Periodic Coxeter matrices and their associated quadratic forms

Masahisa Sato
Mathematical Section in Interdisciplinary, Graduate School of Medicine and Engineering, Yamanashi University, Kofu, Yamanashi 400-8511, Japan
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Abstract

We study relations between the periodicity of a Coxeter matrix and the positivity or the non-negativity of a quadratic form with respect to a Cartan matrix of an artinian ring. We introduce the notion of the weak periodicity of a Coxeter matrix induced by a non-negative quadratic form and we characterize the weak periodicity condition and the periodicity condition in terms of eigenvalues of a Coxeter matrix.

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1. Introduction

A Coxeter matrix with respect to a Cartan matrix of a finite dimensional hereditary algebra of finite representation type is periodic. (For details, refer to Dlab–Ringel [2].) de la Peña [1] discussed some relations between the periodicity of a Coxeter matrix and the non-negativity of a quadratic form with respect to a Cartan matrix of a triangulated algebra.
In this paper, we generalize results in [1,2] for an artinian ring with regular Cartan matrix. But our results seem to be important even for finite dimensional algebras (see for example [5]).

We study eigenvalues of a Coxeter matrix in Section 2 and we investigate relations between the periodicity of a Coxeter matrix and the non-negativity of a quadratic form in Section 3.

It is proved, in Theorem 2.8, that a positive quadratic form induces a periodic Coxeter matrix as a generalization of the work by de la Peña [1] and Dlab–Ringel [2]. But a non-negative quadratic form does not induce a periodic Coxeter matrix in general. So we give some alternative properties of the periodicity of a Coxeter matrix induced by a non-negative quadratic form in Theorem 3.4. One of them is “the weak periodicity” condition. The weak periodicity condition and the periodicity condition are characterized in Theorem 2.6 and Corollary 2.7, respectively. Particularly it is shown that their eigenvalues are primitive roots of unity, thus a Coxeter matrix corresponding to a non-negative quadratic form is a matrix representation of a primitive root of unity.

Concerning with relations between the non-negativity of a quadratic form and the periodicity of a Coxeter matrix studied in [1], we show that they are independent of each other in general by Examples 3.6 and 3.8, but for a diagonalizable Coxeter matrix, we prove that the non-negativity condition implies the periodicity condition in Theorem 3.3. For the relations between our results, refer to the last figure in this paper.

2. Coxeter matrix and quadratic form

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ be the set of complex numbers, real numbers, rational numbers and integers respectively. Throughout this paper, $\mathbf{I}$ always means the unit matrix with a suitable degree. A matrix $\mathbf{M}$ is called “weakly periodic” (resp. “periodic”) if $\mathbf{M}^k - \mathbf{I}$ is nilpotent (resp. zero) for some positive integer $k$. A Coxeter matrix is a matrix over $\mathbb{Z}$ described by $\Phi = -\mathbf{C}^{-1} \cdot \mathbf{t} \mathbf{C}$ with an $n \times n$ regular matrix $\mathbf{C}$ and its transposed matrix $\mathbf{t} \mathbf{C}$ over $\mathbb{Z}$.

In this paper, we always consider a Cartan matrix associated with an artinian ring. That is, for an artinian ring $R$, we consider all the isomorphic classes $[P_1], [P_2], \ldots, [P_n]$ of indecomposable projective modules and the corresponding isomorphic classes $[S_1], [S_2], \ldots, [S_n]$ of simple modules in the Grothendieck group $K_0(R)$ of $R$. Then the Cartan matrix $\mathbf{C}_R$ of $R$ is defined by the equation $([P_1], [P_2], \ldots, [P_n]) = ([S_1], [S_2], \ldots, [S_n]) \mathbf{C}_R$. i.e., For $\mathbf{C}_R = (c_{ij})$, $P_j$ has $c_{ij}$-many $S_i$’s as composition factors of its composition series. We denote $\mathbf{C}_R$ simply by $\mathbf{C}$.

We set a bilinear form $f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \mathbf{C} \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors in the $n$-dimensional $\mathbb{C}$-vector space $\mathbb{C}^n$. 
A quadratic form with respect to \( C \) is defined by \( q(x) = f(x, x) \). i.e., \( q(x) = \frac{1}{2} x C x = \frac{1}{2} x (C + C^T) x \). A quadratic form is said to be "non-negative" (resp. "positive") if \( q(x) \geq 0 \) (resp. \( q(x) > 0 \)) for any non-zero vectors \( x \in \mathbb{Q}^n \).

First we give an example of non-hereditary algebras with \( q(x) > 0 \).

**Example 2.1.** The following two algebras have the same Cartan matrix \( C = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \) and the Coxeter matrix \( A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} \) with \( A^3 = I \) and \( q(x) = x_1 + 3x_1x_2 + 3x_2^2 = (x_1 + \frac{3}{2} x_2)^2 + \frac{3}{4} x_2^2 > 0 \).

1. The algebra given by the following quiver with relations:

   \[
   \begin{array}{ccc}
   0 & \overset{\alpha}{\longrightarrow} & 1 \\
   & \overset{\beta}{\longleftarrow} & \\
   & \overset{\delta}{\longrightarrow} & 2 \\
   \end{array}
   \]

   \( \gamma \alpha = \gamma \beta = \delta \alpha = \delta \beta = 0, \)

   \( \beta \delta = \alpha \delta, \gamma^3 = 0, \delta \gamma = 0. \)

2. The algebra given by the following quiver with relations:

   \[
   \begin{array}{ccc}
   0 & \overset{\alpha}{\longrightarrow} & 1 \\
   & \overset{\beta}{\longleftarrow} & \\
   & \overset{\delta}{\longrightarrow} & 2 \\
   \end{array}
   \]

   \( \delta \alpha = \delta \beta = 0. \)

This algebra has the global dimension 2.

We denote \( \chi(x) = |x I - A| \) the characteristic polynomial of \( A \) and \( G \) the Galois group of the minimal splitting field of \( \chi(x) \) over \( \mathbb{Q} \).

**Proposition 2.2.** Let \( \lambda \) and \( x \) be an eigenvalue and its eigenvector of a Coxeter matrix \( A \), respectively. Then it holds that \( \lambda \lambda^\sigma = 1 \) or \( f(x, x^\sigma) = 0 \) for any \( \sigma \in G \). Particularly \( \lambda^2 = 1 \) or \( q(x) = 0 \).

**Proof.** Since \( Ax = \lambda x \), it holds \( CX = \lambda CX \). By applying \( \sigma \in G \), we have \( C x \sigma = -\lambda \sigma C x \sigma \). Then \( (C x \sigma)^\sigma = (C x)^\sigma = (Ax)^\sigma = -\lambda^\sigma x (C x \sigma) = -\lambda \sigma (C x \sigma) = \lambda \lambda^\sigma (x C x \sigma) \). Thus we get \( f(x, x^\sigma) = \lambda \lambda^\sigma f(x, x^\sigma) \). Particularly \( q(x) = \lambda^2 q(x) \) when \( \sigma = 1 \). It hold that \( \lambda \lambda^\sigma = 1 \) or \( f(x, x^\sigma) = 0 \), particularly \( \lambda^2 = 1 \) or \( q(x) = 0 \). \( \blacksquare \)

**Corollary 2.3.** If \( \lambda \neq -1 \), then \( q(x) = 0 \) for any eigenvector \( x \) of \( A \) with the eigenvalue \( \lambda \).
Remark 2.5. If \( \lambda^2 \neq 1 \), then \( q(x) = 0 \) by Proposition 2.2. Assume \( \lambda = 1 \), then \( \lambda^C x = -Cx \). i.e., \((C + 1)^C x = 0\). Thus \( q(x) = \frac{1}{2}x(C + 1)x = 0 \). □

Corollary 2.4. If \( \lambda \neq 1 \), then \( f(x, x^\sigma) = f(x^\sigma, x) = 0 \). Particularly \( q(x + x^\sigma) = 0 \).

Proof. By Proposition 2.2, \( f(x, x^\sigma) = 0 \) and \( f(x^\sigma, x) = 0 \). Since
\[
q(x + x^\sigma) = q(x) + q(x^\sigma) + f(x, x^\sigma) + f(x^\sigma, x),
\]
we have \( q(x + x^\sigma) = 0 \). □

Remark 2.5. When \( \lambda = -1 \), \( 1^C x = Cx \). i.e., \((C - 1)x = 0\). Let \( W_{-1} \) be the eigenspace with respect to \(-1\). Then \( \dim W_{-1} = \dim \{x | (C - 1)x = 0\} = n - \text{rank}(C - 1) \).

On the other hand, the rank of \( C - 1 \) is even since \( C - 1 \) is a skew symmetric matrix. So if \( n \) is odd, then \( \dim W_{-1} = 2k - 1 \geq 1 \) for some integer \( k \). In the case \( n \) is even and \( |C - 1| = 0 \), we have \( \dim W_{-1} = 2k \geq 2 \) for some integer \( k \).

Theorem 2.6. The following statements are equivalent for a Coxeter matrix \( \Phi \):

1. \( \Phi \) is weakly periodic.
2. \( |\lambda| = 1 \) for any eigenvalue \( \lambda \) of \( \Phi \).
3. \( \chi(x) = |xI - \Phi| \) is a product of cyclotomic polynomials.

Proof. (1) \( \implies \) (2). Let \( \text{Irr}(x, \lambda; \mathbb{Q}) \) be an irreducible polynomial of \( \lambda \) over \( \mathbb{Q} \). Then there are positive integers \( k \) and \( n \) such that \( \text{Irr}(x, \lambda; \mathbb{Q}) \) divides \( x^k - 1 \) over \( \mathbb{Z} \). Thus the root \( \lambda \) of \( \text{Irr}(x, \lambda; \mathbb{Q}) \) satisfies \( \lambda^k = 1 \). Hence \( |\lambda| = 1 \).

(2) \( \implies \) (1). It is well known (by Kronecher [3]) that \( \lambda \) is a root of some cyclotomic polynomial if \( \lambda \) is a root of some polynomial over \( \mathbb{Z} \) such that the leading coefficient and the absolute value of all the roots of the polynomial are 1. Hence there is some natural number \( n \) such that \( \lambda \) is a primitive \( n \)-th root of unity.

Make the Jordan form:
\[
P^{-1}\Phi P = \begin{pmatrix}
\Phi_1 & 0 & \ldots & 0 \\
0 & \Phi_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \Phi_m
\end{pmatrix}, \quad \Phi_i = \begin{pmatrix}
\lambda_i & 1 & 0 & \ldots & 0 \\
0 & \lambda_i & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \lambda_i \\
0 & 0 & \ldots & 0 & \lambda_i
\end{pmatrix}
\]

Take \( k \) as the least common multiple of \( n_{\lambda_1}, n_{\lambda_2}, \ldots, n_{\lambda_m} \), then \((P^{-1}\Phi P)^k - I = P^{-1}(\Phi \Phi^* P - I) = \) an upper triangular matrix with all the diagonals being 0. Hence \( P^{-1}(\Phi \Phi^* P - I)^n = (P^{-1}\Phi \Phi^* P - I)^n = 0 \). That is, \((\Phi \Phi^* P - I)^n = 0 \).
The equivalence of (1) and (2) holds since the multiplicity $\lambda_i$ is 1 (i.e., $\Phi_i = \lambda_i$) for every $i$ in both assertions.

We have the following corollary.

**Corollary 2.7** [1, Theorem 1.1]. The following statements are equivalent for a Coxeter matrix $\Phi$:

1. $\Phi$ is periodic.
2. $\Phi$ is diagonalizable and $|\lambda| = 1$ for any eigenvalue $\lambda$ of $\Phi$.
3. $\Phi$ is diagonalizable and $\chi(x) = |\lambda I - \Phi|$ is a product of cyclotomic polynomials.

**Proof.** The equivalence of (1) and (2) holds since the multiplicity $\lambda_i$ is 1 (i.e., $\Phi_i = \lambda_i$) for every $i$ in both assertions.

The equivalence of (2) and (3) is a direct consequence of Theorem 2.6.

In the following, $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ means the diagonal matrix with the diagonal components $\lambda_1, \lambda_2, \ldots, \lambda_n$.

**Theorem 2.8.** Let $\Phi$ be a Coxeter matrix of a Cartan matrix $C$. Assume that the quadratic form $q(x) = ^t x C x$ is a positive definite. Then it holds that:

1. $\Phi$ is periodic.
2. $|\lambda| = 1$ and $\lambda \neq 1$ for any eigenvalue $\lambda$ of $\Phi$.

**Proof.** (1) Recall that the quadratic form is given by $q(x) = ^t x C x = ^t x(C + 1C)x$. Since $C$ is a symmetric matrix, it is well known that there is a regular matrix $P$ over $\mathbb{Z}$ such that $^t P(C + 1C)P = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for some $\lambda_i \in \mathbb{Z}$.

Take any vector $y = (y_1, y_2, \ldots, y_n)$, then for a vector $x = Py \in \mathbb{Z}^n$, $q(x) = \frac{1}{2}(\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2)$. Since $q(x)$ is a positive definite by assumption, each $\lambda_i$ is a positive integer. Hence for any constant number $c$, there are only finite number of vectors $y \in \mathbb{Z}^n$ such that $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 = c$. That is, there are only finite number of vectors $x \in \frac{1}{\sqrt{c}} \mathbb{Z}^n$ such that $q(x) = c$.

We remark that $q(x) = q(\Phi x)$ since $^t x(-C \cdot (C^{-1}))C(-C^{-1}, 1C)x = ^t x C x$. We fix $i$ ($1 \leq i \leq n$) and denote $e_i$ the vector with $i$th position being 1 and otherwise 0.

Then the set $\{\Phi^m e_i | m \in \mathbb{N}\}$ is a finite set since $q(\Phi^m e_i) = q(e_i) = \lambda_i$.

Thus there is some $m_i$ such that $\Phi^{m_i} e_i = e_i$. Take the least common multiple $k$ of $m_1, \ldots, m_n$, then $\Phi^k e_i = e_i$. That is, $\Phi^k = I$. 

(3) $\implies$ (2) is clear.
(2) $|\lambda| = 1$ is clear since $\Phi$ is periodic. If $\lambda = 1$, then $q(x) = 0$ for an eigenvector $x$ of $\lambda$ by Corollary 2.3. Also $x$ is a non-zero rational vector since $x$ is a solution of the linear equation $(I - \Phi)x = 0$ with integral coefficients. This contradicts the assumption. □

3. Coxeter matrix and quadratic form

In this section, we investigate relations between the non-negativity of a quadratic form of a Cartan matrix and the periodicity of a Coxeter matrix. In the rest of this section, $\Phi$ is a Coxeter matrix and $q(x) = \langle x, Cx \rangle$ is a quadratic form corresponding to a Cartan matrix $C$.

Now we assume that the quadratic form $q(x)$ is non-negative. Then there is an $n \times n$ matrix $P$ over $\mathbb{Z}$ such that

$$^tP(C + ^tC)P = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_t, 0, \ldots, 0)$$

for some $\lambda_1, \ldots, \lambda_t \in \mathbb{N}$.

The following results are well known. (For example, refer to [7].)

**Lemma 3.1.** Let $B$ be a symmetric real matrix.

1. A quadratic form $^txBx$ is non-negative if and only if there is a real matrix $P$ such that $B = ^tPP$.
2. $^txBx = 0$ if and only if $Bx = 0$ for any vector $x$.
3. We denote $B = \begin{pmatrix} B_1 & ^t\alpha \\ \alpha & a \end{pmatrix}$. Then the quadratic form $^txBx$ is positive if and only if the quadratic form $^t\alpha yB_1y$ is positive and $|B| > 0$.

**Lemma 3.2.** Let $x$ be a vector.

1. $(C + ^tC)x = 0$ and $x \neq 0$ if and only if $x$ is an eigenvector of $\Phi$ with an eigenvalue 1.
2. $(C + ^tC)x = 0$ if and only if $^txCx = 0$.

**Proof.** (1) By the equation $C + ^tC = C(I - \Phi)$, $(C + ^tC)x = 0$ if and only if $\Phi x = x$.

(2) holds by $^txCx = \frac{1}{2}^t\alpha x(C + ^tC)x$ and Lemma 3.1. □

Let $W_1$ be the eigenspace over $\mathbb{Z}$ of $\Phi$ with respect to the eigenvalue 1. The set $\text{rad } q = \{ x \in \mathbb{Z}^n | q(x) = 0 \}$ is called a “radical” of a quadratic form $q(x)$. Then $W_1 = \text{rad } q$ by Lemma 3.2.
We summarize that
\[ W_1 = \{ x \in \mathbb{Z}^n | (C + tC)x = 0 \} = \{ x \in \mathbb{Z}^n | q(x) = 0 \} = \{ x \in \mathbb{Z}^n | x = Py, y \in \mathbb{Z}_{e_{t+1} \oplus \cdots \oplus e_n} \}. \]

Hence \( \text{rank}(\text{rad}q) = n - t \) as \( \mathbb{Z} \)-lattice.

We set \( y = y_1e_1 + \cdots + y_ne_n. \) Then we have
\[ q(x) = \frac{1}{2}(\lambda_1y_1^2 + \cdots + \lambda_ty_t^2). \]

We denote the lattice \( W = \mathbb{Z}Pe_1 \oplus \cdots \oplus \mathbb{Z}Pe_t \) whose rank is \( t. \) Then there are only finite number of vectors in \( W \) such that \( q(x) = c \) for a given constant \( c. \) Since \( \mathbb{Z}^n = W_1 \oplus W, \) we can denote \( f_i = f_i + g_i \) with \( f_i \in W_1 \) and \( g_i \in W_1 \) for each \( i = 1, \ldots, n. \)

We assume \( \Phi W_1 = W_1. \) This means that \( W_1 \) is a direct sum of eigenspaces corresponding to eigenvalues except 1. Hence the assumption is equivalent to the condition that \( \Phi \) is diagonalizable.

On the other hand, \( q(\Phi x) = q(x) \) still holds. Thus there is some \( k \) such that \( \Phi^k f_i = f_i \) for any \( i. \) Since \( \Phi g_i = g_i \) for any \( i, \) it follows that \( \Phi^k e_i = e_i \) for any \( i, \) that is, \( \Phi^k = I. \)

From the above discussion, we have the following theorem.

**Theorem 3.3.** A diagonalizable Coxeter matrix \( \Phi \) is periodic if the corresponding quadratic form \( q(x) \) is non-negative.

Furthermore we have the following theorem when we do not assume that a Coxeter matrix is diagonalizable.

**Theorem 3.4.** A Coxeter matrix \( \Phi \) is weakly periodic if the corresponding quadratic form \( q(x) \) is non-negative. Particularly \( |\lambda| = 1 \) for any eigenvalue \( \lambda \) of \( \Phi. \)

**Proof.** Let \( V \) be a direct sum of eigenspaces except the eigenvalue 1. Then \( \Phi^{k_1} : V \rightarrow V \) becomes identity for some \( k_1 \) by the same discussion as above. Hence \( \Phi^{k_2} : V \oplus W_1 \rightarrow V \oplus W_1 \) is an identity map. i.e., \( V \oplus W_1 \) is a subspace of the eigenspace with respect to the eigenvalue 1 of \( \Phi^{k_1}. \)

We take \( k \) such that the dimension of the eigenspace with respect to the eigenvalue 1 of \( \Phi^k \) is the greatest. Then we show that all the eigenvalues of \( \Phi^k \) are 1.

Let \( V' \) be the direct sum of the eigenspaces of \( \Phi^k \) except the eigenvalue 1. Assume that \( V' \) is non-zero. Since \( q(\Phi^{k_2}x) = q(x) \) still holds, we can apply the same way we discussed above. Thus there is some \( k_2 \) such that \( (\Phi^{k_2})^{k_2} : V' \rightarrow V' \) becomes
identity. Hence the dimension of the eigenspace with respect to the eigenvalue 1 of \( \Phi^k \) is greater than the one of \( \Phi^k \). This is a contradiction.

All the eigenvalues of \( \Phi^k \) are given by \( \lambda^k \) for an eigenvalue \( \lambda \) of \( \Phi \), i.e., \( \lambda^k = 1 \) for any eigenvalue \( \lambda \) of \( \Phi \). Hence we conclude the proof by Theorem 2.6. □

**Remark 3.5.** By the following example, we cannot remove the assumption that \( \Phi \) is diagonalizable in Theorem 3.3.

**Example 3.6.** We give an example of algebra whose Cartan matrix \( C \) gives a non-negative quadratic form and \( |\Phi| = 1 \), but the Coxeter matrix \( \Phi \) is neither periodic nor diagonalizable.

Let \( R \) be the algebra given by the following quiver and relations:

\[
\begin{align*}
\delta & \quad \circ \quad \alpha \quad \underset{1}{\overset{\beta}{\rightarrow}} \circ \quad \gamma \\
& \quad \gamma^2 = \delta^2 = \alpha \beta = \beta \alpha = 0, \\
& \quad \alpha \delta = \gamma \alpha = \delta \beta \gamma = 0.
\end{align*}
\]

Then

\[
C = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}
\]

and we have the equation \((\Phi - I)^2 = 0\). But it is easily shown that \( \Phi \) is not periodic.

Also \( \Phi \) is not diagonalizable since the eigenvalue of \( \Phi \) is 1 and its eigenspace is one dimensional.

The quadratic form is given by

\[
q(x) = 2x_1^2 + 4x_1x_2 + 2x_2 = 2(x_1 + x_2)^2 \geq 0.
\]

**Remark 3.7.** The converse of Theorem 3.3 is not true. By the following example, we know the algebra whose Cartan matrix \( C \) gives the quadratic form which takes negative value and \( |C| = -1 \), but the Coxeter matrix \( \Phi \) is periodic (and hence diagonalizable). Also this concludes that Propositions 1.2 and 1.4 in [1] do not hold in general. But it is still open for triangulated algebras.

**Example 3.8.** Let \( R \) be the algebra given by the following quiver and relations:

\[
\begin{align*}
\circ \quad \alpha \quad \underset{1}{\overset{\beta}{\rightarrow}} \circ \quad \gamma \\
& \quad \beta \alpha = \beta \gamma \alpha = \gamma^3 = 0, \quad \gamma^2 = \alpha \beta
\end{align*}
\]

Then

\[
C = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad \Phi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Thus \( |C| = -1 \) and \( \Phi^2 = I \).
The quadratic form is given by
\[ q(x) = x_1^2 + 4x_1x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - x_2^2. \]

For a vector \( x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \), we have \( q(x) = -1 < 0 \).

**Theorem 3.9.** The following statements are equivalent for a quadratic form \( q(x) \):

1. \( q(x) \) is a positive definite.
2. \( q(x) \) is non-negative and \( \Phi \) does not have an eigenvalue 1.
3. \( q(x) \) is non-negative and \( |\Phi + \mathbf{C}| \neq 0 \).
4. \( q(x) > 0 \) for any non-zero vector \( x = \begin{pmatrix} y \\ 0 \end{pmatrix} \) with \( y \in \mathbb{Q}^{n-1} \) and \( |\mathbf{C}| > 0 \).

**Proof.** (1) \( \iff \) (2). This comes from the fact \( \text{rad} \ q = W_1 \).

(2) \( \iff \) (3). This is clear since \( |\mathbf{C}||\mathbf{I} - \Phi| = |\Phi + \mathbf{C}| \neq 0 \) is equivalent to the condition that \( \Phi \) does not have the eigenvalue 1.

(1) \( \iff \) (4). This is a direct consequence of Lemma 3.1 (3). \( \square \)

Our observation in this section indicates that it is important to determine the lattice \( W_1 = \text{rad} \ q \). This is studied by Lenzing and coworkers [4,6]. But the following conjecture still remains.

**Conjecture.** The quadratic form with respect to a Cartan matrix of a triangulated artinian ring is a non-negative definite.

Here an artinian ring is called “triangulated” if its Cartan matrix is a triangular matrix and all of the diagonal components are 1.

We conclude this paper by summarizing the relations between our results by the following figure:

![Diagram](image)

**References**


