A Tauberian Condition and Skew Product Flows with Applications to Integral Equations

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In a recent paper Levin and Shea give sufficient conditions in order that a bounded solution \( x(t) \) of an integral equation can be expressed in the form

\[
x(t) = \sum_{m=1}^{\infty} \psi_m(t) y_m(t) + \eta(t),
\]

where \( \{y_m\} \) is a sequence of solutions of the limiting equations, \( \{\psi_m\} \) is a "\$ sequence" and \( \eta(t) \to 0 \) as \( t \to \infty \). In this paper we show that the expansion (E) is really a consequence of a topological-dynamical principle when one views the solutions of the integral equation as generating a semiflow in the sense of Miller and Sell.

1. INTRODUCTION

In a three part paper [2], Levin and Shea study the asymptotic behavior as \( t \to \infty \) of solutions of certain integral equations. Specifically they seek conditions on these equations in order that a bounded solution \( x(t) \) can be expressed in the form

\[
x(t) = \sum_{m=1}^{\infty} \psi_m(t) y_m(t) + \eta(t),
\]

where \( \eta(t) \to 0 \) as \( t \to \infty \), \( \{y_m\} \) is a sequence of solutions of the limiting equations, and \( \{\psi_m\} \) is a \$ sequence, a concept we shall define in Section 2.

In their paper, Levin and Shea consider several types of integral equations; however, in our applications we shall restrict our attention to three of these, namely, the autonomous equation

\[
x(t) + \int_{-\infty}^{\infty} g(x(t - s)) dA(s) = f(t), \quad -\infty < t < \infty \quad (A)
\]

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the nonautonomous equation
\[ x(t) + \int_{-\infty}^{\infty} g(x(t - s), t - s) \, dA(t, s) = f(t), \quad -\infty < t < \infty \quad \text{(NA)} \]
as well as the Volterra equation
\[ x(t) = f(t) + \int_{0}^{t} a(t, s) \, g(x(s), s) \, ds, \quad 0 \leq t < \infty \quad \text{(V)} \]
which is a special case of (NA). Under certain reasonable assumptions concerning \( f, g, \) and \( A, \) they show that if \( x(t) \) is a bounded solution of (A), (NA), or (V) that satisfies the Tauberian condition
\[ \lim_{t \to \infty, t \to 0} |x(t + \tau) - x(t)| = 0, \quad \text{(T)} \]
then \( x(t) \) can be expanded in the form (E).

We proposed to show in this paper that these results of Levin and Shea follow from a general principle from topological dynamics. The topological-dynamical result, when applied to the skew product flow for integral equations constructed by Miller and Sell [3], lead to the Levin–Shea results. We shall see that the Tauberian condition (T), which is well understood in the theory of linear equations (cf., for example, Karlin [1]), is needed for the general theory in order to guarantee that the trajectory generated by (A), (NA), or (V) be positively compact in an appropriate topology.

In Section 2 we shall give the definition of a \( \psi \) sequence and state a basic lemma proved by Levin and Shea. In Section 3 we shall show how this lemma can be applied to arbitrary continuous flows in a topological space, and in Section 4 we shall present the applications to integral equations.

2. \( \psi \) Sequences

Let \( \{t_m\} \) be an increasing sequence of real numbers that satisfy
\[ \lim_{m \to \infty} (t_m - t_{m-1}) = \infty. \]
A \( \psi \) sequence (associated with \( \{t_m\} \)) is a sequence of real-valued \( C^\infty \) functions \( \psi_m(t) \) that satisfy
\[ \sum_{m=1}^{\infty} \psi_m(t) = 1, \quad -\infty < t < \infty; \quad \text{(1)} \]
\[ \|\psi_m'\|_\infty \to 0 \quad \text{as} \quad m \to \infty; \quad \text{(2)} \]
\[ \sum_{m=1}^{\infty} |\psi_m'(t)| \to 0 \quad \text{as} \quad t \to \infty; \quad \text{(3)} \]
\[ \psi_1(t) \equiv 0 \quad \text{for} \quad t \geq t_2; \quad \text{(4)} \]
for $m = 2, 3, \ldots$, $\psi_m(t)$ has its support in the interval 
$[t_{m-1}, t_{m+1}]$, $\psi_m(t_{m}) = 1$, $\psi'_m(t) \geq 0$ on $[t_{m-1}, t_{m}]$
and $\psi'_m(t) \leq 0$ on $[t_{m}, t_{m+1}]$, \(\text{(5)}\)

The following lemma is a straightforward generalization of Lemma 3.1 of [2].

**LEMMA.** Let $x: \mathbb{R} \rightarrow B$ be continuous, where $B$ is a Banach space with the norm topology, and assume that $\lim \sup_{t \to \infty} ||x(t)|| < \infty$. Let $\Gamma$ be a nonempty collection of bounded continuous functions $y: \mathbb{R} \rightarrow B$, and assume that $\lim \sup_{t \to \infty} \inf_{\{t \in \Gamma | t-r| \leq d\}} ||x(t) - y(t)|| = 0$ holds for every $d > 0$. Then there exist sequences $\{t_m\} \subset \mathbb{R}$, $\{y_m\} \subset \Gamma$, and $\{\psi_m\}$ and a continuous function $\eta: \mathbb{R} \rightarrow B$, where $(t_m - t_{m-1}) \to \infty$ as $m \to \infty$, $\{\psi_m\}$ is a $\psi$ sequence associated with $\{t_m\}$, $||\eta(t)|| \to 0$ as $t \to \infty$, and 

$$x(t) = \sum_{m=1}^{\infty} \psi_m(t) y_m(t) + \eta(t), \quad -\infty < t < \infty. \quad (7)$$

Notice that it follows from the definition of a $\psi$ sequence that the infinite series in (7) contains no more than two terms for any $t \in \mathbb{R}$.

### 3. Continuous Flows

Let $X$ be a Hausdorff topological space and let $\pi$ be a continuous flow (local dynamical system) on $X$. That $\pi$ is a continuous mapping of a set

$$D = \{(x, t) \in X \times \mathbb{R}: \alpha_x < t < \beta_x\}$$

into $X$ satisfying the following properties:

(i) $\alpha_x < 0 < \beta_x$ for all $x \in X$.

(ii) $\pi(x, 0) = x$ for all $x \in X$.

(iii) $\pi(\pi(x, s), t) = \pi(x, s + t)$ in the sense that if the left-hand side is defined, then the right-side is defined and equality holds.

(iv) The intervals $I_x = (\alpha_x, \beta_x)$ are lower semicontinuous, that is, if $x_n \to x$ then $I_x \subset \lim \inf I_{x_n}$.

(v) Each interval $I_x$ is maximal in the sense that either $I_x = \mathbb{R}$ or the set

$$\gamma^+(x) = \{\pi(x, t): 0 \leq t < \beta_x\}$$
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do not lie in a compact set when \( \beta_x < \infty \), and the set

\[ \gamma^{-}(x) = \{ \pi(x, t) : \alpha_x < t \leq 0 \} \]
does not lie in a compact set when \( -\infty < \alpha_x \).

The properties of flows which we will need here are proved, at least when \( X \) is a uniform space, in [5]. The uniform space setting will be adequate for the applications we wish to make to integral equations; however, the properties we require in this section are equally valid in general topological spaces.

A motion \( \pi(x, t) \) is said to be positively compact if \( H^{+}(x) = \text{Cl} \gamma^{+}(x) \) is compact. We see that in this case \( \beta_x = +\infty \) and that the \( \omega \)-limit set

\[ \Omega_x = \bigcap_{\tau \geq 0} H^{+}(\pi(x, \tau)) \]
is a nonempty compact invariant set in \( H^{+}(x) \). Moreover, if \( y \in \Omega_x \), then \( I_y = \mathbb{R} \), that is, \( \pi(y, t) \) is defined for all \( t \in \mathbb{R} \). Also \( \pi(y, t) \in \Omega_x \) for all \( t \in \mathbb{R} \).

We can now state our main theorem.

**Theorem 1.** Let \( \pi(x, t) \) be a positively compact motion in a flow on \( X \). Let \( P : X \to \mathbb{B} \) be a continuous mapping of \( X \) into a Banach space \( \mathbb{B} \). Then there exist sequences \( \{t_m\} \subset \mathbb{R} \), \( \{y_m\} \subset \Omega_x \), \( \{\psi_m\} \), and a continuous function \( \eta : \mathbb{R} \to \mathbb{B} \) such that \( t_m \to \infty \) as \( m \to \infty \), \( \psi_m \) is a sequence associated with \( \{t_m\} \), \( \| \eta(t) \| \to 0 \), as \( t \to \infty \), and

\[ P\pi(x, t) = \sum_{m=1}^{\infty} \psi_m(t) P\pi(y_m, t) + \eta(t), \quad 0 \leq t < \infty. \quad (8) \]

**Proof.** We will apply the Levin–Shea lemma. Let \( x(t) = P\pi(x, t) \) for \( t \geq 0 \) and \( x(t) = Px \) for \( t < 0 \). Since \( \pi(x, t) \) remains in a compact set \( H^{+}(x) \) for all \( t \geq 0 \) and since \( P \) is continuous, we see that \( \| x(t) \| \) is bounded. Define \( \Gamma \) by

\[ \Gamma = \{ P\pi(y, \cdot) : y \in \Omega_x \}. \]

Since \( \Omega_x \) is nonempty compact and invariant we see that \( \Gamma \) is a nonempty collection of bounded continuous functions defined on \( \mathbb{R} \). It remains only to prove that Eq. (6) is satisfied.

Since \( \pi(x, t) \) is positively compact, it follows that for every sequence \( \{\tau_n\} \) with \( \tau_n \to \infty \), there is a subsequence (which we shall denote by \( \{\tau_n\} \)) and a point \( y \in \Omega_x \), such that \( \pi(x, \tau_n + t) \to \pi(y, t) \) uniformly for \( t \) in compact sets. In particular, for every \( d > 0 \), \( \pi(x, \tau_n + t) \to \pi(y, t) \) uniformly for \( | t | \leq d \).
Now assume that Eq. (6) fails. Then for some \( d > 0 \) there is a sequence \( \{s_n\}, s_n \to \infty \), and an \( \epsilon > 0 \) such that for all \( s_n \) one has
\[
\| P\pi(x, s_n + t) - P\pi(y, s_n + t) \| \geq \epsilon
\]  
(9)
for all \( |t| \leq d \) and all \( y \in \Omega_x \). Since \( \pi(y, s_n, t) = \pi(y, s_n + t) \), we see that Eq. (9) can be rewritten as
\[
\| P\pi(x, s_n + t) - P\pi(y, t) \| \geq \epsilon
\]  
(10)
for all \( |t| \leq d \) and all \( y \in \Omega_x \). Now, by selecting a subsequence if necessary, pick a point \( y \in \Omega_x \) such that \( \| P\pi(x, s_n + t) - P\pi(y, t) \| \) uniformly for \( |t| \leq d \). Since \( P \) is continuous, one has \( P\pi(x, s_n + t) \to P\pi(y, t) \) uniformly for \( |t| \leq d \), but this contradicts Eq. (10) and completes the proof of the theorem.

In the applications to integral equations we shall be interested in a semiflow instead of a flow. This differs from a flow in that the mapping \( \pi(x, t) \) is defined only for non-negative time \( 0 \leq t < \beta_x \). Theorem 1 can be extended to include semiflows in several ways. First one observes that Eq. (8) holds for \( t > 0 \), which makes sense for both flows and semiflows. Since the problem of "backing up" a semiflow is a rather messy technical question, it seems that the simplest way to extend Theorem 1 is note that Eq. (6) holds for every \( d > 0 \) if and only if
\[
\lim_{t \to \infty} \{ \inf_{y \in F} \sup_{s \leq t < s+t \leq d} \| x(t) - y(t) \| \} = 0
\]  
(11)
for every \( d > 0 \). Next define \( I \) to be the collection of all functions \( y(t) \) such that for some \( y \in \Omega_x \) one has
\[
y(t) = P\pi(y, t), \quad t \geq 0
\]
\[
y(t) = Py, \quad t < 0.
\]
Now by restricting the time \( t \) to be non-negative, one can use the argument of Theorem 1 to extend the result to semiflows.

4. APPLICATIONS

The method for applying the last theorem to integral equations can perhaps be best understood by considering first the Volterra integral equation (V). The other applications are simply variations on the same theme. We begin with a very brief summary of the essential topological-dynamical theory for (V). For simplicity only we take the point of view used by Miller and Sell in [3]. Among other things, this point of view requires that the solutions of
the Volterra integral equation be uniquely determined. However, recently developments (see [4] and [6]) permit us to drop this uniqueness assumption for the applications of Theorem 1 to integral equations.

We consider (V), where \( f \in C = C(\mathbb{R}^+, \mathbb{R}^n) \) and \( a(t, s) \) and \( g(x, s) \) belong to compatible topological spaces \( A \) and \( G \), respectively. Let \( \varphi(t) \) denote the unique solution of (V) defined on a maximal interval \( 0 < t < \beta \) and define
\[
T_\tau f = T_\tau(f, g, a)
\]
by
\[
T_\tau f(\theta) = f(\tau + \theta) + \int_0^\tau a(\tau + \theta, s) g(\varphi(s), s) \, ds
\]
for \( 0 \leq \tau < \beta \) and \( 0 \leq \theta < \infty \). Then \( T_\tau f \in C \) and at \( \theta = 0 \) one has
\[
T_\tau f(0) = \varphi(\tau).
\]
Let \( g_\tau(x, t) = g(x, \tau + t) \) and \( a_\tau(t, s) = a(\tau + t, \tau + s) \). Then the mapping
\[
\pi(f, g, a; \tau) = (T_\tau f, g_\tau, a_\tau)
\]
defines a semiflow on \( C \times G \times A \), where \( C \) has the topology of uniform convergence on compact sets.

The question of positively compact motions is also studied in [3] and the following theorem is proved.

**Theorem 2.** Assume that \( f, g, \) and \( a \) are chosen so that

(i) \( \{f_\tau: 0 \leq \tau < \infty\} \) lies in a compact set in \( C \),
(ii) \( \{g_\tau: 0 \leq \tau < \infty\} \) lies in a compact set in \( G \), and
(iii) \( \{a_\tau: 0 \leq \tau < \infty\} \) lies in a compact set in \( A \).

If \( \varphi(t), 0 \leq t < \infty, \) is a bounded solution of (V) that satisfies the Tauberian condition (T), then the trajectory \( \pi(f, g, a; t) \) is positively compact.

Now define \( P: C \times G \times A \to \mathbb{R}^n \) by
\[
P(f, g, a) = f(0).
\]
Then \( P\pi(f, g, a; t) = T_t f(0) = \varphi(t) \). If the hypotheses of Theorem 2 are satisfied, then the hypothesis of Theorem 1 is satisfied and we get
\[
\varphi(t) = \sum_{m=1}^\infty \psi_m \varphi_m(t) + \eta(t), \quad 0 \leq t < \infty
\]
where \( \{\varphi_m\} \) is a sequence of solutions of the limiting equations, \( \{\psi_m\} \) is a \( \psi \) sequence, and \( \eta(t) \to 0 \) as \( t \to \infty \). (We refer the reader to [3] for the formula for the limiting equations of (V).)
It is well-known that condition (i) of Theorem 2 is satisfied if and only if $f(t)$ is bounded and uniformly continuous for $t \geq 0$, cf. [4], for example. A special case of condition (i) occurs when $f(t) \to f(\infty)$ as $t \to \infty$.

Conditions (ii) and (iii) of Theorem 2 are studied extensively in the Miller-Sell memoir. In order to complete the connection between our results and those of Levin and Shea we shall present here sufficient conditions that (ii) and (iii) be satisfied in the case that $g(x, t)$ is continuous. One can similarly give sufficient conditions that (ii) and (iii) be satisfied in the case that $g(x, t)$ satisfies a Carathéodory hypothesis.

Conditions (ii) and (iii) in Theorem 2 do, of course, depend on the topologies on $G$ and $A$, respectively. In the notation of [3] we shall now restrict our attention to the case that $G = G^*_\omega$ and $A = A^*_\omega$. [One could formulate similar results for the pairs of spaces $(G^*_\omega, A^*_\omega)$, $1 < p < \infty$, as well as $(G^p, A^p)$, $1 \leq p \leq \infty$, where $g$ is allowed to satisfy a Carathéodory hypothesis.]

If $g(x, t)$ satisfies the hypothesis (I) where

(I): $g(x, t)$ is bounded and uniformly continuous on sets of the form $K \times R^+$, where $K$ is a compact set in $R^n$, then $g$ satisfies condition (ii) of Theorem 2 (see [3], p. 36 and compare with hypothesis $H_{11}(g)$ in [2]).

Similarly $a(t, s)$ will satisfy condition (iii) of Theorem 2 if $a$ satisfies hypothesis (II), see [3], pp. 46–47:

II: (a) For each $t \in R^+$, there is a real number $B$ such that

$$\int_0^{\tau+1} |a(t, s)| ds \leq B$$

for all $\tau \geq 0$, and

(\beta) for each compact set $J \subset R^+$, every locally bounded function $\xi: R^+ \to R^n$, and every $\epsilon > 0$ there is a $\delta > 0$ such that for $|h| \leq \delta$ one has

$$\left| \int_0^{\tau+1} [a(t + h, s) - a(t, s)] \xi(s) ds \right| \leq \epsilon \quad (\tau \geq 0)$$

uniformly for $t \in J$.

It is now easy to give many special cases of hypotheses (I) and (II) which would lead to the Levin–Shea results as applied to (V). We shall give four such illustrations here. Of course, we assume here, as do Levin and Shea, that there exists a bounded solution $q(t)$ of (V) that satisfies the Tauberian condition (T). Each of these cases lead to the expansion formula (E).

Case 1. $\lim_{t \to \infty} f(t) = f(\infty)$ exists, $g$ is independent of $t$, and

$$a(t, s) = \alpha(t - s),$$

where $\alpha(t)$ is locally integrable. (Compare with [2], Section 5).
Case 2. \( \lim_{t \to \infty} f(t) = f(\infty) \) exists, \( g \) is independent of \( t \), and there is an \( \alpha(r) \) in \( L_1[0, \infty) \) such that

\[
\lim_{t \to \infty} \int_0^\infty |a(t, s) - \alpha(s)| \, ds = 0.
\]

(Compare with [2], Section 17).

Case 3 (periodic case). \( f(t) \) is asymptotically periodic, \( g(x, t) \) is periodic in \( t \), and \( a(t, s) \) is periodic in \( t \) with

\[
\sup_{0 \leq t \leq \omega} \int_0^\infty |a(t, s)| \, ds < \infty,
\]

where \( \omega \) is the common period. (Compare with [2], Section 20).

Case 4. \( f \) is bounded and uniformly continuous, \( g \) satisfies (I), and \( a(t, s) = a(t - s) \), where \( a(r) \) is locally integrable. (Compare with [2], Section 20).

Before turning to the Eqs. (A) and (NA) let us briefly comment on the form of the flow for Volterra integral equations in the event that the solutions are not uniquely determined. In this case we observe that the map \( T_r f \) also depends on the particular solution \( \varphi \), so one could write

\[
T_r f = T_r(\varphi, f, g, a).
\]

The semiflow is then given by

\[
\varphi(t, f, g, a; \tau) = (\varphi(\tau + t), T_r f, g_r, a_r),
\]

where \( \varphi(t) = \varphi(\tau + t) \). The phase space for \( \pi \) is constructed using appropriate generalizations of the viewpoint of [6]. The continuity of \( \pi \) follows from standard arguments, see [4], for example. The mapping \( P \) is given as in the case of uniqueness.

The problem of extending this application to Eqs. (A) and (NA) now reduces to the construction of a suitable flow or semiflow for these equations. This construction with all the concomitant topological and analytical details is the subject matter of a forthcoming paper by the author. We will show here only how to formally construct the flow and how to define the mapping \( P \) required by Theorem 1.

For (A) one considers the solution \( x(t) \) as known for all \( s, - \infty < s \leq t \).

Then

\[
x_t(\theta) = x(t + \theta), \quad - \infty < \theta \leq 0
\]
describes the path of the solution in a function space $F_1(-\infty, 0]$. Similarly, $f$ belongs to a related function space $F_2(-\infty, \infty)$. The semiflow is given by

$$\pi(x, f; \tau) = (x_\tau, f_\tau)$$

and the mapping $P$ is given by $P(x, f) = x(0)$.

The semiflow for (NA) is a bit more complicated since one has to translate the terms $g$ and $dA$ too. If one defines

$$g_s(x, t) = g(x, \tau + t) \quad \text{and} \quad dA_s(t, s) = dA(\tau + t, \tau + s),$$

then the semiflow is given by

$$\pi(x, f, g, dA; \tau) = (x_\tau, f_\tau, g_\tau, dA_\tau)$$

and the mapping $P$ is given by $P(x, f, g, dA) = x(0)$.

References