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Multigrid methods for anisotropic BTTB systems

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Abstract

Multigrid methods are highly efficient solution techniques for large sparse multilevel Toeplitz systems which are positive definite and ill-conditioned. In this paper, we develop multigrid methods which are especially designed for anisotropic two-level Toeplitz (BTTB) matrices. First, a method is described for systems with anisotropy along coordinate axes as a suitable combination of semicoarsening and full coarsening steps. Although the basic idea is known from the solution of partial differential equations, we present it here in a more formal way using generating functions and their level curves. This enables us not only to prove the optimal convergence of the two-grid method, but also to carry over the results to systems with anisotropy in other directions. We introduce new coordinates in order to describe these more complicated systems in terms of generating functions. This enables us to solve them with the same efficiency. For the two-level method, we present a convergence proof in this more general case.

AMS classification: 65N55; 65F10; 65F22; 65F35

Keywords: Multilevel Toeplitz; Multigrid methods; Iterative methods; Preconditioners

1. Introduction

Over the last 50 years, iterative methods have been developed for the solution of large sparse linear systems of equations. Already in 1960, Bauer and Householder [2] published their Projected Aggregation Method for the solution of nonsymmetric linear systems. Today, multigrid methods belong to the fastest iterative methods for the solution of large sparse Toeplitz and Block-Toeplitz

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systems. They can either be used as stand-alone solvers or as preconditioners for Krylov subspace methods such as the conjugate gradient algorithm or GMRES. Positive definite systems corresponding to strictly positive generating functions are easily solved with the conjugate gradient algorithm, preconditioned by circulant matrices [6]. If the generating function has up to a finite number of zeros, the use of multigrid methods can be highly efficient, see e.g., [7,5]. However, many applications such as discretization of partial differential equations (PDEs) lead to anisotropic two-level Toeplitz systems. Although for some of these systems the classical convergence theory for (multilevel) Toeplitz matrices [5,3,1] still holds, standard multigrid methods converge so slowly that they become totally impractical. Therefore, we seek to devise multilevel methods, which are especially designed for application to anisotropic problems.

This paper is organized as follows. After giving basic definitions and explaining fundamental properties of multigrid methods for two-level Toeplitz systems in Section 2, we describe the problems arising from anisotropic systems in Section 3. In the rest of this article, multigrid algorithms for two different types of anisotropic problems will be developed. In Section 4 we consider systems where anisotropy occurs along coordinate axes. Some of these results are known from the solution of partial differential equations, but here we present them in a slightly different context, making explicit use of the strong connection between two-level Toeplitz matrices and generating functions with their level curves. This has the advantage that the methods can be extended to more general matrices, and that convergence can be proved in a formal way. The problems considered in Section 5 are more difficult to solve, because anisotropy occurs in other directions. We develop multigrid methods which are suitable for this case by carrying over the results from Section 4. We focus on directions where anisotropy occurs in an angle of $\frac{k}{k+l} \cdot 90°$ towards one of the axes (with k and l being small integers). Although the classical two-level Toeplitz structure with blocks of equal size is lost, the methods still work in this case. We are mostly interested in sparse examples, which arise, e.g., from the discretization of PDEs, but our methods also work in the more general case of dense Toeplitz matrices corresponding to arbitrary generating functions.

2. Multigrid methods for BTTB systems

In this article we solve linear systems of equations corresponding to two-level Toeplitz matrices, i.e., block Toeplitz matrices with Toeplitz blocks (BTTB matrices). Generating functions are closely related to BTTB matrices. They will be used throughout the paper to derive multilevel methods with certain properties.

Definition 1. Let f be a real-valued Lebesgue integrable function which is defined on $[-\pi, \pi]^2$ and periodically extended on the whole plane. The Fourier coefficients of f are given by

$$t_{k,l} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) e^{-ikx-ily} \, \mathrm{d}x \, \mathrm{d}y \quad (k, l \in \mathbb{Z}).$$

We can now define the sequence of matrices $(T_{mn}[f])_{m,n}$. $T_{mn}[f]$ is the mn-by-mn BTTB matrix with entries $(T_{mn}[f])_{(j,k)(p,q)} = t_{j-k,p-q} (0 \leqslant j, k < m, 0 \leqslant p, q < n)$, where (j, k) indicates the block in $T_{mn}[f]$ and (p, q) the position within the block. f is called the generating function of the sequence $(T_{mn}[f])_{m,n}$.

In the following, let $T_n[f]$ denote the one-dimensional n-by-n Toeplitz matrix related to a function $f(x)$, and $T_{mn}[f]$ the mn-by-mn BTTB matrix related to $f(x, y)$. One important

correspondence between BTTB matrices and generating functions is described for example in [11]. If $f_{\min}$ and $f_{\max}$ denote the infimum and supremum values of f (up to zero measure sets), and $f_{\min} < f_{\max}$, then for all $m, n \geqslant 1$, the eigenvalues of $T_{mn}[f]$ lie in the interval $(f_{\min}, f_{\max})$. For $n, m \to \infty$, the extreme eigenvalues tend to $f_{\min}$ and $f_{\max}$. The case $f_{\min} = f_{\max} = c$ is trivial, because then $T_{mn} = cI_{mn}$ with the identity matrix I_{mn}. Therefore, $f(x, y) \geqslant g(x, y)(\forall x, y \in \mathbb{R})$ implies $T_{mn}[f] \geqslant T_{mn}[g]$.

BTTB systems are most efficiently solved with iterative methods. If a system is well-conditioned, i.e., if $f(x, y) > 0$, the preconditioned conjugate gradient (pcg) method works very well with two-level circulant preconditioners [6]. If f has zeros, the corresponding matrix is ill-conditioned, and circulant preconditioners fail in many cases [4,9]. In recent years, multigrid methods turned out to be the most efficient techniques for symmetric positive definite ill-conditioned BTTB systems whose generating function has a single isolated zero of finite order in $[-\pi, \pi[^2$ (see [7,12,8]). They can be used as standalone solvers or as preconditioners for a Krylov subspace method such as pcg. In this paper we will mostly follow the former approach in order to solve systems of the form $T_{mn}[f]x = b$. We will make heavy use of the correspondence between BTTB matrices and generating functions. For the moment let us assume that f has only one zero in $[-\pi, \pi[^2$.

The algebraic multigrid method (AMG) was developed by Ruge and Stüben [10], which as a purely algebraic method does not use real grids. To develop an AMG method we have to define a smoother and a coarse grid correction operator on each level. A smoother such as the Jacobi or Gauss–Seidel method is denoted by $S : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$. To compute the coarse grid correction operator we need to define a restriction matrix $P : \mathbb{R}^{mn} \to \mathbb{R}^{m_C n_C}$ with $n_C m_C$ being the dimension of the coarse grid system matrix T_C. The transpose P^T is chosen to be the prolongation matrix. The matrix $P^T = B \cdot E$ formally consists of two parts. B is defined to deal with the zero of f, whereas E is the two-dimensional elementary restriction matrix. It picks every second column and every second block column of a matrix. E is obtained from the one-dimensional restriction matrices E_m and E_n by $E = E_m \otimes E_n$. The matrix T_C is computed with the Galerkin approach, i.e., as the product

$$T_C = P \cdot T_{mn}[f] \cdot P^T = E^T \cdot (B^T \cdot T_{mn}[f] \cdot B) \cdot E. \tag{1}$$

The coarse grid correction operator can then be written as $X = I_{mn} - P^T \cdot T_C^{-1} \cdot P \cdot T_{mn}[f]$, leading to a global iteration matrix of the two-level method

$$G = S^{\nu_2} \cdot X \cdot S^{\nu_1}, \tag{2}$$

where ν_1 denotes the number of presmoothing steps and ν_2 the number of postsmoothing steps. A multilevel method is defined by using the two-level method recursively to approximate the inverse of T_C.

Since we wish to solve BTTB systems, we can describe prolongation and coarse grid matrices in terms of generating functions. The product $\hat{T} = B^T \cdot T_{mn}[f] \cdot B$ translated into generating functions gives

$$\hat{f}(x, y) = f(x, y) \cdot b(x, y)^2, \tag{3}$$

whereas the elementary projection $T_C = E^T \cdot \hat{T} \cdot E$ becomes

$$f_2(x, y) = \frac{1}{4} \cdot \left[\hat{f}\left(\frac{x}{2}, \frac{y}{2}\right) + \hat{f}\left(\frac{x}{2} + \pi, \frac{y}{2}\right) + \hat{f}\left(\frac{x}{2}, \frac{y}{2} + \pi\right) + \hat{f}\left(\frac{x}{2} + \pi, \frac{y}{2} + \pi\right) \right]. \tag{4}$$

This means f_2 is obtained from the Fourier series of $\hat{f}$ by picking every second coefficient in x and every second coefficient in y. It is important to note that in general the matrices $\hat{T}$ and T_C are not BTTB, but a sum of a BTTB matrix and a matrix of rank $O(n+m)$. In [3,1] different cutting techniques are presented to make sure that the coarse grid matrix still has two-level Toeplitz structure. However, for some important special cases it can be shown that the coarse grid matrix obtained without additional cutting is still BTTB. So far, we have not yet chosen a function $b(x, y)$ for prolongation. If f is zero at (x_0, y_0), Fiorentino and Serra [7] suggest to use a nonnegative function with zeros at the so called mirror points

$$(\pi - x_0, y_0), \quad (x_0, \pi - y_0), \quad \text{and} \quad (\pi - x_0, \pi - y_0). \tag{5}$$

For this choice they showed that T_C is also positive definite, and its generating function has the single zero $(2x_0, 2y_0)$. The choice

$$b(x, y) = (1 + \cos(x - x_0)) \cdot (1 + \cos(y - y_0)), \tag{6}$$

satisfies these properties and corresponds to a matrix which is extremely sparse. In the rest of this paper we can safely assume that the zero of f is located at the origin, because a zero at $(x_0, y_0) \neq (0, 0)$ can be shifted to the origin by diagonal unitary transformation (a phase matrix). For this choice of b the matrix T_C obtained in (1) is guaranteed to be BTTB if m and n are both odd [12].

Convergence proofs have been given for two-level and multilevel methods applied to Toeplitz and multilevel Toeplitz systems, see, e.g., [5,3,1]. All of them are based on general convergence results by Ruge and Stüben [10]. In order to state their theorem for the two-level method we must define, for an arbitrary matrix A, the following inner products in addition to the Euclidean inner product $\langle u, v\rangle$:

$$\langle u, v\rangle_0 = \langle \mathrm{diag}(A)u, v\rangle, \quad \langle u, v\rangle_1 = \langle Au, v\rangle, \quad \langle u, v\rangle_2 = \langle \mathrm{diag}(A)^{-1}Au, Av\rangle. \tag{7}$$

The respective norms, which are derived from these inner products, are denoted $\|\cdot\|_i$, $i = 0, 1, 2$. Moreover, let $\nu_1 = 0$ and $\nu_2 = 1$.

Theorem 1 (Ruge and Stüben [10]). *Let A be a positive definite mn-by-mn matrix, and let S be a smoother satisfying the* smoothing condition, *i.e., there exists an $\alpha > 0$ such that*

$$\|Se^h\|_1^2 \leqslant \|e^h\|_1^2 - \alpha\|e^h\|_2^2, \quad \forall e^h \in \mathbb{R}^n. \tag{8}$$

Furthermore, suppose that the prolongation operator has full rank and that the correcting condition *is satisfied, i.e., there exists a scalar $\beta > 0$ such that*

$$\min_{e^H \in \mathbb{R}^{m_C n_C}} \|e^h - P^T e^H\|_0^2 \leqslant \beta\|e^h\|_1^2, \quad \forall e^h \in \mathbb{R}^{mn}. \tag{9}$$

Then $\beta > \alpha$, and the convergence factor $\|G\|_1$ of the two-level method with G from (2) satisfies

$$\|G\|_1 \leqslant \sqrt{1 - \frac{\alpha}{\beta}}. \tag{10}$$

Sun et al. [12] use this theorem to prove the optimal convergence rate of the TGM iteration and the level-independency for BTTB systems where f has a zero of order at most two in the origin, i.e., where f satisfies

$$\min_{(x,y)\in[-\pi,\pi]^2} \frac{f(x, y)}{2 - \cos x - \cos y} = C > 0 \tag{11}$$

and $b(x, y)$ is chosen as in (6) with $(x_0, y_0) = (0, 0)$. However, apart from these convergence results there is also one important negative observation, which will be crucial for the development of our multigrid methods.

Remark 1. Let $f(x, y)$ be a nonnegative generating function which has a zero at $(0, 0)$. If f has another zero at one of the mirror points from (5), the multigrid method from above fails completely in all numerical experiments. A theoretical reason for this behavior can be found in [3]: The convergence theory for the multigrid method of Serra [7,3] requires that p is nonzero at the origin and zero at all mirror points. Hence, if f is zero at the origin and at one of the mirror point, p cannot meet both requirements. Even if f is close to zero at one of the three points $(0, \pi)$, $(\pi, 0)$, (π, π), convergence of the multigrid method is extremely slow.

3. Problems arising from anisotropic systems

Anisotropic BTTB systems frequently arise from the discretization of partial differential equations. One important model problem for our experiments is obtained from the following equation, which is closely related to the Poisson equation:

$$-\epsilon \cdot u_{xx} - u_{yy} = g. \tag{12}$$

Finite difference discretization of this equation with a five point stencil on a uniform mesh leads to the linear system described in Example 1. The second example shows a matrix, which is not sparse.

Example 1. Let $T_{mn}[f]$ be the BTTB matrix corresponding to the generating function

$$f(x, y) = \alpha \cdot (1 - \cos(x)) + (1 - \cos(y)). \tag{13}$$

If $\alpha = 1$, we get one of the isotropic standard model problems, the discrete Poisson equation. For $\alpha \ll 1$, the problem becomes strongly anisotropic, i.e., the level curves of $f(x, y)$ become extremely flat. This is illustrated in Fig. 1, which depicts the curve $f(x, y) = 0.01$ for three different values of α, i.e., for three different degrees of anisotropy.

Example 2. Let $T_{mn}[f]$ be the BTTB matrix with the underlying generating function

$$f(x, y) = \alpha x^2 + y^2 \quad (\alpha \ll 1). \tag{14}$$

It also has a single zero of order two in the origin, and it can be written as the Fourier sum

$$f(x, y) = (1 + \alpha)\frac{\pi^3}{3} + 4\pi \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} (\alpha \cos(jx) + \cos(jy)). \tag{15}$$

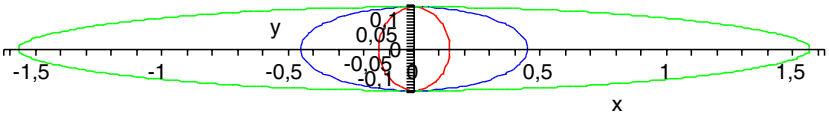

Fig. 1. Curves $f(x, y) = 0.01$ for the function from Example 1 with $\alpha = 1, 0.1, 0.01$.

These two functions are examples where anisotropy occurs along one of the coordinate axes. However, we are also interested in more general functions, where anisotropy occurs in other directions. In most of this article, we restrict ourselves to anisotropic BTTB systems which correspond to generating functions with a single zero in $[-\pi, \pi[^2$ of order two, because these can be described best in a formal way. We will also mention how to solve certain anisotropic systems whose function has several zeros or zeros of higher order. Let us now assume that f has a single zero in the origin of order 2. Then the Taylor expansion of f is of the form

$$f(x, y) = ax^2 + bxy + cy^2 + \cdots = \begin{pmatrix} x & y \end{pmatrix} M \begin{pmatrix} x \\ y \end{pmatrix} + \cdots \tag{16}$$

with $M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. For the analysis of $f(x, y)$ in the neighborhood of the origin we omit all higher order terms and describe f by the symmetric matrix M. Since f is nonnegative, M is positive semidefinite, i.e., its eigenvalues are nonnegative. The eigenvalues λ_1, λ_2 of M give information about the degree of anisotropy, the corresponding orthogonal eigenvectors v_1, v_2 about the direction in which anisotropy occurs. If exactly one of the eigenvalues is close to zero, anisotropy is strong. In the limit case, λ_1 or λ_2 is zero. This means that f is zero along a whole line, which is passing through the origin. In Example 1 the function f can be described by the diagonal matrix $M = \frac{1}{2} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$.

With these two examples we can illustrate why the standard multigrid methods from Section 2 should not be used for the solution of anisotropic systems. Again, generating functions turn out to be a helpful tool for the analysis of multigrid methods. Moderately anisotropic systems can still be solved with standard multigrid, although the number of iterations rises with the degree of anisotropy. This happens because of the weak coupling in one direction, which means that the value of f mainly depends on one of the variables. This implies that even for these systems the design of special multigrid methods for anisotropic systems might lead to faster convergence. However, if the anisotropy is strong, i.e., if $\alpha \ll 1$, standard multigrid cannot be used anymore.

Remark 2. The following problems arise when the standard multigrid method from Section 2 is applied for the solution of strongly anisotropic BTTB systems:

- If anisotropy occurs along one of the axes or in an angle of 45° to the axes, the function f becomes close to zero at one of the mirror points, e.g., at $(\pi, 0)$ in Example 1. From Remark 1 we know that convergence is extremely slow in this case.
- For anisotropy in arbitrary directions f becomes close to zero along a whole line. This means that multigrid methods, which are designed for functions with a single isolated zero are no more applicable.
- Most interesting sparse examples with anisotropy in other directions, i.e., not along the axes, have several zeros in $[-\pi, \pi[$, which means several lines of zero in the limit case.

4. Anisotropy along coordinate axes

BTTB systems where anisotropy occurs along coordinate axes were introduced in Examples 1 and 2. For this type of anisotropic system, we will develop multigrid methods, which do not suffer from the problems described in the previous section. This can be done in two different

ways, either by using semicoarsening combined with a standard smoother such as Jacobi or by applying full coarsening combined with a line smoother such as block Jacobi. In this paper, we focus on the first approach. Although the second approach also leads to satisfactory results, we do not analyze them here. It will be discussed in an upcoming article.

4.1. A two-level method with semicoarsening

One possible way to get rid of the problems described in the previous section is to use semicoarsening in the direction perpendicular to the anisotropy. The smoother is then chosen to be a pointwise one such as the damped Jacobi method. Since we are solving BTTB systems, we wish to describe semicoarsening in terms of generating functions. Let us assume that we have anisotropy along the x-axis (e.g., a function such as f from Example 1 with $\alpha = 0.01$), and that coarsening is done in y-direction only. $P = B \cdot E$ is the product of the following two matrices. B is the BTTB matrix corresponding to a function which is chosen to match the zero of f, for each $x \in [-\pi, \pi]$. The simplest choice is

$$b(x, y) = 1 + \cos(y), \tag{17}$$

corresponding to a matrix of the form

$$B = \begin{pmatrix} 1 & 0.5 & & & \\ 0.5 & 1 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.5 & 1 & 0.5 \\ & & & 0.5 & 1 \end{pmatrix} \otimes I_n. \tag{18}$$

The product $\hat{T} = B^T \cdot T_{mn}[f] \cdot B$, i.e., $\hat{f}(x, y) = f(x, y) \cdot b(x, y)^2$ is computed as in the isotropic case. The elementary restriction matrix E is chosen to be

$$E_{mn} = E_m \otimes I_n \tag{19}$$

with the one-dimensional restriction matrix E_m. Translated to generating functions this becomes

$$f_2(x, y) = \frac{1}{2}\big(\hat{f}(x, y/2) + \hat{f}(x, y/2 + \pi)\big), \tag{20}$$

The result is a coarse grid matrix T_C with half as many blocks as $T_{mn}[f]$, but with the same block size. Again, T_C is BTTB if n is odd and b from (17) is used. In general, additional cutting from [3,1] has to be applied. If anisotropy occurs along the y-axis, semicoarsening is done in x.

Why is this approach superior to the standard coarsening method? Since anisotropy occurs along coordinate axes, we have to focus on the first two problems described in Remark 2. The problem that f is almost zero along a whole axis is overcome by coarsening in only one direction, and therefore by treating the other variable as if it were a constant. With the same argument we get rid of the problem with the zeros at mirror points, because they are all located on the x-axis.

The use of semicoarsening is a good idea for all types of anisotropic problems, not just for the extrem case described above, because it yields a matrix T_C which is less anisotropic than $T_{mn}[f]$. Fig. 2 suggests that generating functions can be used to illustrate this property. Whereas full coarsening does not change the degree of anisotropy, semicoarsening leads to level curves on the coarser level which are less flat. This fact can be described more formally with the following definition.

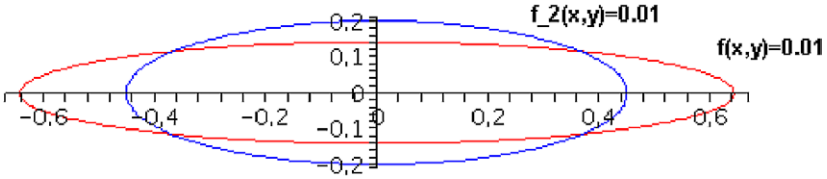

Fig. 2. Level curves $f(x, y) = 0.01$ and $f_2(x, y) = 0.01$ obtained by one semicoarsening step with f from Example 1 and $\alpha = 0.01$.

Definition 2. Let $f(x, y) = c$ be a level curve of f with a sufficiently small, positive real number c, and let f_2 be the function computed in (20). We consider the points $(x_F(c), 0)$ and $(0, y_F(c))$, where the level curve of f intersects the coordinate axes for positive $x_F(c)$ and $y_F(c)$. $x_C(c)$ and $y_C(c)$ are the analogues on the coarser level, i.e., for $f_2(x, y) = c$. For the sake of abbreviation let us omit the parameter c, i.e., let us denote for example $x_F = x_F(c)$. The ratios $r_F = \frac{x_F}{y_F}$ and $r_C = \frac{x_C}{y_C}$ are used as a measure to describe the degree of anisotropy along the coordinate axes x and y for small c.

For the functions in Example 1 we observe that r_F is reduced by a factor 2 after one semicoarsening step, independent of α and for $c \ll 1$. In the following, we prove that this property holds in a more general context.

Theorem 2. *Let f be a nonnegative generating function with a zero of order 2 at the origin which is of the form*

$$f(x, y) = [\lambda_1(1 - \cos(x)) + \lambda_2(1 - \cos(y))] \cdot h(x, y) \tag{21}$$

with $h(x, y) > 0$ and $\lambda_1, \lambda_2 > 0$. Let f_2 be the function obtained by one semicoarsening step with b from (17). Let $r_F = \frac{x_F}{y_F}$ and $r_C = \frac{x_C}{y_C}$ be the ratios described above for f and f_2, respectively. Then the degree of anisotropy is reduced by a factor 2, i.e., $\frac{r_F}{r_C} \to 2$ for $c \to 0$.

Proof

(1) Computation of r_F: Since $h(x, y) > 0$, we can assume that $h_0 := h(0, 0)$ and $h_\pi := h(0, \pi)$ are bounded away from 0. f is approximated in the neighborhood of $(0, 0)$ by the following Taylor expansion with terms of order at most 2:

$$f(x, y) \doteq \left(\lambda_1 \frac{x^2}{2} + \lambda_2 \frac{y^2}{2}\right) \cdot h_0. \tag{22}$$

With this approximation, x_F and y_F can be computed as follows:

$$\begin{aligned} f(x_F, 0) \doteq c &\iff \lambda_1 \frac{x_F^2}{2} h_0 \doteq c, \\ f(0, y_F) \doteq c &\iff \lambda_2 \frac{y_F^2}{2} h_0 \doteq c, \end{aligned} \tag{23}$$

leading to the ratio

$$r_F = \frac{x_F}{y_F} \doteq \sqrt{\frac{\lambda_2}{\lambda_1}}. \tag{24}$$

(2) Computation of r_C: First, we compute $\hat{f}$ with (3) and (17):

$$\hat{f}(x,y) = \left[\lambda_1(1-\cos(x))\left(\frac{3}{2}+2\cos(y)+\frac{1}{2}\cos(2y)\right)\right.$$
$$\left.+\frac{1}{2}\lambda_2\left(1+\cos(y)-\cos(2y)-\cos(y)\cos(2y)\right)\right]\cdot h(x,y). \tag{25}$$

With the abbreviations $\tilde{h}_0 := h(x, y/2)$ and $\tilde{h}_\pi := h(x, y/2+\pi)$ we obtain f_2 from (20) and (25):

$$f_2(x,y) = \frac{\lambda_1}{2}(1-\cos(x))\left(\frac{3}{2}(\tilde{h}_0+\tilde{h}_\pi)+2\cos\left(\frac{y}{2}\right)(\tilde{h}_0-\tilde{h}_\pi)+\frac{1}{2}\cos(y)(\tilde{h}_0+\tilde{h}_\pi)\right)$$
$$+\frac{\lambda_2}{4}\left((\tilde{h}_0+\tilde{h}_\pi)+\cos\left(\frac{y}{2}\right)(\tilde{h}_0-\tilde{h}_\pi)-\cos(y)(\tilde{h}_0+\tilde{h}_\pi)\right.$$
$$\left.-\cos\left(\frac{y}{2}\right)\cos(y)(\tilde{h}_0-\tilde{h}_\pi)\right). \tag{26}$$

With Taylor expansion of the cosine terms at $(0,0)$, and with approximation of $\tilde{h}_0$ and $\tilde{h}_\pi$ by h_0 and h_π this becomes

$$f_2(x,y) \doteq \frac{\lambda_1 x^2}{4}\left(\frac{3}{2}(h_0+h_\pi)+2(h_0-h_\pi)+\frac{1}{2}(h_0+h_\pi)\right)$$
$$+\frac{\lambda_2 y^2}{4}\left((h_0+h_\pi)+\left(1-\frac{y^2}{8}\right)(h_0-h_\pi)-\left(1-\frac{y^2}{2}\right)(h_0+h_\pi)\right.$$
$$\left.-\left(1-\frac{y^2}{2}-\frac{y^2}{8}\right)(h_0-h_\pi)\right)$$
$$=\lambda_1 x^2 h_0+\frac{\lambda_2}{4}y^2 h_0, \tag{27}$$

With this approximation of f_2 we can compute r_C in the same way as r_F above:

$$f_2(x_C,0)\doteq c \iff \lambda_1 x_C^2 h_0 \doteq c,$$
$$f_2(0,y_C)\doteq c \iff \frac{\lambda_2}{4}y_C^2 h_0 \doteq c, \tag{28}$$
$$r_C=\frac{x_C}{y_C}\doteq\sqrt{\frac{\lambda_2}{4\lambda_1}}\doteq\frac{1}{2}r_F. \qquad \square$$

4.2. A multilevel method

We wish to develop a multilevel method for the solution of anisotropic BTTB systems, which combines semicoarsening and full coarsening steps. Therefore, we have to state a criterion for the choice between the two different coarsening stategies. The basic idea is simple: Since we know that the system becomes less anisotropic with each semicoarsening step, we want to apply semicoarsening until the system is not anisotropic anymore, and then switch to full coarsening. So

far, we have proved that the first semicoarsening step reduces the degree of anisotropy by a factor of 2. This shall now be generalized to more than two levels by using the result from the previous subsection on generating functions and their level curves. If the matrix $T_{mn}[f]$ is anisotropic, level curves are flat. The following theorem shows that Theorem 2 can be applied recursively, and therefore that each semicoarsening step reduces the ratio $\frac{x_F}{y_F}$ exactly by a factor 2.

Theorem 3. *Let f be an nonnegative generating function with a zero of order 2 at the origin which is of the form (21). Let $r_F = \frac{x_F}{y_F}$ be the ratio of the intersection points on the finest level.*

Then, the function f_2 obtained after one semicoarsening step is also of the form (21). Therefore, Theorem 2 can be applied recursively, and each semicoarsening step reduces r_F by a factor 2 for small c.

Proof. The coarse grid function $f_2(x, y)$, which has been computed in (26) in the proof of Theorem 2, can be slightly rewritten:

$$f_2(x, y) = \lambda_1(1 - \cos(x))\left(\frac{3}{4}(\tilde{h}_0 + \tilde{h}_\pi) + \cos\left(\frac{y}{2}\right)(\tilde{h}_0 - \tilde{h}_\pi) + \frac{1}{4}\cos(y)(\tilde{h}_0 + \tilde{h}_\pi)\right) + \frac{\lambda_2}{4}(1 - \cos(y))\left((\tilde{h}_0 + \tilde{h}_\pi) + \cos\left(\frac{y}{2}\right)(\tilde{h}_0 - \tilde{h}_\pi)\right). \quad (29)$$

In the neighborhood of the origin, f_2 is approximated by Taylor expansion and by replacing $\tilde{h}_0$ and $\tilde{h}_\pi$ by h_0 and h_π:

$$f_2(x, y) \doteq \lambda_1(1 - \cos(x))2h_0 + \frac{\lambda_2}{4}(1 - \cos(y))2h_0. \quad (30)$$

Thus, f_2 is of the form (21), and Theorem 2 can be applied recursively. □

Remark 3. Theorem 3 has the consequence that level curves become less flat on each level. If we start with the ratio r_F on the finest level, we need $\log_2(r_F)$ semicoarsening steps until the level curves are almost like circles, i.e., until the ratio is close to 1.

To illustrate this coarsening strategy let us again look at level curves of f from Example 1 for $\alpha = 0.01$. Since the ratio r_F is 10 in the beginning, and since it is divided by two in each step, we have almost reached a circle after three steps. Our heuristic suggests that we use three semicoarsening steps and then, if we wish to build a preconditioner with more levels, use standard coarsening for the remaining steps. In our numerical experiments we test different coarsening strategies on the matrices $T_{mn}[f]$ with f from Example 1. The first (denoted y,xy,xy) consists of one semicoarsening step, followed by two full coarsening steps, the second (y,y,y,xy) of three semicoarsening steps and one full coarsening steps, and the third (y,y,y,y,y) of five semicoarsening steps. Each time, the matrix T_C on the coarsest level is of size $\frac{n^2}{32}$-by-$\frac{n^2}{32}$. For all the experiments in this section we use the damped Jacobi smoother. The first part of Table 1 shows that our heuristic works indeed very well. If $\alpha = 0.001$, five semicoarsening steps should be performed to obtain satisfactory results, because r_F is about 30. Again, the numerical results, which are shown in the second part of Table 1, confirm that this is the best choice. Similar results are obtained for the dense matrices $T_{mn}[f]$ from Example 2. Iteration numbers are almost the same as in Table 1, since level curves are very similar to the ones of the first example.

Table 1
Iteration numbers for $T_{nn}[f]$ corresponding to f from Example 1 with $\alpha = 0.01$ and $\alpha = 0.001$

Coarsening	$n = 2^6 - 1$	$n = 2^7 - 1$	$n = 2^8 - 1$
$f(x, y) = 0.01 \cdot (1 - \cos(x)) + (1 - \cos(y))$			
y,xy,xy	63	65	66
y,y,y,xy	16	17	17
y,y,y,y,y	20	20	19
$f(x, y) = 0.001 \cdot (1 - \cos(x)) + (1 - \cos(y))$			
y,xy,xy	125	181	>200
y,y,y,xy	32	45	50
y,y,y,y,y	15	15	15

4.3. Convergence results

In this section, the convergence proofs of [5,12] shall be carried over to anisotropic BTTB systems. For the convergence proofs let us assume that anisotropy occurs along the y-axis, because notation is slightly simpler in this case. Anisotropy along the x-axis is treated similarly. If anisotropy occurs along the y-axis, i.e., if $f(x, y)$ is small for $x = 0$ and all $y \in [0, 2\pi]$, coarsening is done only in x. Instead of (17) and (18) we use $b(x, y) = 1 + \cos(x)$ and

$$P_{mn}^T = I_m \otimes P_n^T \quad \text{with } P_n^T = \begin{pmatrix} 0.5 & 1 & 0.5 & & \\ & & 0.5 & 1 & 0.5 \\ & & & \ddots & \ddots \end{pmatrix}. \tag{31}$$

In this case, f is allowed to be zero on the whole line $x = 0$. Thus, (11) is replaced by

$$\min_{(x,y)\in[-\pi,\pi]^2} \frac{f(x, y)}{1 - \cos x} = C > 0. \tag{32}$$

The following theorem proves convergence of the two-level method.

Theorem 4. *Let $T_{mn}[f]$ be a positive definite BTTB matrix whose generating function is real-valued even and satisfies (32). Let $t_{0,0}$ denote the entries in its main diagonal. Moreover, let the prolongation matrix P_{mn}^T be given by (31), and let the smoother be the damped Jacobi method.*

Then, the convergence factor of the two-level method is uniformly bounded below 1 independent of m and n. The following estimate for the convergence factor holds:

$$\|G\|_1 \leqslant \sqrt{1 - \frac{C}{2\rho(T_{mn}[f])}} \tag{33}$$

Proof. The proof of the smoothing condition (8) is the same as in [5,12]. Therefore, we only have to prove the correcting condition (9). First assume that $n = 2k + 1$ with k being the size of the blocks on the coarse level. Following [5] we define for any

$$e = (e_1, e_2, \ldots, e_m)^T = (e_{1,1}, \ldots, e_{1,n}, e_{2,1}, \ldots, e_{2,n}, \ldots, e_{m,1}, \ldots, e_{m,n})^T \in \mathbb{R}^{mn}$$

the vector

$$e_C = (\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_m)^T = (\tilde{e}_{1,1}, \ldots, \tilde{e}_{1,k}, \tilde{e}_{2,1}, \ldots, \tilde{e}_{2,k}, \ldots, \tilde{e}_{m,1}, \ldots, \tilde{e}_{m,k})^T \in \mathbb{R}^{mk},$$

where $\tilde{e}_{i,j} = e_{i,2j}$. If $j \leqslant 0$ or $j > n$, then we set $e_{i,j} = 0$ in order to complete the notation. For this special choice of e_C we try to find an upper bound for $\|e - P_{mn}^T e_C\|_0^2$ of the form $\beta\|e\|_1$ with β independent of e. Then the correcting condition would follow immediately. With P_{mn}^T from (31), the following upper bound is found:

$$\|e - P_{mn}^T e_C\|_0^2 = \|e - (I_m \otimes P_n^T)e_C\|_0^2 = \sum_{i=1}^{m} \|e_i - P_n^T \tilde{e}_i\|_0^2$$

$$\leqslant \sum_{i=1}^{m} t_{0,0}\langle e_i, T_n[1 - \cos(x)]e_i\rangle = t_{0,0}\langle e, (I_m \otimes T_n[1 - \cos(x)])e\rangle. \quad (34)$$

The m one-dimensional inequalities hold because of a result in [5]. It remains to show that there exists a β independent of e such that

$$t_{0,0}\langle e, (I_m \otimes T_n[1 - \cos x])e\rangle \leqslant \beta\langle e, T_{mn}[f]e\rangle, \quad \forall e \in \mathbb{R}^{mn}. \quad (35)$$

Because of the comments following Definition 1, condition (32) has the consequence

$$C \cdot (I_m \otimes T_n[1 - \cos x]) \leqslant T_{mn}[f]. \quad (36)$$

This implies that (35) is satisfied with

$$\beta = \frac{t_{0,0}}{C},$$

and the correcting condition is proved for the case $n = 2k + 1$. For $n = 2k$ the vector e is embedded into the vector $\hat{e}$ of size $m\hat{n} = m(2k + 1)$ by filling zeros into the additional positions. Then the correction condition also holds because of

$$\|e - P_{mn}^T e_C\|_0^2 \leqslant \|\hat{e} - \hat{P}_{m\hat{n}}^T e_C\|_0^2 \quad (37)$$

and

$$\langle \hat{e}, I_m \otimes T_{\hat{n}}[1 - \cos x]\hat{e}\rangle = \langle e, I_m \otimes T_n[1 - \cos x]e\rangle. \qquad \square$$

To obtain a result for the multilevel method we prove that if (32) holds on some level, it also holds on the next coarser level after one semicoarsening step. Again, the proof is obtained by extending the one from [12]. Let T^h and T^H denote the BTTB matrices on the finer and on the coarser level, and n_h and n_H the respective block sizes. Since semicoarsening is used, the number of blocks is constant on all levels. Furthermore, $t_{0,0}^h$ and $t_{0,0}^H$ are the main diagonal entries of T^h and T^H.

Theorem 5. *Let T^h be a positive definite BTTB matrix of size mn_h satisfying*

$$T^h \geqslant \frac{t_{0,0}^h}{\beta^h} T_{mn_h}[1 - \cos x] \quad (38)$$

for some β^h from (9) independent of mn_h, and let the restriction be defined with (31). Then

$$T^H \geqslant \frac{t_{0,0}^H}{\beta^H} T_{mn_H}[1 - \cos x] \quad (39)$$

with

$$\beta^H = 2\frac{t_{0,0}^H \beta^h}{t_{0,0}^h}. \quad (40)$$

This implies that Theorem 4 can be applied on each level, stating that the correcting condition is also satisfied on coarser levels. If q levels are used, the following estimate for the convergence factor holds:

$$\|G^q\|_1 \leqslant \sqrt{1-\frac{\alpha^q}{\beta^q}} = \sqrt{1-\frac{\alpha^h}{4^{q-1}\beta^h}}. \tag{41}$$

Proof. Define the (n_H+1)-by-n_H matrix

$$K = \frac{1}{2}\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & \ddots & \ddots \end{bmatrix}.$$

Then there exists a permutation matrix Q such that

$$Q \cdot P_{mn_h}^T = I_m \otimes \begin{pmatrix} I_{n_H} \\ K \end{pmatrix} \tag{42}$$

and

$$Q \cdot T_{mn_h}[1-\cos x] \cdot Q^T = I_m \otimes \begin{pmatrix} I_{n_H} & -K \\ -K^T & I_{n_H+1} \end{pmatrix}. \tag{43}$$

With these prerequisites we can derive the lower bound (39) for T^H. By (32) and (38) we have

$$T^H = P_{mn_h} T^h P_{mn_h}^T \geqslant \frac{t_{0,0}^h}{\beta^h} P_{mn_h} T_{mn_h}[1-\cos x] P_{mn_h}^T. \tag{44}$$

With (42) and (43) the right-hand side can be simplified in the following way:

$$\begin{aligned} &\frac{t_{0,0}^h}{\beta^h}\left[I_m \otimes \left(\left(I_{n_H}, K^T\right)\begin{pmatrix} I_{n_H} & -K \\ -K^T & I_{n_H+1} \end{pmatrix}\begin{pmatrix} I_{n_H} \\ K \end{pmatrix}\right)\right] \\ &\quad = \frac{t_{0,0}^h}{\beta^h}\left[I_m \otimes (I_{n_H} - K^T K)\right] = \frac{t_{0,0}^h}{2\beta^h}\left[I_m \otimes T_{n_H}[1-\cos x]\right] \\ &\quad = \frac{t_{0,0}^h}{2\beta^h} T_{mn_H}[1-\cos x], \end{aligned} \tag{45}$$

where the third line follows from the second by the definition of K. (39) and (40) are immediate consequences of (45). □

Remark 4. If the anisotropy is moderate, our heuristic suggests the use of a multigrid method which consists of some semicoarsening steps followed by full coarsening on the coarser levels. In this case, we can combine the convergence results on full coarsening from [12] with our results. This is done by computing β^H in (40) as in [12] if full coarsening is used at some level. As a consequence we obtain an estimate such as (41).

5. Anisotropy in other directions

In the previous section, we have developed multigrid methods for anisotropic problems where anisotropy occurs along coordinate axes. Generating functions and their level curves were used for a theoretical analysis of the methods. Now the same techniques shall also be used in the more general case where anisotropy occurs in other directions. The following two functions illustrate what further problems arise from such matrices.

Example 3. Let $T_{mn}[f]$ and $T_{mn}[g]$ be the BTTB matrices corresponding to the functions

$$\begin{aligned} f(x, y) &= \alpha \cdot (1 - \cos(x + y)) + (1 - \cos(x - y)), \\ g(x, y) &= (1 - \cos(2x + y)) + \alpha \cdot (1 - \cos(x - 2y)). \end{aligned} \tag{46}$$

The left picture in Fig. 3 shows how the two functions behave in the neighborhood of their zero at the origin, i.e., with what kind of anisotropy we have to deal with. f is anisotropic along the line $y = x$, which means it is rotated by an angle of 45° from the x-axis. The anisotropy of g occurs along $y = -2x$, which corresponds to an angle of 30° from the y-axis.

In both cases neither standard multigrid nor the methods from Section 4 work properly, and for $\alpha \to 0$ they fail completely. Semicoarsening along an axis does not help to treat anisotropy in other directions well. Moreover, we have to take into account the third problem mentioned in Remark 2. $f(x, y)$ has another zero at (π, π), independent of α, which is another obstacle to a convergent multigrid method, see Remark 1. In the following, we try to modify the ideas from Section 4 in order to get rid of all three problems from Remark 2. This shall be done in the following way:

- Define a new coordinate system (s, t) such that anisotropy occurs along one of the new axes.
- Apply either semicoarsening with a standard smoother or full coarsening with a line smoother in the appropriate direction.
- BTTB structure with blocks of equal size will be lost, but we can generalize the methods from the previous section such that they work for these matrices as well.

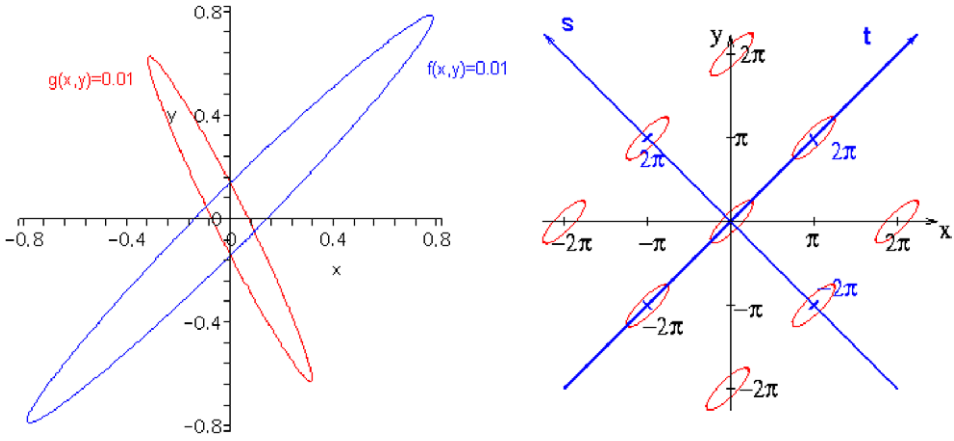

Fig. 3. Level curves $f(x, y) = 0.01$ and $g(x, y) = 0.01$ for the functions from Example 3 with $\alpha = 0.01$ (left); introduction of new coordinates for $f(x, y)$ (right).

5.1. Semicoarsening in the 45° case

We start with the case where anisotropy occurs in an angle of 45° to the coordinate axes, i.e., along the line $y = x$ or $y = -x$. With this case we can illustrate all ideas which are important for the design of a multilevel method for angles of the form $\frac{k}{k+l} \cdot 90°$. The new coordinate system (s, t) must be chosen such that anisotropy is in s- or t-direction. For anisotropy along the line $y = x$ this is done by defining

$$s := x - y \quad \text{and} \quad t := x + y. \tag{47}$$

Then, the function f from Example 3 becomes

$$f(s, t) = (1 - \cos(s)) + \alpha \cdot (1 - \cos(t)). \tag{48}$$

The right picture of Fig. 3 illustrates the consequences of this transformation. For $(s, t) \in [-\pi, \pi[^2$, the function f has only one zero, and anisotropy occurs along the t-axis. Thus, we wish to apply a multilevel method which combines semicoarsening and full coarsening similar to the one from Section 4. Some semicoarsening steps shall be performed to reduce the degree of anisotropy. Then, full coarsening steps are applied to reduce the size of the coarse level matrix. This shall be described with generating functions in the same way as it was done in Section 4. Semicoarsening must be performed in s-direction, for example with the function

$$b(s, t) = 1 + \cos(s). \tag{49}$$

If full coarsening should be applied, the simplest choice for b is

$$b(s, t) = (1 + \cos(s)) \cdot (1 + \cos(t)). \tag{50}$$

In both cases $\hat{b}$ is computed with

$$\hat{f}(s, t) = f(s, t) \cdot b(s, t)^2. \tag{51}$$

Elementary projection within a semicoarsening step is done with

$$f_2(s, t) = \frac{1}{2} \cdot \left[\hat{f}\left(\frac{s}{2}, t\right) + \hat{f}\left(\frac{s}{2} + \pi, t\right) \right], \tag{52}$$

whereas within a full coarsening step with

$$f_2(s, t) = \frac{1}{4} \cdot \left[\hat{f}\left(\frac{s}{2}, \frac{t}{2}\right) + \hat{f}\left(\frac{s}{2} + \pi, \frac{t}{2}\right) + \hat{f}\left(\frac{s}{2}, \frac{t}{2} + \pi\right) + \hat{f}\left(\frac{s}{2} + \pi, \frac{t}{2} + \pi\right) \right]. \tag{53}$$

So far, we have used generating functions to develop our multigrid method. Now these ideas must be translated into matrices. To simplify notation, we choose $m = n$. We start with a two-grid method which consists of one semicoarsening step in s-direction. This step can be divided into three parts.

(1) Defining new coordinates corresponds to permuting rows and columns of $T_{nn}[f]$ and partitioning the resulting matrix into blocks. Since anisotropy occurs in an angle of 45°, this is done as shown in the left picture of Fig. 4. Each block of the matrix corresponds to one diagonal in the picture. This means permutation must be done by the permutation vector

$$\begin{aligned}&(1, 2, n + 1, 3, n + 2, 2n + 1, \ldots, n, n + n - 1, 2n + n - 2, \ldots,\\ &\quad (n - 1)n + 1, \ldots, (n - 1)n, n^2 - 1, n^2),\end{aligned} \tag{54}$$

and the blocks are of size $1, 2, 3, \ldots, n-1, n, n-1, \ldots, 2, 1$. The resulting matrix is denoted $\tilde{T}$.

(2) The matrix B_S must be chosen corresponding to $b(s,t)$ from (49). With the ordering obtained after applying (54), B_S is the block diagonal matrix

$$B_S = \mathrm{diag}(B_1, B_2, \ldots, B_n, \ldots, B_2, B_1), \tag{55}$$

where $B_1 = 1$ and all other blocks are $B_k = \mathrm{tridiag}_k(0.5, 1, 0.5)$ of size k.

(3) The coarse grid matix T_C is again computed by applying elementary projection to the matrix $\hat{T} = B_S \cdot \tilde{T} \cdot B_S$. This means we leave the number of blocks unchanged, and within each block we pick every second row and every second column.

This semicoarsening step in s-direction reduces the degree of anisotropy in the same way as one semicoarsening step in y-direction did in Section 4. This is because after defining the new coordinates we carry out the same calculations in s and t as we did in x and y. The result is summarized in the following theorem. The positive numbers s_F and t_F are the points where s- and t-axis are intersected by the level curves $f(s,t) = c$ for some small positive c.

Theorem 6. *Let f be a nonnegative generating function with a zero of order 2 at the origin which is of the form*

$$f(s,t) = [\lambda_1(1 - \cos(s)) + \lambda_2(1 - \cos(t))] \cdot h(s,t) \tag{56}$$

with the trigonometric polynomial $h(s,t) > 0$ and $\lambda_1, \lambda_2 > 0$. Let f_2 be the coarse level function obtained by one semicoarsening step with b from (49). Let $r_F = \frac{s_F}{t_F}$ and $r_C = \frac{s_C}{t_C}$ be the ratios described above for f and f_2, respectively. Then the degree of anisotropy is reduced by a factor 2, i.e., $\frac{r_F}{r_C} \to 2$ for $c \to 0$. Since the coarse level function is of the form (56), if only terms of order at most 2 are considered in the Taylor expansion, the two-level result can be applied recursively. Then, the degree of anisotropy is reduced by a factor 2 on each level.

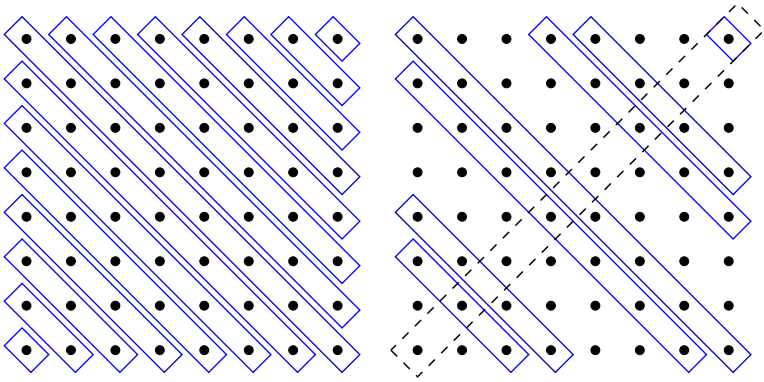

Fig. 4. Partitioning of the original matrix into blocks for semicoarsening (left) and full coarsening (right).

5.2. Full coarsening and the development of a multilevel method

Since we wish to develop a multilevel method as a combination of semicoarsening steps in s and full coarsening steps, we must first describe how a single full coarsening step in s and t is carried out. A two-level method with full coarsening is defined by translating (50) and (53) into matrices. Permutation of $T_{mn}[f]$ and partitioning of $\tilde{T}$ are done in the same way as for a semicoarsening step. The matrix

$$B_F = B_S + B_T \tag{57}$$

is chosen to be the sum of B_S from (55) and B_T, which is defined as

$$B_T = \begin{pmatrix} 0 & B_{1,2} & & & & \\ B_{2,1} & 0 & B_{2,3} & & & \\ & B_{3,2} & 0 & B_{3,4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{2,3} & 0 & B_{2,1} \\ & & & & B_{1,2} & 0 \end{pmatrix}, \tag{58}$$

where $B_{k,k+1}$ is the matrix $tridiag(0.25, 0.5, 0.25)$ of size k-by-$(k+1)$, and $B_{k+1,k}$ the same matrix of size $(k+1)$-by-k. After $\hat{T} = B_F \cdot \tilde{T} \cdot B_F$ is computed, T_C is obtained by elementary projection. Within each block we pick every second row and every second column. On the block level, we pick two rows, eliminate the next two, pick another two rows and so on. This procedure is shown in the right picture of Fig. 4. The solid lines mark the blocks, which are retained on the coarser level, whereas the other blocks are eliminated. The dashed line explains why precisely these blocks have to be chosen. Elimination on the block level must be done such that within the dashed line every second element is retained and the other elements are eliminated.

We now define a multilevel method similar to the one from Section 4, i.e., as a suitable combination of semicoarsening steps followed by some full coarsening steps. The prolongation/restriction matrices and the elementary projection matrices are defined as described above, the change of coordinates has of course only to be done before the first step. Again, we use the same heuristic as in Section 4. Theorem 6 states that the ratio r_F is reduced by a factor 2 in each semicoarsening step. Therefore, semicoarsening steps are applied until level curves are close to circles, i.e., until r_F is almost 1. Then we continue with full coarsening. We wish to test our multilevel method with the function $f(x, y)$ from Example 3, where α takes the values 0.01 and 0.001. In the first case, our heuristic suggests to use three semicoarsening steps, whereas in the second case at least five semicoarsening steps should be used. The numerical results confirm that the method works well. Table 2 shows that the number of iterations is similar to the simpler case where anisotropy occurs along coordinate axes.

5.3. 30° and other directions

The case where anisotropy occurs in an angle of 45° with respect to the coordinate axes is best suited to explain our method. Although systems with this angle arise in many applications, this is not the only important case. Therefore, we want to describe how to solve systems where anisotropy occurs in other directions. The function $g(x, y)$ from Example 3 is anisotropic in an angle of 30° with respect to the y-axis. Furthermore, it has the following zeros in the interval $[-\pi, \pi[^2$:

Table 2
Iteration numbers for $T_{mn}[f]$ corresponding to f from Example 3 with $\alpha = 0.01$ and $\alpha = 0.001$

Coarsening	$n = 2^6 - 1$	$n = 2^7 - 1$	$n = 2^8 - 1$
$f(x, y) = 0.01 \cdot (1 - \cos(x + y)) + (1 - \cos(x - y))$			
s,st,st	43	45	45
s,s,s,st	17	18	18
s,s,s,s,s	21	21	21
$f(x, y) = 0.001 \cdot (1 - \cos(x + y)) + (1 - \cos(x - y))$			
s,st,st	82	97	104
s,s,s,st	28	32	34
s,s,s,s,s	17	17	17

$$(0, 0), \quad \left(\frac{2}{5}\pi, -\frac{4}{5}\pi\right), \quad \left(\frac{4}{5}\pi, \frac{2}{5}\pi\right), \quad \left(-\frac{4}{5}\pi, -\frac{2}{5}\pi\right), \quad \left(-\frac{2}{5}\pi, \frac{4}{5}\pi\right). \tag{59}$$

Again, we wish to define a multilevel method as a combination of semicoarsening and full coarsening steps. As in the 45° case we have to define new coordinates s and t such that anisotropy occurs along coordinate axes and then apply coarsening along s and t. For a problem with an angle of 30° towards the y-axis such as $g(x, y)$ from Example 3 the new coordinates are $s := 2x + y$ and $t := -x + 2y$. Then we choose $b(s, t)$ either as in (49) or in (50), and proceed as in the 45° case. The coordinate transformation is translated into matrices by permuting rows and columns and then partitioning the matrix into blocks. The grid points in Fig. 5 which are highlighted by a solid circle show how two example blocks of the matrix are built. Since we have anisotropy in an angle of 30°, the points corresponding to one block are obtained by moving two steps in x-direction and one step in y-direction. The matrices B_S and B_T are defined as in (55) and (58), just the size of the blocks is different. Elementary projection within a semicoarsening step is done exactly as in the 45° case by eliminating every second row and column within each block, leaving the number of blocks the same. If we apply full coarsening, we eliminate every second row and column within each block, and on the block level, eliminate five consecutive block rows, then pick the next five block rows, eliminate five block rows and so on. The reason for this is explained in Fig. 5, where we have to eliminate every second grid point with a dashed circle. This is the equivalent to the dashed line in Fig. 4 (right). There are three other directions where anisotropy occurs in an angle of 30° to one of the coordinate axes. Each of them is treated as the one we have described here by an appropriate choice of s and t.

Finally, let us take a look at anisotropy which occurs yet in other directions. Functions of the form

$$f(x, y) = \alpha \cdot (1 - \cos(k \cdot x + l \cdot y)) + (1 - \cos(l \cdot x - k \cdot y)) \tag{60}$$

are examples representing classes of problems where anisotropy occurs in an angle of $\frac{k}{k+l} \cdot 90°$ to one of the coordinate axes. In this general case, transformation to new coordinates is done with

$$s := k \cdot x + l \cdot y \quad \text{and} \quad t := l \cdot x - k \cdot y. \tag{61}$$

If $|k|$ and $|l|$ are small, our method works very well for these systems. However, this approach is limited to small $|k|$ and $|l|$, because the block sizes which have to be used in the multigrid method become large if $|k|$ and $|l|$ increase. For $|k| = |l| = 1$, the size was 2, for $|k| = 1$, $|l| = 2$, it was 5, and in general it is $\det \begin{pmatrix} l & -k \\ k & l \end{pmatrix}$, i.e., $k^2 + l^2$. Table 3 illustrates what blocksize we have to use

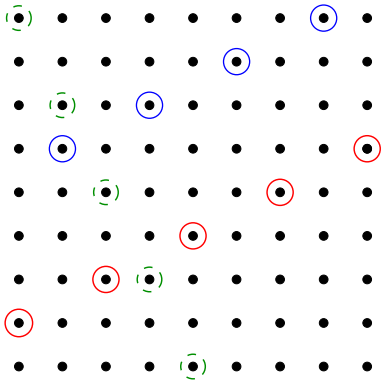

Fig. 5. Partitioning of the original matrix corresponding to g into blocks.

Table 3
Angles and block size for different choices of k and l in (60)

k	l	Block size	Angle (deg)
1	2	5	30
1	3	10	22.5
1	4	17	18
1	5	26	15
2	3	13	36

and what angle we get for different choices of k and l. For each line in Table 3 four problems with different angles can be obtained by interchanging k and l, and by moving α to the term $(1 - \cos(l \cdot x - k \cdot y))$ in (60). Most other angles lead to a block size which becomes too large for practical computations. In theory however, if the size of $T_{mn}[f]$ is large enough, any rational angle can be described by k and l, allowing our method to be applied. Thus, we suggest to treat the given problem as if anisotropy occured along a direction from (60) with a similar angle.

5.4. Convergence results

Finally, we wish to give convergence results for systems with anisotropy in arbitrary directions. Let us assume that anisotropy of a matrix $T_{mn}[f]$ occurs in a direction where the new coordinates for our method can be defined with (61). We prove the following theorem for the case where f is anisotropic along the t-axis. It is required that in the new coordinates, f satisfies the condition

$$\min_{(s,t)\in[-\pi,\pi]^2} \frac{f(s,t)}{1-\cos s} = C > 0, \tag{62}$$

The coordinate transformation with (61) corresponds to a permutation of the rows and columns of $T_{mn}[f]$ with a vector *perm* such as the one in (54), i.e., $\tilde{T} = T_{mn}[f](perm, perm)$. The inverse permutation is defined by the vector *iperm*. The restriction matrix for a semicoarsening step in s-direction is defined as the product of B_S from (55) and an elementary projection matrix. Then, we can prove this more general version of the two-level result from Theorem 4.

Theorem 7. *Let $T_{mn}[f]$ be a positive definite BTTB matrix whose generating function $f(x, y)$ is real-valued even. Assume that introduction of new coordinates s and t by (61) leads to a generating function $f(s, t)$ which satisfies (62). Let $\tilde{T}$ be the permuted matrix defined above. Furthermore, let the restriction matrix $\tilde{P}$ be constructed with B from (55), and let the smoother be the damped Jacobi method.*

Then, the correcting condition (9) is satisfied for $\tilde{T}$ and $\tilde{P}$, and the convergence factor of the two-level method is uniformly bounded from below 1 independently of n.

Proof. The proof is similar to the one of Theorem 4, but this time we have to consider blocks of variable size. First, assume that all diagonal blocks are of odd size, i.e., $n_j = 2k_j + 1$ for some integer k_j. For any

$$e = (e_{1,1}, \ldots, e_{1,n_1}, e_{2,1}, \ldots, e_{2,n_2}, \ldots, e_{b,1}, \ldots, e_{b,n_b})^T$$

we define

$$e_C = (\tilde{e}_{1,1}, \ldots, \tilde{e}_{1,k_1}, \tilde{e}_{2,1}, \ldots, \tilde{e}_{2,k_2}, \ldots, \tilde{e}_{b,1}, \ldots, \tilde{e}_{b,k_b})^T,$$

where $\tilde{e}_{i,j} = e_{i,2j}$. If $j \leqslant 0$ or $j > n_k$, then we set $e_{i,j} = 0$ in order to complete the notation. For this special choice of e_C we try to find an upper bound for $\|e - \tilde{P}^T e_C\|_0^2$ of the form $\beta\|e\|_1$ with β independent of e. Similar to [12] we obtain

$$\begin{aligned}\left\|e - \tilde{P}^T e_C\right\|_0^2 &= t_{0,0} \sum_{i=1}^{b} \sum_{j=0}^{k_i} \left\{ e_{i,2j+1} - \frac{1}{2} e_{i,2j+2} - \frac{1}{2} e_{i,2j} \right\}^2 \\ &\leqslant t_{0,0} \sum_{i=1}^{b} \sum_{j=0}^{n_i} (e_{i,j}^2 - e_{i,j} e_{i,j+1}) = t_{0,0} \langle e, \operatorname{diag}(T_{n_1}, \ldots, T_{n_b}) \cdot e \rangle \end{aligned} \tag{63}$$

with $T_{n_j} = T_{n_j}[1 - \cos(s)]$. Again, we have to find a parameter β independent of e such that

$$t_{0,0} \langle e, \operatorname{diag}(T_{n_1}, \ldots, T_{n_b}) \cdot e \rangle \leqslant \beta \langle e, \tilde{T} e \rangle, \quad \forall e \in \mathbb{R}^{mn}. \tag{64}$$

First, we permute the left-hand side back to x- and y-coordinates with the vector *iperm*, i.e., $e_{iperm} = e(iperm)$ and

$$T_{mn}[1 - \cos(kx + ly)] = \operatorname{diag}(T_{n_1}, \ldots, T_{n_b})(iperm, iperm).$$

As in the proof of Theorem 4 the inequality in the following expression is a consequence of (62):

$$\begin{aligned} t_{0,0} \langle e, \operatorname{diag}(T_{n_1}, \ldots, T_{n_b}) \cdot e \rangle &= t_{0,0} \langle e_{iperm}, T_{mn}[1 - \cos(kx + ly)] \cdot e_{iperm} \rangle \\ &\leqslant t_{0,0} \langle e_{iperm}, T_{mn}[f(x, y)] \cdot e_{iperm} \rangle = t_{0,0} \langle e, \tilde{T} e \rangle. \end{aligned} \tag{65}$$

The last equality is obtained by permutation with the vector *perm*, i.e., by transformation to s- and t-coordinates. From (65) we obtain the parameter $\beta = \frac{t_{0,0}}{C}$ in (64).

Finally, we must get rid of the assumption that the block sizes be odd. Let us therefore assume that the j-th block is of even size n_j. The vector $(e_{j,1}, \ldots, e_{j,n_j})$ is embedded into a vector of size $n_j + 1$ by filling 0 into the additional position. If this is done for all parts of e corresponding to a block of even size, we obtain a vector $\hat{e}$ which is slightly larger than e. Then, with (37) and

$$\langle \hat{e}, \operatorname{diag}(T_{\hat{n}_1}, \ldots, T_{\hat{n}_b}) \hat{e} \rangle = \langle e, \operatorname{diag}(T_{n_1}, \ldots, T_{n_b}) e \rangle.$$

the correcting condition still holds. □

6. Conclusions

This article was devoted to the analysis of anisotropic problems in the context of BTTB systems, generating functions, and their level curves. This point of view allows the development of multilevel methods also for systems where anisotropy occurs in arbitrary directions and not only along coordinate axes. These more complicated systems are solved with the same efficiency. We have not considered functions where the zero is of higher order than 2. These can be solved with prolongation matrices which are a power of the ones corresponding to (17), but theoretical results are harder to obtain (some matrix theoretic tools for the anisotropic case can be found in [3]). Moreover, V-cycle optimality for Toeplitz and BTTB systems has not yet been proved. If a zero of f is not in the origin, the corresponding matrix can be diagonally scaled to shift the zero to the origin. This also makes notation more complicated.

Several applications where anisotropic systems of both types need to be solved will be subject of future research. The most interesting of them seems to be the solution of systems corresponding to functions with a whole zero curve such as

$$f(x, y) = (\rho - \cos(x) - \cos(y))^2 (\rho < 2),$$

which arise when Helmholtz equations are solved. Multigrid preconditioners and solvers for these problems will be presented by the authors in an upcoming paper. Anisotropic systems of the form (60), which approximate the zero curve at several of its points, are essential building blocks for these methods.

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