This is a continuation of an earlier paper by the authors on generalized inverses over integral domains. The main results consist of necessary and sufficient conditions for the existence of a group inverse, a new formula for a group inverse when it exists, and necessary and sufficient conditions for the existence of a Drazin inverse. We show that a square matrix $A$ of rank $r$ over an integral domain $\mathcal{R}$ has a group inverse if and only if the sum of all $r \times r$ principal minors of $A$ is an invertible element of $\mathcal{R}$. We also show that the group inverse of $A$ when it exists is a polynomial in $A$ with coefficients from $\mathcal{R}$.

1. INTRODUCTION

Let $\mathcal{R}$ be an integral domain, i.e., a commutative ring with no zero divisors and with unity. In this paper we consider matrices over $\mathcal{R}$, unless indicated otherwise.
Let $A$ be an $m \times n$ matrix, and consider the Moore-Penrose equations:

1. $AGA = A$
2. $GAG = G$
3. $(AG)^T = AG$
4. $(GA)^T = GA$

where the superscript $T$ denotes transpose. If $G$ is an $n \times m$ matrix satisfying (1), then $G$ is called a generalized inverse (g-inverse, l-inverse) of $A$. A matrix $A$ is said to be regular if it has a g-inverse. If $G$ satisfies (1) and (2), it is said to be a reflexive g-inverse of $A$, whereas it is said to be a Moore-Penrose inverse of $A$ if it satisfies (1)-(4).

Consider the following equations applicable to square matrices:

5. $AG = GA$

\[(1^k) \quad A^k = A^{k+1}G.\]

Borrowing the definition from real matrices (see [2, Chapter 4]), for a square matrix $A$ over an integral domain $R$, a matrix $G$ over $R$ is said to be a group inverse of $A$ if (1), (2), and (5) are satisfied, and a matrix $G$ over $R$ is said to be a Drazin inverse of $A$ if (2), (5), and $(1^k)$ (for some positive integer $k$) are satisfied. We denote a group inverse of $A$ by $A^*$. A matrix $G$ over $R$ satisfying conditions (1) and (5) is called a commuting g-inverse of $A$.

It is well known that over the field of real numbers a square matrix $A$ has a group inverse if and only if $\text{Rank } A = \text{Rank } A^2$ and that every matrix has a Drazin inverse (see [2] and [4]).

The main results of this paper consist of, for a square matrix over an integral domain,

1. necessary and sufficient conditions for the existence of a group inverse,
2. a new formula for finding a group inverse when it exists, and
3. necessary and sufficient conditions for the existence of a Drazin inverse.

For the existence of a group inverse we find necessary and sufficient conditions in terms of its $r \times r$ minors akin to the results of [1] and [3].

We also generalize some results from Rao and Mitra [4, Chapter 4] for matrices over integral domains. Incidentally we give a necessary and sufficient condition for $\text{Rank } A = \text{Rank } A^2$.

Let $A$ be an $m \times n$ matrix, and let $\alpha = \{i_1, \ldots, i_r\}$, $\beta = \{j_1, \ldots, j_r\}$ be subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. We denote by $A_{\alpha\beta}$ the submatrix of $A$ determined by rows indexed by $\alpha$ and columns indexed by $\beta$.

The determinant of a square matrix $A$ is denoted by $|A|$, and $\frac{\partial}{\partial a_{ij}} |A|$
denotes the cofactor of $a_{ij}$ in the expansion of $A$. The determinantal rank (the largest nonvanishing minor) is denoted by $\rho(A)$. $C_r(A)$ is the $r$th compound matrix of $A$ with rows indexed by $r$-element subsets of $\{1, \ldots, m\}$ and columns indexed by $r$-element subsets of $\{1, \ldots, n\}$. At several places in this paper, $\alpha, \beta, \gamma$ are assumed to be $r$-element subsets of $\{1, 2, \ldots, n\}$ without its being stated explicitly.

The relevant properties of $A^g$ and $C_r(A)$ from [1] and [3] that will be used are listed below:

(i) Let $A$ be an $m \times n$ matrix, with $\rho(A) = r$. Then

$$\rho(C_r(A)) = 1.$$  \hspace{1cm} (1.1)

See [3, Lemma 9].

(ii) Let $A$ be an $m \times n$ matrix over the integral domain $\mathbb{R}$ with $\rho(A) = r$. Then $A$ is regular if and only if there exists $c^g_\alpha \in \mathbb{R}$ such that

$$\sum_{\alpha, \beta} c^g_\alpha | A^g_\beta | = 1,$$  \hspace{1cm} (1.2)

where the summation is over all $r$-element subsets $\alpha, \beta$ of $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$ respectively. Furthermore, if $c^g_\alpha$ satisfies (1.2), then $G = (g_{ij})$ is a $g$-inverse of $A$, where

$$g_{ij} = \sum_{\alpha, \beta} c^g_\alpha \frac{\partial}{\partial a_{ij}} | A^g_\beta |.$$  \hspace{1cm} (1.3)

See [3, Theorem 8].

(iii) Let $A$ be an $m \times n$ matrix of rank $r$ over the integral domain $\mathbb{R}$. Let $G$ be a reflexive $g$-inverse of $A$. Then for all $i, j$

$$e_{ij} = \sum_{\alpha, \beta} | G^g_\alpha | \frac{\partial}{\partial a_{ij}} | A^g_\beta |,$$  \hspace{1cm} (1.4)

where $\alpha, \beta$ run over all $r$-element subsets of $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$ respectively. See [1, Theorem 3].

We now introduce some notation. For an $m \times n$ matrix $A$, let $A^-$ be a generalized inverse of $A$, $\mathcal{V}(A)$ be the module generated by columns of $A$, and $\mathcal{B}(A)$ be the module generated by rows of $A$. Borrowing the notation from Rao and Mitra [4], for a matrix $A$,

(1) let $A^-_x$ be a $g$-inverse of $A$ with $\mathcal{V}(A^-_x) = \mathcal{V}(A)$ [equivalently $\mathcal{V}(A^-_x) \subset \mathcal{V}(A)$],
(2) let $A_\rho^-$ be a g-inverse of $A$ with $\mathcal{R}(A_\rho^-) = \mathcal{R}(A)$ [equivalently $\mathcal{R}(A_\rho^-) \subseteq \mathcal{R}(A)$], and

(3) let $A_{\rho\chi}^-$ be a g-inverse of $A$ with $\mathcal{C}(A_{\rho\chi}^-) = \mathcal{C}(A)$ and $\mathcal{R}(A_{\rho\chi}^-) = \mathcal{R}(A)$ [equivalently $\mathcal{C}(A_{\rho\chi}^-) \subseteq \mathcal{C}(A)$ and $\mathcal{R}(A_{\rho\chi}^-) \subseteq \mathcal{R}(A)$].

2. EXISTENCE OF $A_{\rho\chi}^-$

Theorem 2 below gives necessary and sufficient conditions for the existence of $A_{\rho\chi}^-$. We need a crucial result (Lemma 1, below) which generalizes Lemma 4.1.1 of [4]. Let $\mathbb{R}$ be an integral domain. We consider matrices over $\mathbb{R}$.

**LEMMA 1.** Let $A$, $P$, and $Q$ be matrices over the integral domain $\mathbb{R}$. Then $A$ has a g-inverse of the form $PCQ$ for some $C$ if and only if

(i) $\rho(QAP) = \rho(A)$ and

(ii) $QAP$ is regular,

in which case $C$ is a g-inverse of $QAP$. A g-inverse with the above properties is unique whenever $\rho(A) = \rho(P) = \rho(Q)$.

**Proof.** "Only if" part: First note that the Cauchy-Binet formula gives us that $\rho(DE) \leq \min(\rho(D), \rho(E))$. Let $PCQ$ for some $C$ be a g-inverse of $A$. Then $A = A(PCQ)A = A(PCQ)A(PCQ)A$. So $\rho(A) \leq \rho(QAP)$. Again, since $A = A(PCQ)A$, we have that $QAP = QAPCQAP$. So $\rho(QAP) \leq \rho(A)$. Thus we have (i) and (ii).

"If" part: Let $C$ be a g-inverse of $QAP$. So $(QAP)C(QAP) = QAP$. Since $\rho(QAP) = \rho(A)$, we have that $\rho(A) = \rho(QA) = \rho(AP)$. If $A$ and $QA$ are considered as matrices over the field of quotients $\mathbb{F}$ of $\mathbb{R}$, then $\rho(A) = \rho(QA)$ gives us a matrix $D$ over $\mathbb{F}$ such that $A = DQA$. Similarly there exists a matrix $E$ over $\mathbb{F}$ such that $A = APE$. Now $(QAP)C(QAP) = QAP$ gives us $APCQA = DQPCQAPE = DQA = A$. So we are done.

A similar argument gives the uniqueness also (see the last part of the proof of Lemma 4.1.1 of [4]).

**THEOREM 2.** The following statements are equivalent for a square matrix $A$:

(i) $A_\chi^-$ exists.

(ii) $A_\rho^-$ exists.

(iii) $A_{\rho\chi}^-$ exists.
(iv) $\rho(A) = \rho(A^2)$ and $A^2$ is regular.
(v) $\rho(A) = \rho(A^2)$ and $A^3$ is regular.
(vi) $\rho(A) = \rho(A^3)$ and $A^3$ is regular.

Proof. (i) $\Rightarrow$ (iv) follows from Lemma 1 by taking $P = A$ and $Q = I$.
(iv) $\Rightarrow$ (v): Let us verify that $(A^2)^-A(A^2)^-$ is a g-inverse of $A^3$. Since $\rho(A) = \rho(A^2)$, there exists a matrix $E$ over the quotient field of $F$ such that $A = A^2E$. So

\[
A^3(A^2)^-A(A^2)^-A^3 = AA^2(A^2)^-A^2E(A^2)^-A^3 \\
= AA^2E(A^2)^-A^3 \\
= A^2(A^2)^-A^3 \\
= A^3.
\]

So $(A^2)^-A(A^2)^-$ is a g-inverse of $A^3$.
(v) $\Rightarrow$ (vi) is clear.
(vi) $\Rightarrow$ (iii) follows from Lemma 1. In fact $A_{\rho x}^- = A(A^3)^-A$.
(iii) $\Rightarrow$ (i) is trivial.
(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iii) $\Rightarrow$ (ii) hold by similar arguments.

Remark 1. The two concepts $A^*$ and $A_{\rho x}^-$ are identical, and $A^*$ (and so $A_{\rho x}^-$) is unique. For, firstly, that $A^*$ is an $A_{\rho x}^-$ follows because

\[
A^* = A^*AA^* \\
= A^*AA^*AA^* \\
= AA^*A.
\]

Secondly that $A_{\rho x}^-$ is an $A^*$ follows because, as observed in the proof of (vi) $\Rightarrow$ (iii), it is enough to verify Equations (1), (2) and (5) for $G = A(A^3)^-A$. Equation (1) is clear by the definition of a g-inverse. For Equation (2), if $E$ is a matrix over the field of quotients such that $A = A^3E$, then

\[
GAG = A(A^3)^-A^3(A^3)^-A^3E \\
= A(A^3)^-A^3E \\
= A(A^3)^-A \\
= G.
\]
Equation (5), i.e., \( AG = GA \), also follows similarly. The uniqueness of \( A^* \) is easily proven from its definition.

**Remark 2.** From Remark 1, the existence of \( A^* \) is equivalent to all the six statements of Theorem 2. Also, since trivially the existence of a commuting \( g \)-inverse of \( A \) is equivalent to the existence of \( A^* \), the six statements of Theorem 2 are equivalent to the existence of a commuting \( g \)-inverse of \( A \).

**Remark 3.** \( A_{\rho x} = A_{\rho x}^{-} AA_{\rho x}^{-} \) when \( \rho(A) = \rho(A^2) \) and \( A^2 \) is regular.

**Remark 4.** More generally, for a regular matrix \( A \), there is a \( g \)-inverse of the form \( PCQ \) for some \( C \) if and only if there are \( g \)-inverses \( G_1 \) and \( G_2 \) of the form \( PD \) and \( EQ \) respectively. In fact \( G_1 AG_2 \) serves our purpose.

3. Existence of the Group Inverse of \( A \) in Terms of Its Minors

In [1], it was shown that a matrix \( A \) of rank \( r \) over \( \mathbb{R} \) is regular if and only if a linear combination of all the \( r \times r \) minors is one. In [2] we showed that a matrix of rank \( r \) over \( \mathbb{R} \) has a Moore–Penrose inverse if and only if a particular linear combination of all the \( r \times r \) minors is one (namely \( \sum_{\alpha, \beta} u_{\alpha} | A_{\alpha, \beta}^\alpha | A_{\beta}^\alpha | = 1 \), where \( (\sum_{\alpha \beta} | A_{\alpha, \beta}^\alpha |)^{-1} \) exists and equals \( u \)). The aim of this section is to give a similar condition for the existence of \( A^* \). We shall show that \( A^* \) exists if and only if \( \sum_{\alpha} u_{\alpha} | A_{\alpha}^\alpha | = 1 \), where \( (\sum_{\alpha} | A_{\alpha}^\alpha |)^{-1} \) exists and equals \( u \).

First we shall prove the condition for matrices of rank \( 1 \).

**Lemma 3.** If \( A \) is a square matrix of rank \( 1 \) over an integral domain \( \mathbb{R} \), then \( A \) has a group inverse if and only if the trace of \( A \) (\( Tr \) \( A \) for short) is invertible in \( \mathbb{R} \). In this case the group inverse \( A^* = (Tr \ A)^{-2}A \).

**Proof.** Let \( A \) be a matrix of rank \( 1 \) over \( \mathbb{R} \). Over the field of quotients we can write \( A = xy^T \), where \( x \) and \( y \) are \( n \times 1 \) matrices over the field. Note that \( y^Tx \) is the trace of \( A \).

"If" part: Suppose \( Tr \ A \) is invertible in \( \mathbb{R} \). Then we shall prove that \( G = (Tr \ A)^{-2}A \) is the group inverse of \( A \):

\[
AGA = A(Tr \ A)^{-2}AA = xy^T(y^Tx)^{-2}xy^T = xy^T = A.
\]
Similarly we can prove that $GAG = G$ and $AG = GA$. So $G = (\text{Tr } A)^{-2}A$ is the group inverse of $A$.

"Only if" part: Suppose that $A$ has a group inverse. Then $\rho(A) = \rho(A^2) = 1$ and $A^2$ is regular (by Lemma 2). If $B = A^2$, then the $(i, j)$th element of $B$ is

$$b_{ij} = \sum_k a_{ik}a_{kj}. \quad (3.1)$$

Since $B$ is regular and $\rho(B) = 1$, by Lemma 7 of [1] there exists $g_{ji} \in \mathbb{R}$ such that

$$\sum_{i,j} g_{ji}b_{ij} = 1. \quad (3.2)$$

Substituting (3.1) in (3.2), we get

$$\sum_{i,j,k} g_{ji}a_{ik}a_{kj} = 1.$$ 

Since $\rho(A) = 1$, we have $a_{ik}a_{kj} = a_{kk}a_{ij}$. So

$$\left( \sum_k a_{kk} \right) \left( \sum_{i,j} g_{ji}a_{ij} \right) = 1. \quad (3.3)$$

(3.3) now implies that $\sum_k a_{kk} = \text{Tr } A$ is invertible in $\mathbb{R}$.

**Theorem 4.** Let $A$ be an $n \times n$ matrix of rank $r$ over an integral domain $\mathbb{R}$. Then $\rho(A) = \rho(A^2)$, and $A^2$ is regular if and only if

$$\sum_{\gamma} |A_{\gamma}^2|,$$

where $\gamma$ runs over all $r$-element subsets of $\{1, 2, \ldots, n\}$, is invertible in $\mathbb{R}$.

**Proof:** "Only if" part: Let $\rho(A) = \rho(A^2) = r$, and $A^2$ be regular. So $\rho([C_r(A)]^2) = \rho(C_r(A^2)) = 1$ and $C_r(A^2)$ is regular. From the "only if" part of Lemma 3 we get that $\text{Tr } C_r(A) = \sum_{\gamma} |A_{\gamma}^2|$ is invertible in $\mathbb{R}$.

"If" part: Let $\sum |A_{\gamma}^2|$ be invertible in $\mathbb{R}$. First we shall prove that $\rho(A) = \rho(A^2) = r$. Suppose $\rho(A) \neq \rho(A^2)$. Then $\rho(A^2) < r$ (since $\rho(A) = r$), and

$$\left| \left( A^2 \right)^{\gamma}_{\beta} \right| = 0 \quad (3.4)$$
for all r-element subsets $\alpha, \beta$ of $(1, 2, \ldots, n)$. But

$$
|\binom{A^2}{\alpha}_\beta| = \sum_{\gamma} |A^\alpha_{\gamma}| \cdot |A^\beta_{\gamma}|
$$

$$
= \sum_{\gamma} |A^\alpha_{\gamma}| \cdot |A^\alpha_{\beta}| \quad \text{[since } \rho(C_r(A)) = 1] \quad (3.5)
$$

$$
= |A^\alpha_{\beta}| \sum_{\gamma} |A^\alpha_{\gamma}|.
$$

Since $\sum_{\gamma} |A^\alpha_{\gamma}|$ is invertible in $\mathbb{R}$, from (3.4) and (3.5) we get $|A^\alpha_{\beta}| = 0$ for all r-element subsets $\alpha$ and $\beta$. This contradicts the fact that $\rho(A) = r$. So we must have $\rho(A) = \rho(A^2) = r$.

Now it remains to prove that $A^2$ is regular. Since $\sum_{\gamma} |A^\alpha_{\gamma}| = u$ is invertible in $\mathbb{R}$, we have

$$
\left( \sum_{\alpha} |A^\alpha_{\alpha}| \right) \left( \sum_{\beta} |A^\alpha_{\beta}| \right) = 1;
$$

i.e.,

$$
\sum_{\alpha, \beta} u^{-2} |A^\alpha_{\alpha}| \cdot |A^\alpha_{\beta}| = 1,
$$

and

$$
\sum_{\alpha, \beta} u^{-2} |A^\beta_{\alpha}| \cdot |A^\alpha_{\beta}| = 1, \quad \text{because } \rho(C_r(A)) = 1. \quad (3.6)
$$

By the Cauchy-Binet formula we have

$$
|\binom{A^2}{\alpha}_{\alpha}| = \sum_{\beta} |A^\alpha_{\beta}| \cdot |A^\beta_{\alpha}|. \quad (3.7)
$$

By substituting (3.7) into (3.6) we get

$$
\sum_{\alpha} u^{-2} \left| \binom{A^2}{\alpha}_{\alpha} \right| = 1. \quad (3.8)
$$

So, from (1.2), by taking $c^\beta_{\alpha} = 0$ for $\alpha \neq \beta$ and $u^{-2}$ for $\alpha = \beta$, it is clear that $A^2$ is regular. Hence we have proved the theorem.
Theorem 5. Let $A$ be an $n \times n$ matrix over $\mathbb{R}$ such that $\rho(A) = r$. Then the following are equivalent:

(i) $A$ has a group inverse.
(ii) $C_r(A)$ has a group inverse.
(iii) $\sum_\gamma |A_\gamma^r|$ is invertible in $\mathbb{R}$.
(iv) $\rho(A) = \rho(A^2)$, and $A^2$ is regular.

Proof. (i) $\Rightarrow$ (ii) is trivial from the properties of compound matrices, (ii) $\Rightarrow$ (iii) follows from Lemma 3, (iii) $\Rightarrow$ (iv) is a part of Theorem 4, and (iv) $\Rightarrow$ (i) follows from Lemma 2.

We know that over a field the group inverse of a matrix $A$, whenever it exists, can be written as a polynomial in $A$ with coefficients from the field (see [2] and [4]). We shall prove this result in the case of integral domains also.

Theorem 6. Let $A$ be a square matrix of order $n$ over $\mathbb{R}$ for which $A^*$ exists over $\mathbb{R}$. Then $A^*$ is a polynomial in $A$ with coefficients from $\mathbb{R}$.

Proof. Let the characteristic polynomial of $A$ be

$$|\lambda I - A| = p_r \lambda^{n-r} + p_{r+1} \lambda^{n-r+1} + \cdots + \lambda^n,$$

where $r$ is the rank of $A$ and $(-1)^k p_k$ is the sum of all the principal minors of order $k$. Observe that $(-1)^r p_r$ is the sum of all the $r \times r$ principal minors of $A$, which is invertible in $\mathbb{R}$ (by our Theorem 5 above). Now by the Cayley-Hamilton theorem,

$$p_r A^{n-r} + \cdots + A^n = 0,$$

so

$$A^{n-r} = q_{r+1} A^{n-r+1} + q_{r+2} A^{n-r+2} + \cdots + q_n A^n,$$  \hspace{1cm} (3.9)

where $q_k = -p_k/p_r$ for $k \leq n-1$ and $q_n = -1/p_r$. (Observe that $q_k, \ldots, q_n$ are elements of $\mathbb{R}$.)

Multiplying both sides of (3.9) by $(A^*)^{n-r+1}$, we get

$$A^* = q_{r+1} AA^* + q_{r+2} A + \cdots + q_n A^{n-r-1},$$  \hspace{1cm} (3.10)

and multiplying both sides of (3.10) by $A$, we get

$$A^* A = q_{r+1} A + q_{r+2} A^2 + \cdots + q_n A^{n-r}.$$  \hspace{1cm} (3.11)
Substituting (3.11) into (3.10) gives us

\[ A^* = \left( q_{r+1}^2 + q_{r+2} \right) A + \left( q_{r+1} q_{r+2} + q_{r+3} \right) A^2 + \cdots + q_n A^{n-r}, \]

and this is a polynomial in \( A \) over \( \mathbb{R} \).

Incidentally, from the proof of equivalence of (iii) and (iv) of Theorem 5 we can give a condition for \( \rho(A) \) to be equal to \( \rho(A^2) \).

**THEOREM 7.** Let \( A \) be a square matrix of rank \( r \) over \( \mathbb{R} \). Then \( \rho(A) = \rho(A^2) = r \) if and only if the sum of all the \( r \times r \) principal minors of \( A \) is nonzero.

**Proof.** We observe that for any \( \alpha \) and \( \beta \) \((r\)-element subsets of \( \{1, 2, \ldots, n\})\)

\[ |A_\alpha^2 \beta| = \left( \sum_\gamma |A_\gamma^\alpha| \right) |A_\beta^\alpha|, \quad (3.12) \]

where \( \gamma \) runs over all \( r \)-element subsets of \( \{1, 2, \ldots, n\} \). Since \( \mathbb{R} \) is an integral domain, we get that

\[ \rho(A) = \rho(A^2) \quad \text{if and only if} \quad \sum_\gamma |A_\gamma^\alpha| \neq 0. \]

4. **NEW FORMULAE**

We have seen in the previous section that if \( \Sigma_\gamma |A_\gamma^\alpha| \) is invertible in \( \mathbb{R} \), then \( A^* \) exists. We shall give in this section a method of finding \( A^* \) whenever it exists.

First of all observe that from Remark 2 at the end of Section 2, it follows that \( A \) has a commuting \( g \)-inverse if and only if \( \Sigma_\gamma |A_\gamma^\alpha| \) is invertible.

**THEOREM 8.** Let \( A \) be a matrix of rank \( r \) over \( \mathbb{R} \). Then:

(i) If \( u = \Sigma_\gamma |A_\gamma^\alpha| \) is invertible in \( \mathbb{R} \), then \( G = (g_{ij}) \) defined by

\[ g_{ji} = \sum_\gamma u^{-1} \frac{\partial}{\partial a_{ij}} |A_\gamma^\alpha| \]

is a commuting \( g \)-inverse of \( A \).
(ii) If $u = \sum_\gamma |A_\gamma^\gamma|$ is invertible in $\mathbb{R}$, then $G = (g_{ij})$, where

$$g_{ji} = \sum_{\alpha, \beta} u^{-2} \left| A_\alpha^\beta \right| \frac{\partial}{\partial a_{ij}} \left| A_\beta^\alpha \right|,$$

is the group inverse of $A$.

**Proof.** (i): First we shall prove that $G = (g_{ij})$ obtained from the formula

$$g_{ji} = \sum_{\gamma} u^{-1} \frac{\partial}{\partial a_{ij}} \left| A_\gamma^\gamma \right|$$

(4.1)

is a commuting $g$-inverse of $A$. Note that $G$ is a $g$-inverse of $A$ over $\mathbb{R}$ [by (1.2) and (1.3), taking $c_\alpha^\beta = 0$ for $\alpha \neq \beta$ and $c_\alpha^\alpha = u^{-1}$].

Now we shall prove that $G$ commutes with $A$, i.e.,

$$(AG)_{ij} = (GA)_{ij} \quad \text{for all } i, j.$$  

(4.2)

For $i = j$,

$$(AG)_{ii} = \sum_{k=1}^{n} a_{ik} g_{kl},$$

$$= \sum_{k} a_{ik} \sum_{\gamma : i \in \gamma} u^{-1} \frac{\partial}{\partial a_{ik}} \left| A_\gamma^\gamma \right|$$

$$= \left( \sum_{\gamma : i \in \gamma} \left| A_\gamma^\gamma \right| \right) u^{-1}.$$  

(4.3)

Similarly we get $(GA)_{ii} = (\sum_{\gamma : i \in \gamma} \left| A_\gamma^\gamma \right|) u^{-1}$. So $(GA)_{ii} = (AG)_{ii}$. For $i \neq j$,

$$(AG)_{ij} = \sum_{k} a_{ik} g_{kj},$$

$$= \sum_{k} a_{ik} \sum_{\gamma : j \in \gamma} u^{-1} \frac{\partial}{\partial a_{kj}} \left| A_\gamma^\gamma \right|$$

$$= \sum_{\gamma : j \in \gamma} \left( \sum_{k} u^{-1} a_{ik} \frac{\partial}{\partial a_{kj}} \left| A_\gamma^\gamma \right| \right)$$

$$= \sum_{\gamma : j \in \gamma, \ i \not\in \gamma} \left| A_\gamma^\gamma \backslash (j) \cup (i) \right| u^{-1},$$  

(4.4)
because for \( i \in \gamma \)
\[
\sum_k a_{ik} \frac{\partial}{\partial a_{jk}} | A_i^\beta | = 0.
\]

So
\[
(AG)_{ij} = u^{-1} \sum_{\alpha: \text{is}, j \neq \alpha \beta: j \neq \beta} | A_\alpha^\beta |.
\]

Similarly we get
\[
(GA)_{ij} = u^{-1} \sum_{\gamma: i \neq \gamma \text{ or } j \neq \gamma} | A_i^\gamma \setminus \{i\cup\{j\}| = u^{-1} \sum_{\alpha: i \neq \alpha, j \neq \beta \beta: j \neq \beta} | A_\alpha^\beta | = (AG)_{ij}.
\]

Hence \( G \) commutes with \( A \).

Now we shall prove part (ii) of the theorem. Since \( u = \sum_{\gamma} | A_i^\gamma | \) is invertible in \( R \),

\[
\left( \sum_{\gamma} u^{-1} | A_i^\gamma | \right)^2 = u^{-2} \sum_{\alpha, \beta} | A_\alpha^\alpha | | A_\beta^\beta | = 1.
\]

Since \( \rho(C_r(A)) = 1 \), we have \( | A_\alpha^\alpha | | A_\beta^\beta | = | A_\alpha^\alpha | | A_\beta^\beta | \), so

\[
\sum_{\alpha, \beta} u^{-2} | A_\beta^\alpha | | A_\alpha^\beta | = 1.
\]

We claim that the matrix \( G = (g_{ij}) \) obtained from the formula
\[
g_{ji} = \sum_{\alpha, \beta} u^{-2} | A_\alpha^\beta | \frac{\partial}{\partial a_{ij}} | A_\beta^\alpha |
\]
is the group inverse of \( A \).

Note that \( C_r(A^*) \) is the group inverse of \( C_r(A) \). But, by Lemma 3, we get
\[
C_r(A)^* = \left[ \text{Tr } C_r(A) \right]^{-2} C_r(A)
= u^{-2} C_r(A).
\]
Therefore \( |A^*| = u^{-2} |A_\alpha^\beta| \). Since \( A^* \) is a reflexive \( g \)-inverse of \( A \), by (1.4) we get

\[
(A^*)_{ij} = \sum_{\alpha, \beta} u^{-2} |A_\alpha^\beta| \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|
\]

So we get \( G = A^* \). Hence the proof.

**Remark.** Theorem 7 provides a direct proof of (iii) \( \Rightarrow \) (iv) of Theorem 5.

5. DRAZIN INVERSE

In this section we shall give necessary and sufficient conditions for a square matrix over \( \mathbb{R} \) to have a Drazin inverse over \( \mathbb{R} \).

**Theorem 9.** Let \( A \) be a matrix over \( \mathbb{R} \). Then \( A \) has a Drazin inverse over \( \mathbb{R} \) (satisfying (2), (5), and (1k)) if and only if for that \( k \), \( \rho(A^k) = \rho(A^{k+1}) \) and \( A^{2k+1} \) is regular. Also, the Drazin inverse, when it exists, is unique.

**Proof.** “Only if” part: Let \( A \) have a Drazin inverse, say \( G \), over \( \mathbb{R} \). Condition (1k) gives us that \( \rho(A^k) = \rho(A^{k+1}) \) and also

\[
A^{2k+1} = A^{2k+2} G
\]

\[
= A^{2k+1} GA \quad \text{[from condition (5)]}
\]

\[
= A^{2k+1} G^{2k+1} A^{2k+1} \quad \text{[from condition (2)].}
\]

So \( A^{2k+1} \) is regular.

“If” part: Let \( k \) be a positive integer for which \( \rho(A^k) = \rho(A^{k+1}) \) and \( A^{2k+1} \) is regular. We shall prove that \( G = A^k (A^{2k+1})^{-1} A^k \) is a Drazin inverse of \( A \).
Since \( p(A^k) = p(A^{k+j}) \) for all positive integers \( j \), there exists matrices \( D, E, \) and \( F \) over the field of quotients of \( R \) such that

\[
\begin{align*}
A^{k+1} &= DA^{2k+1}, \\
A^k &= EA^{2k+1}, \\
A^k &= A^{2k+1}F.
\end{align*}
\]

So

\[
AG = AA^k(A^{2k+1})^{-1}A^k
= D(A^{2k+1})F = A^{k+1}F.
\]

Similarly

\[
GA = A^k(A^{2k+1})^{-1}A^kA
= A^kA^{2k+1}AA^k
= A^{k+1}F.
\]

Hence \( AG = GA \), i.e. (5) holds. Also,

\[
\begin{align*}
A^{k+1}G &= A^{k+1}A^k(A^{2k+1})^{-1}A^k \\
&= A^{2k+1}(A^{2k+1})^{-1}A^{2k+1}F \\
&= A^{2k+1}F \\
&= A^k.
\end{align*}
\]

Hence \( (1^k) \). Finally,

\[
G^2A = G(GA) = [A^k(A^{2k+1})^{-1}A^k]A^{k+1}F \quad \text{(since \( GA = A^{k+1}F \))}
= A^k(A^{2k+1})A^k = G.
\]

Hence (2). Thus \( G \) is a Drazin inverse of \( A \).

Now we shall prove that the Drazin inverse is unique when it exists. First observe that if \( G \) satisfies \( (1^k) \) then \( G \) satisfies \( (1^m) \) for all \( m \geq k \). If \( F \) and \( G \)
are two Drazin inverses of $A$, we can choose a $k$ such that $F$ and $G$ both satisfy conditions (2), (5), and (1'). By repeated applications of (5) and (1k) we get
\[ G_{k+1}A_{k+1}F_{k+1} = G_{k+1}A_{k} = G \]
and
\[ G_{k+1}A_{2k+1}F_{k+1} = A_{k+1}F_{k+1} = F. \]
So $F = G$.

**Remark 1.** Let us observe that if $A$ has a Drazin inverse over $\mathbb{R}$ and if the index of $A$ is $p$, then $\rho(A^p) = \rho(A^{p+1})$ and $A^{2p+1}$ is regular. If $A$ has a Drazin inverse $H$ over $\mathbb{R}$, then considering $A$ as a matrix over the field of quotients of $\mathbb{R}$, $A^{2p+1}$ has a $g$-inverse over this field. So $A$ has a Drazin inverse $G$ over this field, and $G$ satisfies (2), (5), and (1'). By the uniqueness of the Drazin inverse over the field, we have then $G = H$. So $H$ satisfies (2), (5), and (1'). Theorem 9 gives our statement. Also we have the following result: $A$ has a Drazin inverse over the integral domain $\mathbb{R}$ if and only if $A^{2p+1}$ is regular, where $p$ is the index of $A$.

**Remark 2.** Given an integral domain $\mathbb{R}$, for a given matrix $A$ over $\mathbb{R}$ there need not exist an integer $k$ such that $\rho(A^k) = \rho(A^{k+1})$ and $A^{2k+1}$ is regular over $\mathbb{R}$. For example, take

\[ \mathbb{R} = \mathbb{Z} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

$\rho(A) = \rho(A^k) = 1$ for all positive integers $k$. But $A^k$ is not regular for $k \geq 2$.

**Remark 3.** Note that $A$ has a Drazin inverse with index $p$ if and only if $A^p$ has a group inverse (it is easy to verify that $G^p$ is a group inverse of $A^p$, and in fact $p$ is the smallest positive integer for which $A^p$ has a group inverse). Conversely, if $A^p$ has a group inverse, then $A^{3p}$ is regular with $\rho(A) = \rho(A^{3p})$, which implies that $A^{2p+1}$ is regular.

Now we shall prove the following theorem

**Theorem 10.** Let $A$ be a matrix over $\mathbb{R}$ with index $p$, and $\rho(A^p) = s$. Then the following are equivalent:

(i) $A$ has a Drazin inverse.

(ii) $C_u(A)$ has a Drazin inverse.
(iii) \( C_s(A^p) \) has a group inverse.

(iv) \( \text{Tr } C_s(A^p) \) is invertible over \( \mathbb{R} \), and \( A^p \) is regular.

(v) \( A^p \) has a group inverse.

(vi) \( A^{p+n} \) is regular for all positive integers \( n \).

(vii) \( A^{2n} \) is regular.

**Proof.** (i) \( \Rightarrow \) (ii) follows from the properties of \( C_s(A) \).

(ii) \( \Rightarrow \) (iii): Since \( C_s(A) \) has a Drazin inverse with index \( \leq k \), \( C_s(A^k) \) has a group inverse (from Remark 3 following Theorem 9).

(iii) \( \Rightarrow \) (iv) is trivial by Lemma 3.

(iv) \( \Rightarrow \) (v) holds from Theorem 4.

(v) \( \Rightarrow \) (vi): If \( n \) is a positive integer, then choosing \( m \) such that \( n \leq (m - 1)p \), we have \( \rho(A^p) = \rho(A^{mp}) = \rho(A^{p+n}) \), and since \( A^{mp} \) is regular, \( A^{k+n} \) is regular.

(vi) \( \Rightarrow \) (vii) is obvious.

(vii) \( \Rightarrow \) (i) holds from Remark 1 following Theorem 9.

It is known that over a field every matrix has a decomposition (see [2, Chapter 4] and [4, Chapter 4]) of the form

\[
A = A_1 + A_2
\]

with the properties

(i) \( \rho(A_1) = \rho(A_2^2) \),

(ii) \( A_2 \) is nilpotent, and

(iii) \( A_1 A_2 = A_2 A_1 = 0 \).

We shall now investigate whether over an integral domain a similar decomposition also exists for every matrix.

Observe that over a field, condition (i) is equivalent to

(i') \( A_1 \) has a group inverse.

In the following theorem we shall give a necessary and sufficient condition for a square matrix over an integral domain to have a decomposition satisfying properties (i'), (ii), and (iii).

**Theorem 11.** A square matrix \( A \) over \( \mathbb{R} \) has a decomposition \( A = A_1 + A_2 \) satisfying (i'), (ii), and (iii) if and only if \( A \) has a Drazin inverse. Such a decomposition is unique.

**Proof.** "If" part: If \( A \) has a Drazin inverse \( K \) over \( \mathbb{R} \), then, by defining \( A_1 = AKA = K^* \) and \( A_2 = A - A_1 \), one can check, as in the proof of Theorem 10 in Chapter 4 of [2], that \( A_1 \) and \( A_2 \) satisfy (i'), (ii), and (iii).
"Only if" part: Suppose that $A$ has a decomposition of the form $A = A_1 + A_2$ with (i'), (ii), and (iii). Then there is a positive integer $m$ such that $A_2^n = 0$. For this $m$, $A^m = A_1^m$. Since $A_1^m = A^m$, the index of $A$ is $d \leq m$. Since $A_1$ has a group inverse, $A_1^m = A^m$ has a group inverse. Since some power of $A$ has a group inverse, $A$ has a Drazin inverse.

Uniqueness of the decomposition follows as in the real case. 

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REFERENCES


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