Two-color Rado numbers for $x + y + c = kz$

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Abstract

For integers $c \geq 0$ and $k \geq 1$, let $R = R(c, k)$ be the least integer, provided it exists, such that every 2-coloring of the positive integers up to $R$ admits a monochromatic solution to $x_1 + x_2 + c = kx_3$. $R$ exists if and only if $k$ is odd or $c$ is even. If $k = 4$ and $c$ is even, then $R = \lceil (3c/2)/8 \rceil + \varepsilon$ where $\varepsilon \in \{0, 1, 2, 3\}$.

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1. Introduction

In this paper we consider 2-colorings of the positive integers and ask whether the existence of certain “monochromatic” configurations is guaranteed. Specifically, we ask if, for a given equation, $L$, there is an integer, $R(L)$, such that every 2-coloring of the positive integers up to $R(L)$ contains a solution to $L$ among elements of the same color. Such a solution is referred to as a monochromatic solution. The analogous question can be asked for any finite number of colors. Schur [6] answered the existence question affirmatively for the equation $x + y = z$ and for any finite number of colors. An equation having this property is referred to as regular. Rado [4] later characterized all regular linear equations.

If $R(L)$ exists (here and henceforth we will be speaking of 2-colorings), then what is the least value that it can take on? Such a value is referred to as the 2-color Rado number for $L$.

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If $R(L)$ does not exist, then the 2-color Rado number is defined to be infinite. Those classes of equations that naturally contain the Schur equation have received the most attention (see, for example, [1,5,7]). Harborth and Maasberg [2] completely determined the 2-color Rado numbers for $a(x + y) = bz$ for all positive integer constants $a$ and $b$.

Consider the family of equations of the form $L(c, k) : x_1 + x_2 + c = kx_3$ for integers $c \geq 0$ and $k \geq 1$, where the $x_i$’s are not necessarily distinct. We use $R(c, k)$ to denote the 2-color Rado number for $L(c, k)$ and denote the set $\{1, \ldots, n\}$ using $[n]$. The integers and positive integers are denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively.

In Section 2, the instances where $R(c, k)$ is finite are characterized. In addition, the 2-colorings of $\mathbb{N}$ avoiding monochromatic solutions to $L(c, k)$ are characterized. In Section 3, we survey numerical results on $R(c, k)$ and estimate the values of $R(c, 4)$ to within at most three.

2. The existence question

Burr et al. [1] considered, among others, the 2-color Rado numbers for equations of the form $L(c, 1)$ and $L(0, k)$. In both cases, the Rado numbers are finite. Indeed, $R(c, 1)$ is linear in $c$ and $R(0, k)$ is quadratic in $k$. What can be said of the finiteness of $R(c, k)$ in general? The following recursion is a first step.

Lemma 1. For all integers $c, b \geq 0$ and $k \geq 3$, we have $R(c, k) \leq R(c + b(2 - k), k) + b$.

Proof. If $R(c + b(2 - k), k) + b = \infty$, then there is nothing to prove. Assume otherwise. Fix an arbitrary 2-coloring $\Delta : [R(c + b(2 - k), k) + b] \to [2]$. Induce a coloring $\Delta' : [R(c + b(2 - k), k)] \to [2]$ by the rule

$$\Delta'(y) = \Delta(y + b).$$

By definition, there exists in $[R(c + b(2 - k), k)]$ a triple $(y_1, y_2, y_3)$ which is a monochromatic solution to $L(c + b(2 - k), k)$. Define the triple $(x_1, x_2, x_3)$ from $[R(c + b(2 - k), k) + b]$ by $x_1 = y_1 + b$, $x_2 = y_2 + b$, and $x_3 = y_3 + b$. Now the $x_i$ are monochromatic via the original coloring. Moreover, $x_1 + x_2 + c = y_1 + y_2 + y_3 + b + c = (y_1 + y_2 + c + b(2 - k)) + k(b) = k(y_3 + b) = kx_3$. Hence, we have a monochromatic solution to $L(c, k)$ under an arbitrary 2-coloring of $[R(c + b(2 - k), k) + b]$. \(\square\)

On the question of the finiteness of $R(c, k)$, the following result is an easy consequence of Lemma 1.

Corollary 2. If $3 \leq k \in \mathbb{N}$ and $c \in \mathbb{N}$ satisfy $(k - 2)|c$, then $R(c, k) \leq c/(k - 2)$.

Proof. Observe that $R(k - 2, k) = 1$ since $(1, 1, 1)$ is trivially a monochromatic solution to $L(k - 2, k)$. Define $m = c/(k - 2) - 1$. By Lemma 1, $R(c, k) \leq R(c + m(2 - k), k) + m = R(k - 2, k) + c/(k - 2) - 1 = c/(k - 2)$. \(\square\)

The following result is another useful consequence.
Corollary 3. Given \( k \geq 3 \), if \( R(c, k) \) is finite for some \( c_0 \in [k - 2] \), then \( R(c, k) \) is finite for all \( c \equiv c_0 (\text{mod} \ k - 2) \).

This arises from repeatedly applying Lemma 1. Corollary 3 is of no use for \( k \in \{1, 2\} \). However, the existence question for \( R(c, 1) \) is already answered in the affirmative. In [3], it was shown that \( R(c, 2) \) is finite for all even \( c \) (it is linear in \( c \)) and infinite for all odd \( c \).

The question of existence for higher values of \( k \) can be approached on a case-by-case basis using Corollary 3. An exhaustive computer search was performed to compute \( R(c, k) \) for all \( k \in \{4, 5, \ldots, 12\} \) and \( c \in \{1, 2, \ldots, k - 3\} \). Table 1 displays the results.

From the results shown, we may apply Corollaries 2 (specifically in that \( R(k - 2, k) = 1 \)) and 3 to conclude that for \( 3 \leq k \leq 12 \), the Rado number \( R(c, k) \) is finite if \( k \) is odd or \( c \) is even. This condition is easily seen to be necessary for the existence of \( R(c, k) \). Remarkably, it is also sufficient.

Theorem 4. For \( c \geq 0 \) and \( k \geq 1 \), \( R(c, k) \) is finite if and only if \( k \) is odd or \( c \) is even.

Proof. Necessity was shown in [3]. For example, the coloring \( A : \mathbb{N} \to \{2\} \), where \( A(x) = 1 \) if \( x \) is odd and \( A(x) = 2 \) if \( x \) is even, avoids monochromatic solutions to \( L(c, k) \) when \( k \) is even and \( c \) is odd.

The sufficiency is established for \( k \in \{1, 2\} \) and needs only to be proven for \( k \geq 3 \). Suppose that \( A \) is a 2-coloring of the positive integers admitting no monochromatic solutions to \( L(c, k) \) for \( k \geq 3 \). We show that \( k \) is even and \( c \) is odd.

We start with a property that \( A \) obviously has; Property 1: for every \( d > 0 \), there exist \( d_1, d_2 \geq d \) such that statements (1) and (2) below hold.

\[
\begin{align*}
A(d_1) &= 1 & A(d_1 + 1) &= 2, \\
A(d_2) &= 2 & A(d_2 + 1) &= 1.
\end{align*}
\]

Otherwise, if either of (1) and (2) fail there is an infinite block of monochromatic positive integers, certainly containing a solution to \( L(c, k) \) as well. Now let \( i \) be any positive integer. By Property 1, there is an integer \( j \geq (i + c + 1)/k \) such that \( A(j) = A(i) \) and \( A(j + 1) \neq A(i) \).
Since \((jk - c - i, i, j)\) is a solution, we have that \(\Delta(jk - c - i) = \Delta(j + 1)\). Since \((jk - c - i, i + k, j + 1)\) is a solution, it is also required that \(\Delta(i + k) = \Delta(i).\) With this in mind, we arrive at another property of \(\Delta\): Property 2: if integers \(i, j > 0\) are congruent modulo \(k\), then they are colored the same.

Evidently, \(L(c, k)\) has no solutions among the same residue class modulo \(k\). Such a solution would look like \((a_1k + i) + (a_2k + i) + c = k(a_3k + i)\) for some \(a_1, a_2, a_3 \geq 0\) and \(i \in [k]\). By setting \(b = a_1 + a_2 - ka_3\) we may write \(kb + c = ki - 2i\). Since \(k \geq 3\), it follows that we avoid such solutions only if

for all \(i \in [k]\) we have \(-2i \not\equiv c \text{(mod } k)\). \(\square\)

If \(k\) is odd, then \(-2\) generates the additive group \(\mathbb{Z}_k\) and statement (3) fails. So \(k\) must be even. If \(k\) is even, then \(-2\) generates the additive group of even numbers modulo \(k\), and (3) fails if \(c\) is even. So \(k\) must be even and \(c\) must be odd. \(\square\)

Aside from the characterization, something about the type of colorings that avoid solutions has come to light; that is, if \(\Delta\) is a 2-coloring of \(\mathbb{N}\) avoiding monochromatic solutions, then it is an extension of a coloring of \(\mathbb{Z}_k\) which also avoids monochromatic solutions to \(L(c, k)\): \(x + y + c \equiv 0 \text{(mod } k)\). It is easy to check that the converse holds. When \(k\) is even and \(c\) is odd, the “even-odd” coloring observed in [3] is a perfect example of such an extension. However, this is not the only coloring that does the trick. Consider the following property that a coloring \(\Delta\) may have.

**Multicolored pairs (MP) property:** If \(k \geq 2\) is even and \(c \geq 1\) is odd, then for all integers \(i, j > 0\) with \(j \equiv -(i + c) \text{(mod } k)\) we have \(\Delta(i) \not\equiv \Delta(j)\).

It is easy to check that the constraints on \(c\) and \(k\) ensure that no \(i\) will satisfy \(i \equiv -(i + c) \text{(mod } k)\). Moreover, the congruence is symmetric in \(i\) and \(j\). The MP property characterizes all of the “infinite” 2-colorings.

**Theorem 5.** The coloring \(\Delta : \mathbb{N} \to [2]\) avoids monochromatic solutions to \(L(c, k)\) if and only if the MP property holds for \(\Delta\).

**Proof.** To see sufficiency, let \((i, j, l)\) be a solution. Then \(i \equiv -(j + c) \text{(mod } k)\), and by the MP property \(i\) and \(j\) are not colored the same. So \(\Delta\) avoids monochromatic solutions.

To see necessity, let \(i \equiv (1 + c)/(k - 1)\). Then setting \(j = ki - (i + c)\), the triple \((i, j, i)\) is a solution. Since \(\Delta\) avoids monochromatic solutions, \(\Delta(i) \not\equiv \Delta(j)\). With respect to Property 2 from the proof of Theorem 4, it follows easily that for all \(i' \equiv i \text{(mod } k)\) and \(j' \equiv j \equiv -(i' + c) \text{(mod } k)\), we have \(\Delta(i') = \Delta(i) \not\equiv \Delta(j) = \Delta(j')\). \(\square\)

Of course, when \(k = 2\), only an “even-odd” coloring works. By counting pairs, we see that for even \(k \geq 2\) and odd \(c \geq 1\), there are, up to permuting colors, exactly \(2^{k/2 - 1}\) colorings of \(\mathbb{N}\) avoiding monochromatic solutions to \(L(c, k)\). Note that this count does not depend on \(c\).

Colorings which extend a coloring of \(\mathbb{Z}_n\) play an important role in Rado numbers. It seems that whenever the Rado number for a given equation and a given number of colors is found to be infinite, the color construction is always an extension coloring. Indeed, in Rado’s characterization of regular homogeneous equations [4], all colorings used in the
proof are extensions of a coloring of a field \( \mathbb{Z}_p \) for some \( p \). Is there a nontrivial proof of an infinite \( t \)-color Rado number for some linear equation that does not require the construction of an extension coloring?

3. Numerical results

It was shown in [1] that \( R(c, 1) = 4c + 5 \) and that

\[
R(0, k) = \begin{cases} 
5 & \text{if } k = 1, \\
1 & \text{if } k = 2, \\
9 & \text{if } k = 3, \\
k(k + 1)/2 & \text{if } k \geq 4.
\end{cases}
\]

This has inspired an effort to simultaneously generalize both results through the study of \( R(c, k) \) (see, for example, [3]). However, these attempts have been unable to determine \( R(c, k) \) exactly. The number-theoretic interplay between \( c \) and \( k \) seems to prevent us from arriving at an explicit formula. In [3], it was observed that the lower bound

\[
R(c, k) \geq \text{LB}(c, k) = \left\lceil \frac{2 [(c + 2)/k + c]}{k} \right\rceil
\]

holds for all integers \( c \geq 0 \) and \( k \geq 1 \). In addition, it was found that \( R(c, 2) \) agrees with \( \text{LB}(c, 2) \) when \( c \) is even and, of course, is infinite when \( c \) is odd. An upper bound of \( c \) was observed for the values of \( R(c, 3) \) and it was conjectured that for \( c \) sufficiently large, \( R(c, 3) = \text{LB}(c, 3) \). Empirical evidence suggests that for sufficiently large \( c \), the value of \( R(c, k) \) may agree with \( \text{LB}(c, k) \) (this will be discussed further at the end of this section).

Extending previous investigations of \( R(c, 0) \), \( R(c, 1) \), \( R(c, 2) \), and \( R(c, 3) \), we now present a result on \( R(c, 4) \). We already know that \( R(c, 4) = \infty \) when \( c \) is odd, so it is henceforth assumed that \( c \) is even. To start, we use Lemma 1 and arrive at the following result.

**Corollary 6.** For all even \( i \), the relation \( R(c, 4) \leq R(c - i, 4) + i/2 \) holds.

**Proof.** This follows by taking \( k = 4 \) and \( b = i/2 \) in Lemma 1. \( \square \)

A quick look at (4) reveals the following bound:

\[
R(c, 4) \geq \text{LB}(c, 4) = \begin{cases} 
(3c + 2)/8 & \text{if } c \equiv 2(\text{mod } 8), \\
(3c + 4)/8 & \text{if } c \equiv 4(\text{mod } 8), \\
(3c + 6)/8 & \text{if } c \equiv 6(\text{mod } 8), \\
(3c + 8)/8 & \text{if } c \equiv 0(\text{mod } 8).
\end{cases}
\]

Note that \( \text{LB}(c, 4) = \lceil (3c + 2)/8 \rceil \). The main numerical result, that \( R(c, 4) \) differs from \( \text{LB}(c, 4) \) by no more than three, is given below.
**Theorem 7.** For all even \( c \geq 2 \),

\[
LB(c, 4) \leq R(c, 4) \leq \begin{cases} 
LB(c, 4) & \text{if } c \equiv 2, 4, 6, 8 \text{(mod 32)}, \\
LB(c, 4) + 1 & \text{if } c \equiv 10, 12, 14, 16 \text{(mod 32)}, \\
LB(c, 4) + 2 & \text{if } c \equiv 18, 20, 22, 24 \text{(mod 32)}, \\
LB(c, 4) + 3 & \text{if } c \equiv 26, 28, 30, 0 \text{(mod 32)}. 
\end{cases}
\]

**Proof.** The first inequality was stated already in (5). We nevertheless spell out how it is realized by noting that the coloring \( \delta \) below avoids monochromatic solutions to \( L(c, 4) \):

\[
\delta(x) = \begin{cases} 
1 & \text{if } x \in \left\{ 1, 2, \ldots, \left\lceil \frac{c+2}{4} \right\rceil - 1 \right\}, \\
2 & \text{if } x \in \left\{ \left\lceil \frac{c+2}{4} \right\rceil , \ldots, \left\lceil \frac{2(c+2)+c}{4} \right\rceil - 1 \right\}. 
\end{cases}
\]

It remains to establish the second inequality, the upper bound on \( R(c, 4) \). Two cases need to be considered.

**Case 1:** \( c \equiv 2 \text{ (mod 32)} \).

Fix an arbitrary 2-coloring \( \Delta : [(3c + 2)/8] \to [2] \). Without loss of generality, \( \Delta(1) = 1 \). Since \((1, 1, (c+2)/4)\) is a solution to \( L(c, 4) \), we may assume that \( \Delta((c+2)/4) = 2 \). Otherwise we would have a monochromatic solution and would be done. Similarly, since \((c+2)/4, (c+2)/4, (3c+2)/8\) is a solution, we may assume that \( \Delta((3c+2)/8) = 1 \). Since \((c+6)/8, (3c+2)/8, (3c+2)/8\) is a solution, we may assume that \( \Delta((c+6)/8) = 2 \). We now have

\[
\Delta(1) = 1, \quad \Delta \left( \frac{c+6}{8} \right) = 2, \quad \Delta \left( \frac{c+2}{4} \right) = 2, \quad \text{and } \Delta \left( \frac{3c+2}{8} \right) = 1.
\]

Note that these numbers are positive integers as \( c \equiv 2 \text{ (mod 32)} \). On the one hand, if \( \Delta((11c+10)/32) = 1 \), then \((1, (3c+2)/8, (11c+10)/32)\) is a monochromatic solution to \( L(c, 4) \). On the other hand, if \( \Delta((11c+10)/32) = 2 \), then \((c+6)/8, (c+2)/4, (11c+10)/32)\) is a monochromatic solution. In either case, we confirm the existence of a monochromatic solution. Hence, \( R(c, 4) \leq (3c+2)/8 = LB(c, 4) \).

**Case 2:** \( c \geq 4, c \not\equiv 2 \text{ (mod 32)} \).

Let \( i = \min\{n \geq 0 : (c-n) \equiv 2 \text{ (mod 32)} \} \). Surely \( i \) is even. Hence, by Corollary 6,

\[
R(c, 4) \leq R(c-i, 4) + i/2.
\]

Since \( c-i \equiv 2 \text{ (mod 32)} \), Case 1 implies that \( R(c-i, 4) = (3(c-i) + 2)/8 \). Then

\[
R(c, 4) \leq \frac{3(c-i) + 2}{8} + \frac{i}{2} = \frac{3c + (i+2)}{8}.
\]

Now, in light of (5), we see that

\[
c \equiv 10, 18, 26 \text{(mod 32)} \Rightarrow R(c, 4) \leq LB(c, 4) + i/8, \\
c \equiv 4, 12, 20, 28 \text{(mod 32)} \Rightarrow R(c, 4) \leq LB(c, 4) + (i-2)/8, \\
c \equiv 6, 14, 22, 30 \text{(mod 32)} \Rightarrow R(c, 4) \leq LB(c, 4) + (i-4)/8, \\
\text{and } c \equiv 0, 8, 16, 24 \text{(mod 32)} \Rightarrow R(c, 4) \leq LB(c, 4) + (i-6)/8.
\]

The conclusion of Case 2 now follows from these inequalities, and the union of these cases completes the proof. \( \square \)
We were unable to narrow the remaining gap between the upper and lower bounds theoretically. However, using an exhaustive computer search, the difference between the lower bound $\text{LB}(c,4)$ and the actual Rado number $R(c,4)$ was computed for even $c$ between 2 and 72. Table 2 summarizes the findings. Notice that when $30 \leq c \leq 72$, the difference between the lower bound and the actual Rado number is zero. Such agreement was observed in the $k = 3$ case as well. It is probably true that for $k \geq 3$ and $c$ sufficiently large, the finite values of $R(c, k)$ agree with $\text{LB}(c, k)$.

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References