X, \ xRy \text{ iff } \Gamma(y) \subseteq \Gamma(x) \cup \{x\}
\]

is a total preorder (i.e. reflexive and transitive, and \( xRy \Rightarrow yRx \)).

Proof. All along this proof, when proving \((i)(a)\) we assume that \( G = (X, U) \) and \( \varphi(G) = (X, E) \), and when proving \((i)(b)\) we assume that \( G = (X, E) \) and \( G' = (X, U) \).

(1)(a) The comment related to Theorem 1 shows that we can assume that \( v^+ = v^- \) and that, for every \( x \in X \), \( v^+(x) > 0 \); let \( v = v^+ \); suppose first that \( S \subseteq X \) is not an independent set for \( \varphi(G) \); then there exist \( x, y \in S \) such that \( x \neq y \) and \( xy, yx \in U \); therefore \( v(S) \equiv v(x) + v(y) = v^+(x) + v^-(y) > t \). Suppose now that \( S \) is an independent set for \( \varphi(G) \) such that \( v(S) > t \); let \( S' \subset S \) be maximal in regard to

\[
\begin{align*}
\text{2}_K & \quad \text{P}_4 \quad \text{C}_4 \\
\text{Fig. 6.}
\end{align*}
\]
inclusion such that \( v(S') \leq t \). Now let \( x \in S - S' \); then \( v(S' \cup \{x\}) = v^+(x) + v^-(S') \leq t \) since \( v^+(x), v^-(S') \leq t \) and \( \{x\} \times S' \) is an empty rectangle. This contradicts the maximality of \( S' \).

(1)(b) If \( G, v \) and \( t \) satisfy (T1), then obviously \( G, v \) and \( t \) satisfy (T2); as shown below (see (2)(b)), there exists \( G' \in \varphi^{-1}(G) \) satisfying (F2) and therefore (F1) (it did not seem to us that \( G, v \) and \( t \) induced some \( G' \in \varphi^{-1}(G), v^+, v^- \) and \( t' \) satisfying (F1) straightforwardly).

(2)(a) Let \( v = v^+ + v^- \) and \( t' = 2t \); then \( \varphi(G), v \) and \( t' \) satisfy (T2).

(2)(b) Let \( v = v^+ = v^- = G' \in \varphi^{-1}(G) \) such that, for every \( x \in X, xx \in U \) iff \( 2v(x) > t \); then \( G', v^+, v^- \) and \( t \) satisfy (F2).

(3)(a) For every vertex \( x \), if \( \delta^+(x) \leq \lceil k'/2 \rceil \), then \( d(x) = d^+(x) \). otherwise \( d(x) = d^+(x) - 1 \); so if \( d^+_{\lceil k'/2 \rceil} > d^+_{\lceil k'/2 \rceil} + 1 \), then \( k = k^+ \), \( D_i = D^+_i \) for every \( i \in [0, k] \) and to complete the proof is obvious.

(3)(b) Let \( G' \in \varphi^{-1}(G) \) be such that, for every \( x \in X, xx \in U \) iff \( 2d^+(x) > k \); the proof is the same as in (a) when \( k = k^+ \).

(4)(a) By induction on \( k'' = k' - 1 \). If \( k'' = 1 \), then \( \varphi(G) \) clearly satisfies (T4).

Assume that \( k'' = 2 \); one has \( d^+_i = |D_{k'}^+| \) and it is clear that \( d^+_{k''} = \sum_{i=1}^{k''} |D_i^+| \); therefore, for every \( x \in D^+_{k''}, \Gamma^+(x) = X - D^-_{k''} \) and, for every \( x \in D^-_{k''}, \Gamma^+(x) = D^+_{k''} \).

Let \( G' \) be the digraph obtained from \( G \) by suppressing the vertices of \( D_{k''}^+ \cup D_{k''}^- \); let \( d^+_0, d^+_{i}, \ldots, d^+_k \) be the sequence of outdegrees of \( G' \) and define \( d^-_{i}, D^+_i, D^-_i, \ldots \) accordingly; then \( k'' = k'' - 2 \) and, for every \( i \in [0, k''] \), \( D^+_i = D^-_{i+1} \); therefore \( G' \) satisfies (F4) (for every \( i \in [0, k''], \)

\[
d^+_{i+1} = d^+_i - |D^+_i| = d^+_{i+1} + |D^-_{i+1}| - |D^+_i| = d^+_i + |D^-_{i+1}|
\]

and thus \( \varphi(G') \) satisfies (T4). Now, \( \varphi(G) \) is easily reconstructible from \( \varphi(G') \); add every vertex \( x \in D_{k} \) so that \( \Gamma(x) = X - D_{0} - \{x\} \), and every vertex \( x \in D_{0} \) so that
\( \Gamma(z) = \emptyset \) (therefore, \( k = k' + 2 \) while, \( \forall \) every \( i \in [1, k-1] \), \( D_i = D_i' - 1 \)) then:

1. \( d_1 = |D_k| \);

2. for every \( i \in [1, \lceil \frac{k}{2} \rceil \cup \lceil \frac{k}{2} \rceil , k-1] \), \( d_{i+1} = d_i' + |D_k| = d_i' - 1 + |D_k' - 1| + |D_k| = d_i + |D_k| \) (note that \( i \neq \lceil \frac{k}{2} \rceil - 1 \iff i \neq \lceil \frac{k}{2} \rceil \));

3. \( d_k = |X - D_0| - 1 = d_k' + |D_k| + |D_k| = d_{k-1} + |D_k| \) (for it is clear from (T4) that \( d_{k'} = |\bigcup_{j=1}^{k'} D_j| - 1 \));

4. \( d_{1+2j} - 1 = d_{k+2j} - 1 + |D_k| - d_{k+2j} + |D_k'| - 1 - 1 + |D_k| = d_{k+2j} + |D_k'| - 1 - 1 \) and \( \varphi(G) \) satisfies (T4).

4(b) Define \( G' \) as in (3)(b). The proof is obvious.

5(a) Let \( X_1 = \{ x \in X : f_1(x) > f_2(x) \} \) and \( X_2 = \{ x \in X : f_1(x) \leq f_2(x) \} \); then \( f(x) = f_1(x) \) for every \( x \in X_1 \) and \( f(x) = f_2(x) \) for every \( x \in X_2 \);

1. \( X_1 \) is an independent set for \( \varphi(G) \) otherwise, for some \( x, y \in X_1 : f_1(y) \leq f_2(x) \) and \( f(x) < f_1(y) \).

2. \( X_2 \) is a complete set for \( \varphi(G) \) otherwise, for some \( x, y \in X_2 : f_2(y) < f_1(x) \).

3. For every \( x \in X_1 \) and every \( y \in X_2 \), \( xy \in E \) iff \( f_1(x) \leq f_2(y) \) iff \( f(x) < f(y) \).

5(b) Let \( G' \in \varphi^{-1}(G) \) be such that, for every \( x \in X \), \( xx \in U \) iff \( x \in X_2 \). Assume that \( f(x) = \{ a_1, a_2, \ldots, a_p \} \) with \( a_1 < a_2 < \cdots < a_p \). Let \( (\emptyset', \leq') \) be a linear extension of \( (\emptyset, \leq) \) such that, for every \( i \in [1, p] \), \( h_i \in \emptyset' \) and such that \( b_p < b_{p-1} < \cdots < b_1 < a_1 \). Let \( x \in X \) and \( f(x) = a_1 \); if \( x \in X_1 \) (resp. \( X_2 \)) let \( f_1(x) \) (resp. \( f_2(x) \)) = \( f(x) \) and \( f_2(x) \) (resp. \( f_1(x) \)) = \( h_2 \); then \( G', f_1, f_2 \) \( (\emptyset', \leq') \) satisfy (F5).

6(a) Obvious.

6(b) Let \( G' \in \varphi^{-1}(G) \) be such that, for every \( x \in X \), \( xx \in U \) iff there exists a construction of \( G \) such that \( x \) 'appears' while applying (2) to a graph of \( T \) different from the empty graph; notice that there are no two constructions of \( G \) such that \( x \) 'enters' \( G \) while applying (2) in one of them and (11) in the other one, in both cases not to the empty graph. Now let \( G_1 = ([x_1], \emptyset) \), \( G_2, \ldots, G_n = G \) be a sequence such that \( G_i \) is deduced from \( G_{i-1} \) and \( x_i \) by applying \( (n_i) \), with \( n_i \in \{ 1, 2 \} ; \) if \( n_2 = 1 \) \( (G_2 = ([x_1, x_2], \emptyset)) \), let \( G'_1 = ([x_1], \emptyset) \) otherwise let \( G'_1 = ([x_1]. \{ x_1, x_i \}) \). Now let \( G_1, G_2, \ldots, G_n \) be the sequence such that \( G_i' \) is deduced from \( G_{i-1} \) and \( x_i \) by applying \( (n_i) \), where, if \( n_i = 1 \), then \( A \) is empty, otherwise \( A \) is the set of vertices of \( G_{i-1} \); it is easy to see that \( G_n' \) is equal to \( G' \).

7(bis) see (7bis), as (7bis) can be viewed as a restatement of (7).

7(bis)(a) Obviously, if \( \varphi(C) \) induces a 4-alternated-cycle, then \( G \) induces a 4-alternated-anticycle.

7(bis)(b) Let \( X_1 = \{ x \in X : \exists y z \in E, x y, x z \notin E \} \),

\[ X = \{ x \in X - X_1 : \exists y \in X - X_1, x y \notin E \} \],

\( X_3 = X - X_1 - X_2 \) and let \( G' \in \varphi^{-1}(G) \) be such that, for every \( x \in X \), \( xx \in U \) iff \( x \in X_1 \);

1. there is no subgraph of \( G' \) isomorphic to a digraph of \( \mathcal{G}_4 \), otherwise \( G \) induces a 4-alternated-cycle (4-a-c for short);

2. if there exist \( x, y, z \in X \) such that \( x \neq y \neq z \neq x \) and \( y z \in U \) while \( y x, z x \notin U \),
then \( x \in X_1 \) and \( xx \notin U \); therefore no subgraph of \( G' \) is isomorphic to a digraph of \( \mathcal{G}_2' \):

(iii) therefore \( x \in X_3 \), then \( xx \in U \) and no subgraph of \( G' \) is isomorphic to a digraph of \( \mathcal{G}_2' \).

(4) Suppose there exist \( x, y \in X \) such that \( x \neq y \) and \( xy, yx \in U \) while \( xx \notin U \); then \( x \in X_1 \) and \( x \in X_3 \):

(i) suppose there exist \( x, y, z \in X \) such that \( x \neq y \neq z \neq x \) and \( yz \notin U \) while \( yx \in U \):

(ii) \( x \notin X_1 \) otherwise there exists \( it' \in E \) such that \( xt, xt' \notin E \) and if \( zt \in E \), then \( (xt, tz, yz, yx) \) is a 4-a-c induced by \( G \) while if \( zt \notin E \), then \( (xz, zt, tt', t'x) \) is a 4-a-c induced by \( G \), a contradiction in both cases;

(iii) \( x \in X_2 \) otherwise there exists \( i \in X - X_1 \) such that \( xi \notin E \), then \( iz \in E \) and \( (xt, tz, yz, yx) \) is a 4-a-c induced by \( G \), a contradiction:

(iv) therefore \( x \in X_3 \), then \( xx \in U \) and no subgraph of \( G' \) is isomorphic to a digraph of \( \mathcal{G}_2' \).

(5) Suppose there exist \( x, y \in X \) such that \( x \neq y \) and \( xy, yx \in U \) while \( xx \notin U \); then \( x \in X_3 \), and thus \( y \in X_1 \); therefore \( yy \notin U \) and no subgraph of \( G' \) is isomorphic to the only digraph of \( \mathcal{G}_2' \).

(6)(a) Assume that \( x, y \in X \) such that \( x \neq y \) and \( \Gamma^{-1}(x) \subseteq \Gamma^{-1}(y) \); then \( \Gamma(x) = \Gamma^{-1}(x) \subseteq \Gamma^{-1}(y) \subseteq \Gamma(y) \cup \{y\} \). (note that, given any graph, \( R \) is a preorder).

(6)(b) Let \( X = \{x_1, x_2, \ldots, x_n\} \) such that \( \Gamma(x_i) \subseteq \Gamma^*(x_i) \) for every \( i \in [1, n] \) and let \( G' \in \varphi^{-1}(G) \) be such that, for every \( i \in [1, n] \), \( x_i, x_i \in U \) iff \( x_i \in \Gamma(x_{i-1}) \); then, for each \( i \in [1, n] \), \( \Gamma(x_i) \subseteq \Gamma^*(x_i) \) and moreover, if \( x_{i-1} \in \Gamma^*(x_{i-1}) \), then \( x_{i-1} \in \Gamma(x_{i-1}) \subseteq \Gamma(x_i) \) and \( \Gamma(x_i) \subseteq \Gamma^*(x_i) \); therefore \( \Gamma^*(x_{i-1}) \subseteq \Gamma^*(x_i) \), which completes the proof.

From Theorems 1 and 2 we deduce a set of characterizations of threshold graphs.

**Theorem 3.** For any graph \( G = (X, E) \), conditions (T1), . . . , (T8) of Theorem 2 are equivalent.

**Remarks.** (1) Theorem 3 is 'covered' by results that can be found in [2, 4, 5].

(2) The fact that \( G \) is a threshold graph iff \( \bar{G} \) is a threshold graph [2] can, also,
be viewed as a corollary of the fact that such a result holds for Ferrers digraphs (see [8]; anyway, it is clear from Theorem 1).

(3) The comment related to Theorem 1, together with the (1)(a) and/or the (2)(a) parts of the Theorem 2 proof, shows that one can demand that \( v : X \to \mathbb{N}^+ \) and \( t \in \mathbb{N}^+ \) in (T1) and (T2) of Theorem 2 as shown is [2].

Now, when dealing with a symmetric digraph, the algorithm given in Section 2 can be slightly modified: replace Step 1 by

**Step 1’.** If \( p = q \), then stop.

Assume that the so modified algorithm terminates on Step 1’. \( G \) being a symmetric digraph. Let \( P = \{x_{\lambda(1)}, \ldots, x_{\lambda(p)}\} \) and \( Q = \{x_{\mu(q+1)}, \ldots, x_{\mu(n)}\} \) (if \( q = 0 \), then \( P = \emptyset \) and if \( Q = n \), then \( Q = \emptyset \)).

Obviously \( \lambda \) and \( \mu \) can be constructed equal in Step 0 and then \( X = P \cup Q \) while \( P \cap Q = \emptyset \).

Then \( G \) is indeed a Ferrers digraph since, if we let \( f_2(x) = 2n + 1 - f_1(x) \) for each \( x \in P \) and \( f_1(x) = 2n + 1 - f_2(x) \) for each \( x \in Q \), then:

\[ \forall x, y \in X \quad xy \in U \Leftrightarrow f_1(x) \leq f_2(y) \]

(for the modified algorithm terminates with \( h = n \) and statement (5) of the proof of the algorithm, in Section 2, is still valid).

**Comments.** Let \( f(x) = n + 1 - f_1(x) \) for each \( x \in P \) and \( f(x) = n + 1 - f_2(x) \) for each \( x \in Q \); then \( \varphi(G) \), \( f \), \( Q \) and \( P \) satisfy (T5). Furthermore, \( f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(n) \) is the ordering of \( X \), regarding \( \varphi(G) \), computed by the algorithm given in [2] and which also computes \( P - \{f^{-1}(1)\} \) and \( Q - \{f^{-1}(1)\} \). In fact, replacing Step 1 of our algorithm by ‘Step 1’. If \( |p - q| = 1 \), then stop:’ would yield the algorithm of [2]; it is easy to see that if our algorithm terminates in Step 1”, then \( G \) is a Ferrers digraph anyway. The use of Step 1’ instead of Step 1” tells us whether \( f^{-1}(1) \) belongs to \( P \) or \( Q \), that is whether there is a loop on it or not. As Theorem 4 shows, this does not affect the fact that \( \varphi(G) \) is a threshold graph.

### 4. Correspondence between Ferrers digraphs and threshold graphs

When proving parts (b) of Theorem 2, given a threshold graph \( G \), we use seemingly different ways to construct a Ferrers digraph \( G' \in \varphi^{-1}(G) \). It turns out that they all lead to the same \( G' \), except in (8)(b) when \( k \) is odd. But there are always different possible choices of \( G' \), as shown by the following Theorem 4.

**Lemma.** If \( G \) is a threshold graph of order \( n > 1 \), then \( |D_{[k/2]}| \geq 2 \).

**Proof.** Use Theorem 3 and (T4)(b) for instance when \( k > 0 \) or note that, if \( G = G_n \) as in the (6)(b) proof, then \( x_1, x_2 \in D_{[k/2]} \).
**Theorem 4.** For any threshold graph $G \in \mathcal{F}$, $|\varphi^{-1}(G) \cap \mathcal{F}| = 1 + |D_{\lfloor k/2 \rfloor}|$; more precisely, if $G \in \varphi^{-1}(G)$, then $G'$ is a Ferrers digraph iff it satisfies the following conditions:

(a) if $\delta(x) < \lceil k/2 \rceil$, then $xx \notin U$,
(b) if $\delta(x) > \lceil k/2 \rceil$, then $xx \in U$,
(c) if $k$ is even, then $xx \notin U$ for every $x \in D_{k/2}$ except, possibly, for one vertex in $D_{k/2}$; if $k$ is odd, then $xx \in U$ for every $x \in D_{\lfloor k/2 \rfloor}$ except, possibly, for one vertex in $D_{\lfloor k/2 \rfloor}$.

**Proof.** First, if $G'$ satisfies (a), (b) and (c), then $G'$ satisfies (F8). Conversely, assume that $G' \in \varphi^{-1}(G)$ is a Ferrers digraph:

(1) Assume that $k$ is even:

(i) if $x, x' \in D_{k/2}$ and $xx, x'x' \in U$, then $G({x, x'}) \in \mathcal{G}_2$; when $n > 1$, as $|D_{k/2}| \geq 2$, let $x_0 \in D_{k/2}$ such that $x_0x_0 \notin U$;

(ii) if $\delta(x) > k/2$, then $xx \notin U$, otherwise $G({x, x_0}) \in \mathcal{G}_2$;

(iii) if $\delta(x) < k/2$, then $xx \in U$, otherwise let $y \in D_{k/2+1} : G({x, y}) \in \mathcal{G}_2$.

(2) Assume that $k$ is odd:

(i) if $x, x' \in D_{\lfloor k/2 \rfloor}$ and $xx, x'x' \notin U$, then $G({x, x'}) \in \mathcal{G}_2$; as $|D_{\lfloor k/2 \rfloor}| \geq 2$, let $x_0 \in D_{\lfloor k/2 \rfloor}$ such that $x_0x_0 \in U$;

(ii) if $\delta(x) < \lfloor k/2 \rfloor$, then $xx \notin U$, otherwise $G({x, x_0}) \in \mathcal{G}_2$;

(iii) if $\delta(x) > \lfloor k/2 \rfloor$, then $xx \in U$, otherwise let $y \in D_{\lfloor k/2 \rfloor+1} : G({x, y}) \in \mathcal{G}_2$; then $|\varphi^{-1}(G) \cap \mathcal{F}| - 1 + |D_{\lfloor k/2 \rfloor}|$.

**Corollary.** For any threshold graph $G$ of order $n$,

$$|\varphi^{-1}(G) \cap \mathcal{F}| = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n > 1. \end{cases}$$

**Proof.** According to Lemma 1 and to Theorem 4.

**References**


