Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems

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Abstract

This work deals with planar polynomial differential systems \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \). We give a set of necessary conditions for a system to have an invariant algebraic curve. These conditions are determined from the value of the cofactor at the singular points of the system, once considered in a compact space. We apply these results to show the non-Liouvillian integrability of several families of quadratic systems with an algebraic limit cycle.

Résumé

Ce travail traite des systèmes différentiels polynomiaux \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \). Nous donnons un ensemble de conditions nécessaires pour l’existence d’une courbe algébrique invariante. Ces conditions sont déterminées à partir de la valeur du cofacteur aux points singuliers du système, une fois exprimé dans un espace compact. Nous utilisons ces résultats pour démontrer la non intégrabilité Liouville de plusieurs familles de systèmes quadratiques avec un cycle limite algébrique.

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1. Definitions and preliminary results

We consider a planar polynomial differential system

\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \]  

where \( P(x, y), Q(x, y) \in \mathbb{R}[x, y] \) are coprime polynomials, that is, there is no non-constant polynomial which divides \( P \) and \( Q \), and \( \dot{\cdot} = \frac{d}{dt} \). Equally, we can consider the equation \( \omega = 0 \), where \( \omega \) is the 1-form defined in \( \mathbb{R}^2 \) by \( \omega = Q(x, y)dx - P(x, y)dy \).

Let \( d \) be the maximum degree of the polynomials \( P \) and \( Q \), we say that system (1) is of degree \( d \).

As usual, we denote by \( \mathbb{R}[x, y] \) the ring of polynomials in two variables with real coefficients and by \( \mathbb{R}(x, y) \) its quotient field. Analogous definitions stand for \( \mathbb{C}[x, y] \) and \( \mathbb{C}(x, y) \).

Given a system (1) we define an invariant algebraic curve as an algebraic curve \( f(x, y) = 0 \), where \( f(x, y) \in \mathbb{R}[x, y] \), such that:

\[ P(x, y) \frac{\partial f}{\partial x}(x, y) + Q(x, y) \frac{\partial f}{\partial y}(x, y) = k(x, y)f(x, y), \]  

where \( k(x, y) \) is a polynomial called the cofactor of \( f(x, y) \). It is easy to see that if \( P \) and \( Q \) are polynomials of degree at most \( d \), then the cofactor is of degree at most \( d - 1 \).

The importance of invariant algebraic curves to understand the dynamics of a system (1) has been remarked by several authors. We refer to [24,25], and the references therein, for an exhaustive survey on this topic as well as being two of the initial works on this subject.

As we always assume a system (1) with real coefficients, we can suppose, without loss of generality, that if \( f(x, y) = 0 \) is an invariant algebraic curve with cofactor \( k(x, y) \), then both \( f(x, y), k(x, y) \in \mathbb{R}[x, y] \). If \( f(x, y) = 0 \) is an invariant algebraic curve with non-null imaginary part and with a certain cofactor \( k(x, y) \), then its conjugate curve \( \overline{f}(x, y) = 0 \) also satisfies (2) with cofactor \( \overline{k}(x, y) \). Hence, the product of these complex polynomials \( f(x, y)\overline{f}(x, y) \in \mathbb{R}[x, y] \) and satisfies (2) with cofactor \( (k(x, y) + \overline{k}(x, y)) \in \mathbb{R}[x, y] \). In \( \mathbb{R}^2 \), the curve given by \( f(x, y)\overline{f}(x, y) = 0 \) may only contain a finite number of isolated singular points or be the null set.

We can also assume, without loss of generality, that \( f(x, y) \) is an irreducible polynomial in \( \mathbb{R}[x, y] \), because if \( f(x, y) \) is reducible, then all its proper factors are invariant algebraic curves.

The function \( g = \exp\{h/f\} \), where \( h, f \) are two coprime polynomials, is called an exponential factor for system (1) provided that for some polynomial \( k \) of degree at most \( d - 1 \) the following relation is satisfied

\[ P(x, y) \frac{\partial g}{\partial x}(x, y) + Q(x, y) \frac{\partial g}{\partial y}(x, y) = k(x, y)g(x, y). \]
We say that \( k(x, y) \) is the cofactor of the exponential factor \( g = \exp[h/f] \).

We remark that an exponential factor \( g = \exp[h/f] \) does not define an invariant curve, but the next proposition, easily proved, gives the relationship between both notions.

**Proposition 1.** If \( g = \exp[h/f] \) is an exponential factor and \( f \) is not a constant, then \( f = 0 \) is an invariant algebraic curve, and \( h \) satisfies the equation \( P \frac{\partial h}{\partial x} + Q \frac{\partial h}{\partial y} = hk_f + fk_g \), where \( k_f \) and \( k_g \) are the cofactors of \( f \) and \( g \), respectively.

Since system (1) is a real system, there is no lack of generality in considering that \( h(x, y), f(x, y) \in \mathbb{R}[x, y] \). If \( g = \exp[h/f] \) is an exponential factor with non-null imaginary part, then its complex conjugate, \( \bar{g} = \exp[\bar{h}/\bar{f}] \) is also an exponential factor, as it can be easily checked by its defining equation. Moreover, the product \( g \bar{g} = \exp[h/f + \bar{h}/\bar{f}] \) is a real exponential factor with a real cofactor.

This paper has one main goal which consists on giving a set of necessary conditions for polynomials of degree lower or equal than \( d - 1 \) to be the cofactor of an invariant algebraic curve or an exponential factor for a system (1). These conditions are the value of the cofactor at a non-degenerate or an elementary degenerate singular point whose ratio of eigenvalues does not equal 1. These results are given in Section 2. A generic system (1) of degree \( d \) has \( d^2 + d + 1 \) non-degenerate and different singular points (finite and infinite) whose ratio of eigenvalues is not a rational number and which are not all contained in an algebraic curve of degree \( \leq d - 1 \). The value of the cofactor at one of these points brings forth a linear equation with \( d(d + 1)/2 + 1 \) unknowns, which correspond to the coefficients of the cofactor and the degree of the curve. Therefore, in general, we have much more independent linear equations than unknowns. Hence, this set of necessary conditions is wide enough to, in general, completely characterize the cofactors and the degree related to invariant algebraic curves of a given system.

The characterization of the invariant algebraic curves of a system (1) gives, in most cases, the dynamics since these curves are usually made of graphics. Moreover, the knowledge of invariant algebraic curves is directly connected with the existence of a Liouvillian first integral. Hence, we apply the aforementioned result in relation to the integrability problem. We now describe how to achieve it.

The integrability problem consists on studying and determining, when possible, the type of a first integral for a system (1).

**Definition 2.** Let \( \mathcal{U} \) be an open subset of \( \mathbb{R}^2 \) and \( \Sigma \) a set of orbits contained in \( \mathcal{U} \) such that \( \mathcal{U} \setminus \Sigma \) is open. We say that a \( C^k \), \( k \geq 0 \), function \( H : \mathcal{U} \setminus \Sigma \rightarrow \mathbb{R} \) is a first integral of class \( k \) for system (1) defined in \( \mathcal{U} \setminus \Sigma \) if \( H(x, y) \) is constant on each solution of system (1) contained in \( \mathcal{U} \setminus \Sigma \), and \( H(x, y) \) is non-constant on any open subset of \( \mathcal{U} \setminus \Sigma \).

If \( k > 0 \), this definition is equivalent to the fact that \( P \frac{\partial H}{\partial x} + Q \frac{\partial H}{\partial y} \) is null on \( \mathcal{U} \setminus \Sigma \) and \( H(x, y) \) non-locally constant.

To look for an inverse integrating factor for a given system (1) is an equivalent problem as to look for a first integral.
Let \( W \) be an open set of \( \mathbb{R}^2 \). A function \( V: W \to \mathbb{R} \) of class \( C^k(W) \), \( k > 1 \), that satisfies the linear partial differential equation

\[
P(x, y) \frac{\partial V}{\partial x}(x, y) + Q(x, y) \frac{\partial V}{\partial y}(x, y) = \left( \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y) \right) V(x, y),
\]

is called an inverse integrating factor of system (1) on \( W \).

We note that \( \{V = 0\} \) is formed by orbits of system (1). The function \( V \) allows the computation of a first integral of the system on \( W \setminus \{V = 0\} \). The first integral \( H \) associated to the inverse integrating factor \( V \) can be computed through the integral

\[
H(x, y) = \int_{(x_0, y_0)}^{(x, y)} \frac{Q(x, y) \, dx - P(x, y) \, dy}{V(x, y)},
\]

where \( (x_0, y_0) \in \mathbb{R}^2 \) is not a critical point and the condition (3) ensures that this line integral is well defined.

We are concerned with the possible functional classes a first integral for a system (1) can belong to, in particular, we are concerned with the existence of a Liouvillian first integral for a system (1). A precise definition of the Liouvillian class of functions is given in [23, 27]. Roughly speaking, a Liouvillian function, or a function which can be expressed by means of quadratures, is a function constructed from rational functions by using algebraic operations, composition, exponentials and integration, applied a finite number of times.

M.F. Singer shows in [27] the narrow relationship between the existence of a Liouvillian first integral for a system (1) and the existence of invariant algebraic curves.

**Theorem 3** [27]. If the system (1) has a Liouvillian first integral, then there is an inverse integrating factor of the form \( V = \exp\left( \int_{(x_0, y_0)}^{(x, y)} \eta \right) \), where \( \eta \) is a rational 1-form such that \( d\eta \equiv 0 \).

Taking into account Theorem 3, C. Christopher [12] gives the following result, which makes precise the form of the inverse integrating factor.

**Theorem 4** [12]. If the system (1) has an inverse integrating factor of the form \( \exp\left( \int_{(x_0, y_0)}^{(x, y)} \eta \right) \), where \( \eta \) is a rational 1-form such that \( d\eta \equiv 0 \), then there exists an inverse integrating factor of system (1) of the form

\[
V = \exp[D/E] \prod C_i^{l_i},
\]

where \( D, E \) and the \( C_i \) are polynomials in \( x \) and \( y \) and \( l_i \in \mathbb{C} \).

We notice that \( C_i = 0 \) are invariant algebraic curves and \( \exp[D/E] \) is an exponential factor for system (1). In fact, since system (1) is a real system, we can assume, without loss of generality, that \( V \) is a real function.

Theorem 4 states that the search for Liouvillian integrals can be reduced to the search of invariant algebraic curves and exponential factors. Therefore, if we characterize the possi-
ble cofactors, we have the invariant algebraic curves of a system and, hence, its Liouvillian or non-Liouville integrability.

In Section 3, we use these results on Liouvillian integrability of a polynomial system (1) to prove the non-existence of a Liouvillian first integral for several families of quadratic systems. We recall that a quadratic system is a system of degree 2. In order to show the power of the method, we first study a family of quadratic Lotka–Volterra systems and we show that no Liouvillian first integral can exist for this family.

We then apply the results given in Section 2 to some families of quadratic systems with an algebraic limit cycle of degrees 2 and 4. We recall that a limit cycle for system (1) is an isolated periodic orbit and a limit cycle is said to be algebraic when it is contained in one of the ovals of an invariant algebraic curve. We show that the first integral of these systems cannot belong to the functional class of Liouvillian functions. To this end we will first prove that there are no other invariant algebraic curves for each system but the one which defines the algebraic limit cycle. As a consequence, and applying Theorem 4, we will conclude with the desired result.

2. Necessary conditions for a cofactor

We notice that the existence of an invariant algebraic curve depends on the existence of its corresponding cofactor. Moreover, while the degree of an invariant algebraic curve is not uniformly bounded for all the systems of degree \( d \), the degree of its cofactor must be lower or equal than \( d - 1 \).

The main result of this work is a set of necessary conditions for a polynomial \( k(x, y) \in \mathbb{R}[x, y] \), of degree lower or equal than \( d - 1 \), to be the cofactor of an invariant algebraic curve or an exponential factor for a planar polynomial system (1) of degree \( d \).

These conditions on the cofactor correspond to its possible values at the critical points of the considered system. A singular point, also denoted by critical point, of system (1) is a point \((x_0, y_0) \in \mathbb{C}^2\) so that \(P(x_0, y_0) = Q(x_0, y_0) = 0\). The hypothesis of coprimality between \(P\) and \(Q\) implies that all the singular points of a system (1) are isolated, that is, there always exists a neighborhood for each singular point so that no other singular point belongs to it.

When \((x_0, y_0) \in \mathbb{R}^2\), we say that it is a real singular point. We notice that since both \(P(x, y)\) and \(Q(x, y)\) have real coefficients, if \((x_0, y_0)\) is a singular point with non-null imaginary part, then its conjugate \((\bar{x}_0, \bar{y}_0)\) is also a singular point. A singular point \((x_0, y_0)\) with non-null imaginary part will be called a complex singular point, to be distinguished from a real one.

Real singular points for a system (1) are classified in terms of the behavior in \(\mathbb{R}^2\) of the solutions of the system in their neighborhood, see for instance [22]. This behavior is, in general, determined by the linear approximation of the system at the point. That is, let \((x_0, y_0)\) be a critical point for system (1) and let \(A(x_0, y_0)\) be the following matrix:

\[
A(x_0, y_0) = \begin{pmatrix}
\frac{\partial P}{\partial x}(x_0, y_0) & \frac{\partial P}{\partial y}(x_0, y_0) \\
\frac{\partial Q}{\partial x}(x_0, y_0) & \frac{\partial Q}{\partial y}(x_0, y_0)
\end{pmatrix},
\]  

(5)
which gives the linear approximation of the system at \((x_0, y_0)\).

A singular point \((x_0, y_0)\) is called degenerate if the matrix \(A(x_0, y_0)\) is degenerate, that is, \(\det A(x_0, y_0) = 0\). Otherwise, we will say that the singular point \((x_0, y_0)\) is non-degenerate. Assume that \((x_0, y_0)\) is a non-degenerate singular point. Let \(\lambda, \mu \in \mathbb{C}\) be the eigenvalues of \(A(x_0, y_0)\). We have that \(\lambda \mu \neq 0\).

If \(\lambda, \mu \in \mathbb{R}\) and \(\lambda/\mu < 0\), the point \((x_0, y_0)\) is called a saddle. If \(\lambda, \mu \in \mathbb{R}\) and \(\lambda/\mu > 0\), the point \((x_0, y_0)\) is called a node.

As we are considering a real system and a real singular point, the characteristic polynomial of the matrix \(A(x_0, y_0)\) is quadratic and with real coefficients. If one of the eigenvalues is a complex number with non-null imaginary part, the other eigenvalue is its conjugate.

If \(\lambda = a + bi\) and \(\mu = a - bi\), where \(i = \sqrt{-1}\) and \(b \neq 0\), then if \(a \neq 0\) the point is called a strong focus and if \(a = 0\) then it can be a center or a weak focus. The so-called center problem consists in distinguishing between these two possibilities. Since this work is not concerned with this problem, we refer the interested reader to [22].

For a degenerate singular point the local study of the solutions in its neighborhood can be made by using the blow-up technique. If \(A(x_0, y_0)\) has only one eigenvalue equal to zero then \((x_0, y_0)\) is an elementary degenerate singular point and its local behavior is studied in [2]. If zero is a double eigenvalue of \(A(x_0, y_0)\) but \(A(x_0, y_0)\) is not identically zero then the degenerate singular point \((x_0, y_0)\) is called nilpotent and a good characterization of its local behavior is given in [1]. Finally, for degenerate singular points \((x_0, y_0)\) with \(A(x_0, y_0)\) identically zero, see [14] for a detailed description of the blow-up technique, which can be rather complicated.

Complex singular points are also classified in degenerate and non-degenerate in the same manner as real singular points. The local behavior of the solutions in a neighborhood of a complex singular point has not the same sense than in the real case. However, the eigenvalues of the matrix \(A(x_0, y_0)\) defined in (5) will be of great importance for our results, both for real and for complex singular points.

We take advantage of the location of each singular point of system (1) and the local behavior of the solutions in its neighborhood. In order to enlarge the set of conditions on the cofactor \(k(x, y)\) at each singular point, we also consider infinite singular points. That is, we extend an equation \(Q(x, y) dx - P(x, y) dy = 0\), to the projective plane \(\mathbb{P}^2\), and we consider all the singular points of the extended equation in \(\mathbb{P}^2\).

The next subsection consists in a brief summary of the process of extending the equation in the affine plane to the projective plane, as well as its most interesting features like invariant algebraic curves and exponential factors.

In Section 2.2 a result due to A. Seidenberg [26], which describes the local behavior of the analytic solutions in a neighborhood of a singular point, is given. By using this result, the main contribution of this paper is given in Section 2.3, as well as its proof.

2.1. Critical points at infinity

We first define a polynomial differential equation in \(\mathbb{P}^2\) and the notion of invariant algebraic curve and critical point for the equation. We describe the planar polynomial systems obtained when taking local coordinates and we show the coherence between immersing an equation in the affine plane to the projective plane and the other way round,
that is, submersing the differential equation of the projective plane in the affine plane by means of taking local coordinates. We only give an introductory summary of all the facts related to differential equations in \( \mathbb{CP}^2 \).

Critical points at infinity may also be studied by submerging our equation in the Poincaré’s sphere \( S^2 \), see [20]. Both ways to study critical points at infinity are equivalent.

We recall that \( \mathbb{CP}^2 = \{ C^3 \setminus \{ (0,0,0) \} \} / \sim \) with the equivalence relation \( [X,Y,Z] \sim [X',Y',Z'] \) if, and only if, there exists \( v \in \mathbb{C} - \{ 0 \} \) such that \( [X', Y', Z'] = v[X, Y, Z] \).

We consider \( P, Q, R \), three homogeneous polynomials of degree \( d + 1 \) in the variables \( (X, Y, Z) \) and the 1-form:

\[
\Omega := P \, dX + Q \, dY + R \, dZ.
\]

We always assume that \( P, Q, R \) are coprime polynomials, that is, that there is no non-constant polynomial which divides \( P, Q \) and \( R \).

**Definition 5.** We say that the 1-form \( \Omega \) is projective if

\[
X \, P + Y \, Q + Z \, R = 0.
\]

**Proposition 6.** The 1-form \( \Omega = P \, dX + Q \, dY + R \, dZ \) is projective if, and only if, there exist polynomials \( L, M, N \) of degree \( d \) such that \( \Omega \) reads for \( \Omega = (MZ - NY) \, dX + (NX - LZ) \, dY + (LY - MX) \, dZ \).

Equally, the 1-form \( \Omega = P \, dX + Q \, dY + R \, dZ \) is projective if, and only if, there exist polynomials \( L, M, N \) of degree \( d \) such that:

\[
\Omega = \det \begin{bmatrix} L & M & N \\ X & Y & Z \\ dX & dY & dZ \end{bmatrix}.
\]

This proposition is proved in [8]. We notice that the polynomials \( L, M, N \) are not uniquely determined by \( P, Q \) and \( R \). These polynomials can be replaced by \( L' = L + \Delta X, M' = M + \Delta Y \) and \( N' = N + \Delta Z \), where \( \Delta \) is any homogeneous polynomial with variables \( X, Y, Z \) and degree \( d - 1 \).

The projective 1-form \( \Omega \) defines a differential equation in \( \mathbb{CP}^2 \) given by \( \Omega = 0 \), which may be written, by Proposition 6,

\[
(MZ - NY) \, dX + (NX - LZ) \, dY + (LY - MX) \, dZ = 0.
\]

In this context, an invariant algebraic curve for Eq. (7) is an algebraic curve \( F(X, Y, Z) = 0 \), where \( F \) is a homogeneous polynomial, such that

\[
L \frac{\partial F}{\partial X} + M \frac{\partial F}{\partial Y} + N \frac{\partial F}{\partial Z} = K \, F.
\]

for a certain homogeneous polynomial \( K(X, Y, Z) \) of degree \( d - 1 \), called the cofactor.

We notice that the cofactor is not uniquely determined since it depends on \( L, M, N \). By changing \( (L, M, N) \) to \( (L + \Delta X, M + \Delta Y, N + \Delta Z) \) with \( \Delta \) a homogeneous polynomial of degree \( d - 1 \), which define the same differential equation (7), an easy application of Euler’s theorem on homogeneous functions shows that \( K \) changes to \( K + n \Delta \) where \( n \) is
the degree of $F$. Let us prove this statement. We apply Euler’s theorem on homogeneous functions to the polynomial $F(X, Y, Z)$, i.e.,

$$X \frac{\partial F}{\partial X} + Y \frac{\partial F}{\partial Y} + Z \frac{\partial F}{\partial Z} = nF.$$ 

So, let $K(X, Y, Z)$ be its cofactor for $L, M$ and $N$, i.e.,

$$L \frac{\partial F}{\partial X} + M \frac{\partial F}{\partial Y} + N \frac{\partial F}{\partial Z} = KF,$$

and we change $L, M$ and $N$ by $L+\Delta X, M+\Delta Y$ and $N+\Delta Z$, where $\Delta$ is a homogeneous polynomial of degree $d-1$. Hence,

$$(L+\Delta X) \frac{\partial F}{\partial X} + (M+\Delta Y) \frac{\partial F}{\partial Y} + (N+\Delta Z) \frac{\partial F}{\partial Z}$$

$$= \left( L \frac{\partial F}{\partial X} + M \frac{\partial F}{\partial Y} + N \frac{\partial F}{\partial Z} \right) + \Delta \left( X \frac{\partial F}{\partial X} + Y \frac{\partial F}{\partial Y} + Z \frac{\partial F}{\partial Z} \right)$$

$$= (K+n\Delta)F.$$

A point $[X_0, Y_0, Z_0] \in \mathbb{CP}(2)$ is a singular point of the projective equation $PdX + QdY + RdZ = 0$ if $P(X_0, Y_0, Z_0) = Q(X_0, Y_0, Z_0) = R(X_0, Y_0, Z_0) = 0$.

An Eq. (7) is a planar polynomial system (1) when taking local coordinates in a chart. Let us consider a point $p := [X_0, Y_0, Z_0] \in \mathbb{CP}(2)$ and, without loss of generality, we assume that $Z_0 \neq 0$. We define the local coordinates in $p$ by $x = X/Z$ and $y = Y/Z$. So, in local coordinates, we have $p = (x_0, y_0)$ with $x_0 = X_0/Z_0$ and $y_0 = Y_0/Z_0$. We consider an Eq. (7) and we define $P(x, y) := L(x, y, 1) - xN(x, y, 1)$ and $Q(x, y) := M(x, y, 1) - yN(x, y, 1)$. We notice that replacing $(L, M, N)$ by $(L+\Delta X, M+\Delta Y, N+\Delta Z)$, where $\Delta$ is any homogeneous polynomial of degree $d-1$, gives the same polynomials $P(x, y)$ and $Q(x, y)$. We say that the equation $Q(x, y)dx - P(x, y)dy = 0$ is the differential equation (7) at the local chart at $p$. An invariant algebraic curve $F(X, Y, Z) = 0$ for Eq. (7) becomes $f(x, y) = 0$ if $F(x, y, 1) = 0$. It is easy to show that if $F(X, Y, Z) = 0$ is an invariant algebraic curve for (7) with cofactor $K(X, Y, Z)$, then $f(x, y) = 0$ is an invariant algebraic curve for $Q(x, y)dx - P(x, y)dy = 0$ with cofactor $k(x, y) = K(x, y, 1) - nN(x, y, 1)$. Moreover, if $p$ is a singular point for (7), then $(x_0, y_0)$ is a singular point for $Q(x, y)dx - P(x, y)dy = 0$.

We describe now the process of extending an equation $Q(x, y)dx - P(x, y)dy = 0$, to the projective space $\mathbb{CP}(2)$. We consider the change to projective coordinates $x = X/Z$ and $y = Y/Z$, from which $dx = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$ are deduced. The coordinates $(x, y)$ are usually called finite coordinates and the set of points $[X, Y, Z] \in \mathbb{CP}(2)$ with $Z = 0$ is called the line at infinity.

Writing $P(x, y) = P_0 + P_1(x, y) + P_2(x, y) + \cdots + P_d(x, y)$ where $P_i(x, y)$ is a homogeneous polynomial of degree $i$, and expressing $(x, y)$ in terms of $(X, Y, Z)$ we have:

$$P \left( \frac{X}{Z}, \frac{Y}{Z} \right) = P_0 + \frac{1}{Z} P_1(X, Y) + \frac{1}{Z^2} P_2(X, Y) + \cdots + \frac{1}{Z^d} P_d(X, Y)$$

$$= \frac{1}{Z^d} (Z^d P_0 + Z^{d-1} P_1(X, Y) + \cdots + P_d(X, Y)).$$
We define $L(X,Y,Z) = Z^d P_0 + Z^{d-1} P_1(X,Y) + \cdots + P_d(X,Y)$, which is a homogeneous polynomial of degree $d$. Analogously, we define the homogeneous polynomial of degree $d$, $M(X,Y,Z)$, from $Q(x,y)$, such that $M(X,Y,Z) = Z^d Q(X/Z,Y/Z)$. Substituting in $Q(x,y) \, dx - P(x,y) \, dy = 0$ we have $L(Y \, dZ - Z \, dY) + M(Z \, dX - X \, dZ) = 0$, which is an Eq. (7) with $N \equiv 0$.

An invariant algebraic curve $f(x,y) = 0$, with cofactor $k(x,y)$, for an equation $Q(x,y) \, dx - P(x,y) \, dy = 0$ defines the invariant algebraic curve $F(X,Y,Z) = 0$ in $\mathbb{CP}^2$ with $F(X,Y,Z) = Z^n f(X/Z,Y/Z)$, where $n$ is the degree of $f$. We have that the associated cofactor of $F(X,Y,Z) = 0$ is $K(X,Y,Z) = Z^{d-1} k(X/Z,Y/Z)$.

2.2. Local behavior of solutions

In this subsection we give a brief summary of definitions and results concerning formal differential equations and their solutions. These results concern the local behavior of solutions in a neighborhood of a critical point of a system (1). Formal differential equations were studied by Seidenberg in [26]. We explicitly state only the necessary result for our aims. In [28], the author states the same result included in his Theorem 2.3. We refer the reader to [26,28] for a further description.

Let $\mathbb{C}[x,y]$ be the ring of formal power series in two variables with coefficients in $\mathbb{C}$, that is,

$$\mathbb{C}[x,y] = \left\{ \varphi(x,y) = \sum_{i,j \geq 0} \varphi_{ij} x^i y^j \mid \varphi_{ij} \in \mathbb{C} \right\},$$

with the usual operations of addition and multiplication. This ring is factorial.

We are also interested in the ring $\mathbb{C}[x,y]$ of convergent power series, that is the subring of $\mathbb{C}[x,y]$ of elements $\varphi$ with a positive radius of convergence. When $\varphi \in \mathbb{C}[x,y]$, we say that it is analytic.

We describe some properties of the elements of $\mathbb{C}[x,y]$. The order of $\varphi(x,y)$ is $\min_{i,j \geq 0} \{ i + j \mid \varphi_{ij} \neq 0 \}$. The set of units for this ring corresponds to all the $\varphi(x,y) \in \mathbb{C}[x,y]$ of order 0, that is, such that $\varphi_{00} \neq 0$. A unit element of this ring will be denoted by $\upsilon(x,y)$.

Two formal series $\varphi(x,y), \psi(x,y) \in \mathbb{C}[x,y]$ are said to be equal, $\varphi = \psi$, when $\varphi_{ij} = \psi_{ij}$ for all $i,j \geq 0$.

A formal power series $\varphi(x,y)$ is said to be constant if $\varphi_{ij} = 0$ for all $i,j \geq 1$. The formal power series whose coefficients are all null is denoted by $0$.

We define the partial derivatives of $\varphi(x,y)$ with respect to $x$ and $y$ in the following way

$$\frac{\partial \varphi}{\partial x} := \sum_{i,j \geq 0} i \varphi_{ij} x^{i-1} y^j, \quad \frac{\partial \varphi}{\partial y} := \sum_{i,j \geq 0} j \varphi_{ij} x^i y^{j-1}.$$

Let $\varphi(x,y) \in \mathbb{C}[x,y] - \{0\}$ be an irreducible non-unit element. An analytic branch centered at $(0,0)$ is the equivalence class of $\varphi$ under the equivalence relation $\varphi \sim \psi$ if $\varphi = \upsilon \psi$. Since $\varphi(x,y)$ is non-unit, we have that its order is greater or equal than 1.

A branch $\varphi(x,y)$ is said to be linear if its order equals 1 and non-linear if its order is strictly greater than 1.
Let \( \varphi(x, y) \in \mathbb{C}[x, y] - \{0\} \) a non-unit element of order \( s \). The homogeneous polynomial of degree \( s \) given by \( \varphi_s := \sum_{i=0}^{s} \varphi_{i,s} - i x^i y^{s-i} \) is called the tangents of \( \varphi \) at the origin. This homogeneous polynomial \( \varphi_s \) factorizes in \( \mathbb{C}[x, y] \) in exactly \( s \) linear factors: \( \varphi_s = \ell_1 \ell_2 \ldots \ell_s \). If all these linear factors are \( x \) (respectively \( y \)), we say that \( \varphi \) has vertical (respectively horizontal) tangent.

Consider the formal differential equation \( Q(x, y) dx - P(x, y) dy = 0 \), where \( Q(x, y), P(x, y) \in \mathbb{C}[x, y] - \{0\} \) are non-unit elements. Equally, this formal differential equation can be given by \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \).

By a solution of this formal differential equation, we mean an analytic branch \( \varphi(x, y) \) centered at the origin such that there exists \( \kappa(x, y) \in \mathbb{C}[x, y] \) satisfying
\[
P \frac{\partial \varphi}{\partial x} + Q \frac{\partial \varphi}{\partial y} = \kappa \varphi.
\]

**Theorem 7** [26,28]. Consider the formal differential system
\[
\dot{x} = \lambda x + X_2(x, y), \quad \dot{y} = \mu y + Y_2(x, y),
\]
(8)
where \( X_2, Y_2 \in \mathbb{C}[x, y] \) are of order greater or equal than two and with \( \lambda \neq 0 \).

(a) If \( \mu \neq 0 \) and \( \lambda, \mu \) are rationally independent or \( \lambda/\mu < 0 \), or \( \mu = 0 \), then there are exactly two solutions, a linear branch with horizontal tangent and a linear branch with vertical tangent.

(b) If \( \mu \neq 0 \) and \( \lambda, \mu \) are rationally dependent and \( \lambda/\mu > 0 \), then
(i) if \( \lambda/\mu = 1 \), then for any direction there is a linear solution,
(ii) if \( \lambda/\mu > 1 \) and \( \lambda/\mu \in \mathbb{N} \), then there is a unique linear solution with horizontal tangent and either there are no solutions with vertical tangent or there are an infinite number of solutions with vertical tangent, all of them linear,
(iii) if \( \lambda/\mu > 1 \) and \( \lambda/\mu \notin \mathbb{N} \), we have that \( \lambda/\mu = p/q \) with \( p, q \in \mathbb{N} \) and \( 1 < q < p \).
Then there is a unique linear solution with horizontal tangent and there is one solution with vertical tangent, which is linear. There are an infinite number of solutions with vertical tangent, all of them are non-linear and their tangents are given by \( \varphi_q = x^q \).

The expressions \( X_2 \) and \( Y_2 \) are formal complex series and \( \lambda \) and \( \mu \) are complex numbers which can be real as a particular case.

We remark that except in the case \( \lambda/\mu = 1 \), the solutions of system (8) can only have horizontal or vertical tangent.

In [26], the proof of this result is given by means of blowing-up the origin of system (8). In [28] system (8) is supposed to be analytic in a neighborhood of the origin and the normal form technique is used to prove Theorem 7.

By using the normal form theory, Walcher in [28] also proves the following result, which lets us distinguish between a unique solution and an infinite number of solutions in case \( \lambda/\mu > 1 \) and \( \lambda/\mu \in \mathbb{N} \). We do not intend to give a survey on normal form theory and we only state the following proposition for the sake of completeness. We refer the reader to
for an exhaustive description of this classical theory due to Poincaré and the proof of the following proposition.

**Proposition 8** (Poincaré). Let us consider a system \( (8) \). Let us assume that \( \lambda/\mu = m \) with \( m \in \mathbb{N} \) and \( m > 1 \). Then, there is a formal change of variables which transforms the system to
\[
\dot{x} = \lambda x + cy^m, \quad \dot{y} = \mu y,
\]
with \( c \in \mathbb{C} \).

Moreover, if system \((8)\) is analytic in a neighborhood of the origin, then the change of variables is analytic in a neighborhood of the origin.

When \( c = 0 \) in \((9)\), we say that system \((8)\) is linearizable. If \( c \neq 0 \), we say that it is non-linearizable.

The distinction between linearizable and non-linearizable systems allows the distinction between an infinite number of solutions or a unique solution, as stated and proved in [28].

**Proposition 9.** Consider a system \((8)\), analytic in a neighborhood of the origin, where \( \lambda/\mu = m \) with \( m \in \mathbb{N} \) and \( m > 1 \).

- If the system is linearizable, then there is a unique solution with horizontal tangent and an infinite number of solutions with vertical tangent, all of them linear.
- If the system is non-linearizable, there is exactly one solution which is linear and with horizontal tangent.

Let \((x_0, y_0)\) be a non-degenerate or elementary degenerate singular point for system \((1)\). Let \( \lambda \) and \( \mu \) be the eigenvalues of the matrix \( A(x_0, y_0) \). Throughout this work we will always assume that \( \lambda \neq 0 \) and \( \lambda \neq \mu \). We notice that by an affine change of coordinates we can always write system \((1)\) in the form \((8)\) with this critical point at the origin. Moreover, the eigenvectors \( v_\lambda \) and \( v_\mu \) of the matrix \( A(x_0, y_0) \) are related to the horizontal and vertical directions for system \((8)\). Therefore, Theorem 7 and Proposition 9 apply for the critical point \((x_0, y_0)\) of system \((1)\).

2.3. The main result

The main result in this section is split up in Theorems 13, 14 and 15, which are consequences of Theorem 7 and Proposition 9. These theorems give necessary conditions for a polynomial of degree lower or equal than \( d - 1 \) to be a cofactor of an invariant algebraic curve.

Given \( p_0 := (x_0, y_0) \) a singular point for Eq. \((1)\) and \( f(x, y) = 0 \) an invariant algebraic curve, only two possibilities hold: either \( f(p_0) \neq 0 \) or \( f(p_0) = 0 \). The next lemma deals with the first possibility and Theorems 13, 14 and 15 with the second one.

**Lemma 10.** Let us consider a system \((1)\), \((x_0, y_0)\) one of its critical points and \( f(x, y) = 0 \) an invariant algebraic curve with cofactor \( k(x, y) \). If \( f(x_0, y_0) \neq 0 \), then \( k(x_0, y_0) = 0 \).
Proof. Assume that \( f(x, y) = 0 \) is an invariant algebraic curve for system (1) with cofactor \( k(x, y) \). Let \((x_0, y_0)\) be a singular point of system (1). Since the left-hand side of the equality
\[
P(x, y) \frac{\partial f}{\partial x}(x, y) + Q(x, y) \frac{\partial f}{\partial y}(x, y) = k(x, y) f(x, y)
\]
is zero at \((x_0, y_0)\) and \( f(x_0, y_0) \neq 0 \), we deduce that \( k(x_0, y_0) = 0 \). \( \square \)

We remark that since we can assume that \( f(x, y) \) and its cofactor \( k(x, y) \) belong to \( \mathbb{R}[x, y] \), then if \((x_0, y_0)\) is a complex singular point such that \( f(x_0, y_0) \neq 0 \), Lemma 10 implies that \( k(x_0, y_0) = 0 \) and \( k(x_0, y_0) = 0 \).

Let \( f(x, y) = 0 \) be an algebraic curve and \((x_0, y_0)\) a point such that \( f(x_0, y_0) = 0 \). We may expand \( f(x, y) \) in powers of \((x - x_0)\) and \((y - y_0)\): \( f(x, y) = f_s(x, y) + f_{s+1}(x, y) + \cdots + f_a(x, y) \), where \( f_j(x, y) \) are homogeneous polynomials of degree \( j \) in powers of \((x - x_0)\) and \((y - y_0)\). Let \( s \) be the lowest degree in this expansion with \( f_s(x, y) \neq 0 \). Since \( f(x_0, y_0) = 0 \), we have \( s \geq 1 \). As \( f_s(x, y) \) is a homogeneous polynomial of degree \( s \) in powers of \((x - x_0)\) and \((y - y_0)\) it factorizes in \( s \) linear homogeneous polynomials, that is, \( f_s(x, y) = \ell_1 \ell_2 \cdots \ell_s \) with \( \ell_i = a_i(x - x_0) + b_i(y - y_0), a_i, b_i \in \mathbb{C}, i = 1, 2, \ldots, s \). We say that \( f_s(x, y) = 0 \) is the equation of the tangents of the curve \( f(x, y) = 0 \) in \((x_0, y_0)\).

Given a polynomial \( f(x, y) \) we denote by \( \nabla f(x, y) \) the gradient vector at the point \((x, y)\), that is \( \nabla f(x, y) := \left( \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right) \). As usual, if \( u, v \in \mathbb{C}^2 \) we denote by \( u \cdot v \) its Euclidean scalar product.

The following three theorems are a consequence of Theorem 7 and deal with a non-degenerate or an elementary degenerate critical point \((x_0, y_0)\), whose ratio of eigenvalues does not equal 1. We provide the tangents of an invariant algebraic curve \( f(x, y) = 0 \) irreducible in \( \mathbb{C}[x, y] \) such that \( f(x_0, y_0) = 0 \). Once these tangents are described we deduce the value of the cofactor \( k(x, y) \) at \((x_0, y_0)\). We also describe the possible existence of another invariant algebraic curve \( f(x, y) = 0 \) irreducible in \( \mathbb{C}[x, y] \), with cofactor \( k(x, y) \) such that \( f(x, y) \) and \( f(x, y) \) are relatively coprime and \( f(x_0, y_0) = 0 \).

As before, we denote by \((x_0, y_0)\) a singular point of system (1) which is supposed to be non-degenerate or elementary degenerate. Let \( A \) be the matrix of the linear approximation of system (1) at \((x_0, y_0)\). Since we fix the point \((x_0, y_0)\) we do not explicit the dependence on it. We denote by \( \lambda \) and \( \mu \) the eigenvalues of the linear approximation at this point and by \( v_\lambda \) and \( v_\mu \) two corresponding eigenvectors. We assume that \( \lambda \neq \mu \) and \( \lambda \neq 0 \).

We denote by \( \ell_\lambda \) any non-null homogeneous polynomial of degree 1 such that \( \nabla \ell_\lambda \cdot v_\lambda = 0 \). An analogous definition stands for \( \ell_\mu \). We use this notation throughout the paper.

Let us consider an invariant algebraic curve \( f(x, y) = 0 \) irreducible in \( \mathbb{C}[x, y] \) such that \( f(x_0, y_0) = 0 \). Let us translate the point \((x_0, y_0)\) to the origin \((0, 0)\). The polynomial \( f(x, y) \) is also an element of the ring \( \mathbb{C}[x, y] \) and, consequently, of \( \mathbb{C}[x, y] \). Since, \( f(0, 0) = 0 \), it is not a unit element. In this ring \( f(x, y) \) can be a reducible element. The following lemma describes the decomposition of \( f(x, y) \) in the ring \( \mathbb{C}[x, y] \) which coincides with its decomposition in \( \mathbb{C}[x, y] \). The proof of the following lemma makes use of Newton–Puiseux algorithm which is described in Chapter 1 of [5], see Corollary 1.5.5 (p. 25) and Theorem 1.8.3 (p. 32).

**Lemma 11.** Let \( f(x, y) \in \mathbb{C}[x, y] \) of positive order. Then,
there are \( m \) irreducible elements \( \varphi_1, \varphi_2, \ldots, \varphi_m \in \mathbb{C}[x, y] \), \( m \geq 0 \), such that \( f \) decomposes in the form

\[
f = \nu x^r \varphi_1 \ldots \varphi_m,
\]

where \( r \in \mathbb{N}, r \geq 0, \) and \( \nu \) a unit element in \( \mathbb{C}[x, y] \).

- The elements \( \varphi_i \) of (10) can be taken in \( \mathbb{C}[x][y] \), that is, they are polynomials in \( y \).

Such a decomposition is uniquely determined, up to order, by \( f \).

If \( f \in \mathbb{C}\{x, y\} \) then the elements \( \nu, \varphi_1, \varphi_2, \ldots, \varphi_m \) of (10) belong to \( \mathbb{C}\{x, y\} \). In fact, \( \varphi_i \in \mathbb{C}\{x\}[y], i = 1, 2, \ldots, m \).

Since we consider algebraic curves given by \( f(x, y) = 0 \) with \( f \) as an irreducible element of \( \mathbb{C}[x, y] \), which is a subring of \( \mathbb{C}\{x, y\} \), we can strengthen the thesis of the previous lemma.

**Lemma 12.** Let \( f(x, y) \) be an irreducible non-constant polynomial in \( \mathbb{C}[x, y] \) such that \( f(0, 0) = 0 \). Then, the decomposition given in (10) is square free.

**Proof.** Taking into account Lemma 11, we only need to prove that there is no repeated element \( \varphi_i \), neither \( r > 1 \) in the decomposition (10).

Assume that either there is a repeated element \( \varphi_i \) or \( r > 1 \) in the decomposition (10). Then, this element divides both \( f \) and \( \frac{\partial f}{\partial y} \) in \( \mathbb{C}[x, y] \). Therefore, \( f \) and \( \frac{\partial f}{\partial y} \) in \( \mathbb{C}[x, y] \) intersect in an infinite number of points inside the disk of convergence of this repeated element.

However, by Bézout’s theorem, if \( f \) and \( \frac{\partial f}{\partial y} \) have an infinite number of intersection points, there is a polynomial both dividing \( f \) and \( \frac{\partial f}{\partial y} \). Since \( f \) is an irreducible polynomial, this divisor must coincide with \( f \). So \( f \) divides \( \frac{\partial f}{\partial y} \), that is, there exists \( g \in \mathbb{C}[x, y] \) such that \( \frac{\partial f}{\partial y} = g f \). Hence, the degree of \( \frac{\partial f}{\partial y} \) equals the sum of degrees of \( f \) and \( g \). But this is not possible because if \( f \) has degree \( n, n \geq 1 \), then \( \frac{\partial f}{\partial y} \) has degree at most \( n - 1 \). \( \square \)

We do not give an explicit statement of Bézout’s theorem since it is a classical and well known result. See, for instance, [19,29] for a rigorous statement.

Let us consider \( f(x, y) = 0 \) an invariant algebraic curve of system (1) with \( f(x, y) \in \mathbb{C}[x, y] \) as an irreducible polynomial. Let \( (x_0, y_0) \) be a non-degenerate or elementary degenerate critical point of system (1) with eigenvalues \( \lambda \) and \( \mu \) such that \( \lambda \neq \mu \) and with \( f(x_0, y_0) = 0 \). Without loss of generality we can translate \( (x_0, y_0) \) to the origin. An easy reasoning, based on the fact that \( \mathbb{C}[x, y] \) is a factorial ring, shows that each of the irreducible elements appearing in the decomposition of \( f(x, y) \) written in (10) is a solution of system (1). Therefore, we notice that \( f(x, y) = 0 \) defines a finite number of branches in \( (x_0, y_0) \) corresponding to its irreducible non-unit factors in \( \mathbb{C}[x, y] \). The tangents of these branches are given by \( f_s(x, y) = 0 \) as defined above. The following theorem describes these tangents and the value of the cofactor at the singular point.

**Theorem 13.** With the described notation, we have that \( f_s(x, y) = (\ell_\lambda)^r (\ell_\mu)^{s-r} \) with \( r, s \in \mathbb{N} \) and \( r \leq s \). Moreover, \( k(x_0, y_0) = r \mu + (s-r) \lambda \).
Theorem 14. Let
\[ fs(x,y) \]
be an invariant algebraic curve irreducible in \( \mathbb{C} \). With associated eigenvalues \( \lambda \) and \( \mu \), we get

\[ \nabla f \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) = k(x,y)f(x,y) \]

in powers of \( x \) and \( y \) and equating the non-null terms of lowest degree, which corresponds to degree \( s \), we have that \( \nabla f_s \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) = k(0,0)f_s \). Since \( f_s(x,y) = (\ell\lambda)^s(\ell\mu)^{-r} \), and dividing both members of the equation by \( (\ell\mu)^{-r}(\ell\mu)^{r-1} \) we get

\[
r\ell\mu \left[ \nabla \ell\lambda \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) \right] + (s-r)\ell\lambda \left[ \nabla \ell\mu \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) \right] = k(0,0)\ell\lambda \ell\mu.
\]

(11)

Let us first assume that \( r \neq 0 \) and \( s \neq r \), then from Eq. (11) we deduce that there exists \( a \in \mathbb{C} \) such that \( \nabla \ell\lambda \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) = a\ell\lambda \). Using the identity \( \ell\lambda = \nabla \ell\lambda \cdot \left( \begin{array}{c} x \\ y \end{array} \right) \) and equating the coefficients of \( x \) and \( y \) we get \( \nabla \ell\lambda \cdot A = a\nabla \ell\lambda \). We multiply both terms of this equality by \( v_\mu \), noticing that since \( \nabla \ell\lambda \cdot v_\lambda = 0 \) then \( \nabla \ell\lambda \cdot v_\mu \neq 0 \). Hence, \( \nabla \ell\lambda \cdot A \cdot v_\mu = a\nabla \ell\lambda \cdot v_\mu \) and this gives \( \mu \nabla \ell\lambda \cdot v_\mu = a\nabla \ell\lambda \cdot v_\mu \) from which we obtain \( a = \mu \). Analogous reasonings show that \( \nabla \ell\mu \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) = \lambda \ell\mu. \) Substituting in Eq. (11) and dividing by \( \ell\mu \), we have that \( k(x_0,y_0) = r\mu + (s-r)\lambda \).

If \( r = 0 \), then \( s-r = s \geq 1 \). From Eq. (11) we get that there exists \( b \in \mathbb{C} \) such that \( \nabla \ell\mu \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) = b\ell\mu \) and \( k(x_0,y_0) = (s-r)b \). As before, it is easy to show that \( b = \lambda \).

If \( s = r \), then \( s \geq 1 \) and from Eq. (11) we have again \( \nabla \ell\lambda \cdot A \cdot \left( \begin{array}{c} x \\ y \end{array} \right) = a\ell\lambda \), with \( a \in \mathbb{C} \) and \( k(x_0,y_0) = ra \). The equality \( a = \mu \) is achieved as before. \( \square \)

The following theorem precises more accurately the form of the equation of the tangents \( fs(x,y) = 0 \).

Theorem 14. Let \( p_0 \) be a singular point for system (1) with associated eigenvalues \( \lambda \) and \( \mu \), with \( \lambda \neq 0 \) and let \( f(x,y) = 0 \) be an invariant algebraic curve irreducible in \( \mathbb{C}[x,y] \) such that \( f(p_0) = 0 \). We assume that \( f_s(x,y) = 0 \) is the equation of the tangents of \( f = 0 \) at \( p_0 \).

- If either \( \mu \neq 0 \) and \( \lambda \) and \( \mu \) are rationally independent or \( \lambda/\mu < 0 \), or \( \mu = 0 \), then
  - either \( s = 2 \) and \( f_2 = \ell_\lambda \ell_\mu \),
  - or \( s = 1 \) and \( f_1 = \ell_\lambda \),
  - or \( s = 1 \) and \( f_1 = \ell_\mu \).

- If \( \mu \neq 0 \) and \( \lambda \) and \( \mu \) are rationally dependent and \( \lambda/\mu > 0 \), we assume that \( \lambda/\mu > 1 \).
  - If \( \lambda/\mu = m \) with \( m \in \mathbb{N} \), \( m > 1 \), and system (1) is linearizable in \( (x_0,y_0) \), then there exists \( r \in \mathbb{N} \), \( r \geq 0 \) and \( \varepsilon \in \{0,1\} \) such that \( s = r + \varepsilon \) and \( f_2 = (\ell_\lambda)^\varepsilon (\ell_\mu)^r \).
If \( \lambda/\mu = m \) with \( m \in \mathbb{N}, m > 1 \), and system (1) is non-linearizable in \((x_0, y_0)\), then \( s = 1 \) and \( f_1 = \ell_\lambda \).

- If \( \lambda/\mu = p/q \) with \( p, q \in \mathbb{N} \) and \( 1 < q < p \), then
  - either \( s = 1 \) and \( f_1 = \ell_\lambda \),
  - or there exists \( r \in \mathbb{N}, r \geq 0 \), and \( \varepsilon \in \{0, 1\} \) such that \( s = rq + \varepsilon \) and \( f_s = (\ell_\lambda)^r(\ell_\mu)^\varepsilon \).
  - or there exists \( r \in \mathbb{N}, r \geq 0 \), and \( \varepsilon \in \{0, 1\} \) such that \( s = rq + 1 + \varepsilon \) and \( f_s = (\ell_\lambda)^r(\ell_\mu)^\varepsilon + 1 \).

In the work [21], J. Moulin-Ollagnier also gives a set of necessary conditions for a system (1) to have an invariant algebraic curve. His set of necessary conditions correspond to the value of the cofactor at singular points with eigenvalues \( \lambda, \mu \) such that \( \lambda, \mu \neq 0 \) and \( \lambda/\mu < 0 \) or \( \lambda/\mu \) rationally independent. His conditions coincide with the ones we give for these values of the eigenvalues. However, his proof uses other techniques, such as Levelt’s method.

**Proof.** The fact that a unique linear branch is defined in a given direction depending on the value of the eigenvalues \( \lambda \) and \( \mu \) plays a fundamental role in this proof. To simplify notation, we consider the point \((x_0, y_0)\) translated to the origin \((0, 0)\). Let us consider the polynomial \( f(x, y) \) and factorize it in linear branches belonging to the ring of formal power series \( \mathcal{C}[x, y] \).

First we assume that \( \mu \neq 0 \) and \( \lambda \) and \( \mu \) are rationally independent or \( \lambda/\mu < 0 \), or \( \mu = 0 \), then \( f(x, y) \) can factorize at most in two linear branches because only two linear branches are defined in a neighborhood of \((x_0, y_0)\). That is, \( f(x, y) \) can be \( f(x, y) = \phi_1\psi_\mu, v \) or \( f(x, y) = \phi_\mu, v \) or \( f(x, y) = \phi_\mu, v \) with \( \phi_1, \psi_\mu, v \in \mathcal{C}[x, y] \), \( v \) is a unit element, \( \phi_1 = \ell_\lambda + \text{h.o.t.} \) and \( \psi_\mu = \ell_\mu + \text{h.o.t.} \) where h.o.t. denotes terms of order greater or equal than two. These factorizations give the form of \( f_s \).

Now, we assume that \( \mu \neq 0 \) and \( \lambda \) and \( \mu \) are rationally dependent and \( \lambda/\mu > 1 \). In this case non-linear branches can also appear in the factorization of \( f(x, y) \) in \( \mathcal{C}[x, y] \). However, all the irreducible non-linear branches which may appear are of the form \( (\ell_\mu)^q + \text{h.o.t.} \), where here h.o.t. means terms of order greater or equal than \( q + 1 \). Let \( s \) be the number of irreducible non-unit linear branches which appear in the factorization of \( f(x, y) \) plus the number of non-linear branches each one multiplied by its order \( q \) at the origin. We have that \( s \in \mathbb{N} \) and \( s \geq 1 \). Since only one linear branch is defined with tangent \( \ell_\lambda \) by Theorem 7, we have that either \( f(x, y) = \phi_\lambda\psi_1 \phi_\mu \psi_\mu, v \) or \( f(x, y) = \phi_\lambda\psi_1 \phi_\mu, v \) or \( f(x, y) = \phi_\mu, v \) or \( f(x, y) = \phi_\mu, v \) with \( \phi_1 \) a linear branch with tangent \( \ell_\lambda \), \( \phi_\mu \) a non-linear branch with tangent \( (\ell_\mu)^q \) and \( v \) a unit element. These factorizations give the described form of \( f_s \). \( \Box \)

In the following theorem we assume the existence of an invariant algebraic curve \( f(x, y) = 0 \) irreducible in \( \mathcal{C}[x, y] \) and such that \( f(x_0, y_0) = 0 \). We study the possible existence of another invariant algebraic curve \( \tilde{f}(x, y) \) irreducible in \( \mathcal{C}[x, y] \), such that \( f \) and \( \tilde{f} \) are relatively coprime and \( \tilde{f}(x_0, y_0) = 0 \). We assume that this curve \( \tilde{f}(x, y) = 0 \) exists and we describe the form of its tangents at \((x_0, y_0)\), that is \( \tilde{f}_s(x, y) = 0 \), depending on the tangents of \( f(x, y) = 0 \) at this point.
Theorem 15. Let $p_0$ be a singular point for Eq. (1) with associated eigenvalues $\lambda$ and $\mu$, with $\lambda \neq 0$ and $\lambda \neq \mu$. We assume that $f = 0$ and $\tilde{f} = 0$ are two coprime invariant algebraic curves irreducibles in $\mathbb{C}[x, y]$ for system (1) such that $f(p_0) = \tilde{f}(p_0) = 0$. Let $f_s = 0$ and $\tilde{f}_s = 0$ be the equations of the tangents of these curves at $p_0$.

- If either $\mu \neq 0$ and $\lambda$ and $\mu$ are rationally independent or $\lambda/\mu < 0$, or $\mu = 0$, then
  - if $s = 2$, then no such $f(x, y) = 0$ can exist.
  - if $s = 1$ and $f_1 = \ell_\lambda$, then $\tilde{s} = 1$ and $f_1 = \ell_\mu$.
  - if $s = 1$ and $f_1 = \ell_\mu$, then $\tilde{s} = 1$ and $\tilde{f}_1 = \ell_\lambda$.
- If $\mu \neq 0$ and $\lambda$ and $\mu$ are rationally dependent and $\lambda/\mu > 0$, we assume that $\lambda/\mu > 1$. Then there exists $\tilde{s} \in \mathbb{N}$, $\tilde{s} > 1$, such that
  - if $\lambda/\mu = m$ with $m \in \mathbb{N}$, $m > 1$, and system (1) is linearizable in $(x_0, y_0)$, then
    - if $f_0 = \ell_\lambda(\ell_\mu)^{y-1}$, then $\tilde{f}_0 = (\ell_\mu)^{y}$.
    - if $f_0 = (\ell_\mu)^{y}$, then $\tilde{f}_0 = (\ell_\mu)^{y-1}$ with $\epsilon \in [0, 1]$.
  - if $\lambda/\mu = m$ with $m \in \mathbb{N}$, $m > 1$, and system (1) is non-linearizable in $(x_0, y_0)$, then no such $f$ can exist.
  - if $\lambda/\mu = p/q$ with $p, q \in \mathbb{N}$, $1 < q < p$, then
    - if $f_0 = \ell_\lambda(\ell_\mu)^{y-1}$, then there exist $\tilde{f} \in \mathbb{N}$ and $\epsilon \in [0, 1]$ such that $\tilde{s} = q + \epsilon$ and $\tilde{f}_0 = (\ell_\mu)^{\tilde{f}}$.
    - if $f_0 = (\ell_\mu)^{y}$, then there exist $\tilde{f} \in \mathbb{N}$ and $\epsilon, \tilde{\epsilon} \in [0, 1]$ such that $\tilde{s} = q + \epsilon + \tilde{s}$ and $\tilde{f}_0 = (\ell_\mu)^{\tilde{f}}$.

Proof. The proof is a straightforward consequence of Theorem 7. For instance, if $\lambda$ and $\mu$ are rationally independent or $\lambda/\mu < 0$ and $s = 2$, no other $f(x, y)$ can exist as long as it owns the described features since it would define another linear branch in $(x_0, y_0)$. The other cases use exactly the same reasoning with the addition of the result given in Theorem 14. □

In case of considering an invariant algebraic curve $f(x, y) = 0$ irreducible in $\mathbb{R}[x, y]$ and $(x_0, y_0)$ a real singular point with eigenvalues $a \pm bi$ with $b \neq 0$, we can reduce the number of possibilities.

Lemma 16. Let $f(x, y) = 0$ be an invariant algebraic curve irreversible in $\mathbb{R}[x, y]$ and $(x_0, y_0)$ a real singular point with eigenvalues $a \pm bi$ with $b \neq 0$, then $s = 2$. $f_2 = \ell_\ell k_{x_{i}, \mu}$, $k(x_0, y_0) = \lambda + \mu$ and no other invariant algebraic curve $\tilde{f}(x, y) = 0$ irreversible in $\mathbb{R}[x, y]$ with $\tilde{f}(x_0, y_0) = 0$ can exist.

Proof. Let us consider the singular point $(x_0, y_0)$ such that the matrix $A(x_0, y_0)$, with $A(x, y)$ previously defined, has eigenvalues with non-null real and imaginary part: $\lambda = a + bi$, $\mu = a - bi$ with $a, b \in \mathbb{R}$ and $b \neq 0$. Let us consider a real, irreversible in $\mathbb{R}[x, y]$, invariant algebraic curve $f(x, y) = 0$ with cofactor $k(x, y)$ and such that $f(x_0, y_0) = 0$. If this polynomial was irreducible in the ring $\mathbb{C}[x, y]$ by Theorems 13 and 14 its tangents in $(x_0, y_0)$ would be $a_0(x - x_0) + b_0(y - y_0)$ with $(-b_0, a_0)$ one of the eigenvectors of $A(x_0, y_0)$. The eigenvectors of $A(x_0, y_0)$ have non-null imaginary part because $A(x_0, y_0)$
is a matrix with real coefficients and eigenvalues with non-null imaginary part. Hence, the coefficients of \(a_0(x-x_0) + b_0(y-y_0)\) have non-null imaginary part, in contradiction to the fact that \(f(x, y)\) is a reducible element of \(C[x, y]\) and equals the product of two complex conjugate branches. Its tangents in \((x_0, y_0)\) equal the sum of the two eigenvalues, that is \(k(x_0, y_0) = \lambda + \mu = 2a\). Theorem 15 shows that no other invariant algebraic curve can pass through \((x_0, y_0)\).

Lemma 10 and Theorems 13, 14 and 15 give the possible values of the cofactor \(k(x, y)\) of an invariant algebraic curve at a non-degenerate or elementary degenerate singular point \((x_0, y_0)\) whose ratio of eigenvalues does not equal one. Given an Eq. (1) of degree \(d\), we can extend this equation to \(\mathbb{CP}^2\). If \(p_0 := [X_0, Y_0, Z_0]\) is a singular point of the extended equation, we can take local coordinates at this point and the hypothesis of Lemma 10 and Theorems 13, 14 and 15 are satisfied. We obtain, in this way, a condition on the value at \(p_0\) of a polynomial \(k(x, y)\) of degree \(\leq d - 1\) to be a cofactor of an invariant algebraic curve. For an infinite point, the coefficients of the cofactor also depend on the degree \(n\) of the curve. So, we give also conditions on the degree of the algebraic curve. Therefore, the union of all these conditions for each non-degenerate or elementary degenerate singular point, finite or infinite, gives a set of necessary conditions on \(k(x, y)\) to be a possible cofactor, and on the degree of the curve.

The following two lemmas also give conditions on a polynomial of degree \(\leq d - 1\) to be a cofactor, but associated with an exponential factor instead of an invariant algebraic curve.

Lemma 17. Let \(g = \exp[h/f]\) be an exponential factor for system (1) with cofactor \(k_g(x, y)\) and let \((x_0, y_0)\) be a critical point such that \(f(x_0, y_0) \neq 0\), then \(k_g(x_0, y_0) = 0\).

Proof. The left-hand side of the defining equation of the exponential factor

\[
P(x, y) \frac{\partial g}{\partial x}(x, y) + Q(x, y) \frac{\partial g}{\partial y}(x, y) = k_g(x, y)g(x, y)
\]

equals zero at \((x_0, y_0)\) and since \(f(x_0, y_0) \neq 0\), we have that \(g(x, y)\) is a non-null well defined function in a neighborhood of this point. Hence, \(k_g(x_0, y_0) = 0\).

The following lemma is a generalization of Lemma 17 and gives the form of an exponential factor for Eq. (1) in any chart of the extended differential equation \(\Omega = 0\), where \(\Omega = L(Y dZ - Z dY) + M(Z dX - X dZ)\) as formerly defined.

Lemma 18. Let \(g(x, y) = \exp[\phi(x, y)]\) be an exponential factor of an equation \(\omega = 0\), where \(\omega = Q(x, y) dx - P(x, y) dy\) is a 1-form of degree \(d\), where \(\phi(x, y)\) is either a polynomial or a rational function. Let

\[
k_g(x, y) := P(x, y) \frac{\partial \phi}{\partial x}(x, y) + Q(x, y) \frac{\partial \phi}{\partial y}(x, y)
\]
be the cofactor of this exponential factor \( g \) and we define

\[
K_G(X, Y, Z) := Z^{d-1} k_g \left( \frac{X}{Z}, \frac{Y}{Z} \right), \quad \Phi(X, Y, Z) := \phi \left( \frac{X}{Z}, \frac{Y}{Z} \right).
\]

Let \([X_0, Y_0, Z_0] \in \mathbb{CP}(2)\) be a critical point of the 1-form \( \Omega \) such that \( \Phi(X_0, Y_0, Z_0) \) is well-defined (it is not a point vanishing its denominator) then \( K_G(X_0, Y_0, Z_0) = 0 \).

**Proof.** We define \( G(X, Y, Z) := \exp \{ \Phi(X, Y, Z) \} \). We take local coordinates at the point \([X_0, Y_0, Z_0] \). At least one of \( X_0, Y_0 \) and \( Z_0 \) is not null. We first assume that \( Z_0 \neq 0 \) and we show how this result coincides with the one given in Lemma 17. We assume, for instance, that \( Y_0 \neq 0 \) (if it is \( X_0 \neq 0 \) analogous reasonings work).

If \( Z_0 \neq 0 \), we define \( x_0 = X_0 / Z_0 \) and \( y_0 = Y_0 / Z_0 \). We have that \((x_0, y_0)\) is a critical point for the 1-form \( \omega \) and \( g(x, y) = G(x, y, 1) \) is an exponential factor with cofactor \( k_g(x, y) = K_G(x, y, 1) \). If \( \Phi(X_0, Y_0, Z_0) \) is well-defined then \( \phi(x, y) = \Phi(x, y, 1) \) is well defined in \((x_0, y_0)\) and the same proof of Lemma 17 shows that \( k_g(x_0, y_0) = 0 \) and then, \( K_G(X_0, Y_0, Z_0) = 0 \).

If \( Y_0 \neq 0 \), we define \( u_0 = X_0 / Y_0 \), \( v_0 = Z_0 / Y_0 \), \( u = X / Y \) and \( v = Z / Y \). We consider \( \tilde{P}(u, v) = L(u, 1, v) - uM(u, 1, v) \), \( \tilde{Q}(u, v) = -vM(u, 1, v) \) and \( \tilde{Q}(u, v) du - \tilde{P}(u, v) dv = 0 \) is equation \( \Omega = 0 \) in this local chart. The point \((u_0, v_0)\) is a critical point for this equation. We define \( \tilde{\phi}(u, v) := \Phi(u, 1, v) \) and \( \tilde{g}(u, v) := G(u, 1, v) \), which is an exponential factor of system \( \dot{u} = \tilde{P}(u, v), \dot{v} = \tilde{Q}(u, v) \), with cofactor \( \tilde{k}_g(u, v) := K_G(u, 1, v) \). Let us prove this statement. We have that

\[
L(X, Y, Z) \frac{\partial \Phi}{\partial X}(X, Y, Z) + M(X, Y, Z) \frac{\partial \Phi}{\partial Y}(X, Y, Z) = K_G(X, Y, Z),
\]

for the definition of \( \Phi \) and \( k_g \). Since \( \Phi(X, Y, Z) \) is a homogeneous function of degree 0, then

\[
\left( L(X, Y, Z) - \frac{X}{Y} M(X, Y, Z) \right) \frac{\partial \Phi}{\partial X}(X, Y, Z)
- \frac{Z}{Y} M(X, Y, Z) \frac{\partial \Phi}{\partial Z}(X, Y, Z) = K_G(X, Y, Z).
\]

Taking local coordinates \( u = X / Y \) and \( v = Z / Y \) we deduce that

\[
\tilde{P}(u, v) \frac{\partial \tilde{\phi}}{\partial u}(u, v) + \tilde{Q}(u, v) \frac{\partial \tilde{\phi}}{\partial v}(u, v) = \tilde{k}_g(u, v),
\]

which implies that \( \tilde{g}(u, v) \) is an exponential factor for \( \tilde{Q}(u, v) du - \tilde{P}(u, v) dv = 0 \) with cofactor \( \tilde{k}_g(u, v) \). The same reasoning given in Lemma 17 shows that \( \tilde{k}_g(u_0, v_0) = 0 \) and then, \( K_G(X_0, Y_0, Z_0) = 0 \). \( \square \)
3. Applications of the Results

3.1. A Lotka–Volterra system

Let us consider the following Lotka–Volterra system

\[ \dot{x} = x(ax + by + 1), \quad \dot{y} = y(x + y), \quad (12) \]

with \(0 < a < 1\) and \(b > 1\). This family of quadratic systems is shown to have no Liouvillian first integral in [4]. However, once more, we prove this fact in this paper in order to show the power of Theorems 13, 14 and 15. Indeed, in [4] the integrability of the system (12) for all \(a, b \in \mathbb{R}\) is studied. We focus on parameters satisfying \(0 < a < 1\) and \(b > 1\) because in this parameter region the nature of the six singular points of the system does not change and we can directly apply our results without considering repetitive cases. Our aim is to show how easy computations show that no other invariant algebraic curve different from \(x = 0\) and \(y = 0\) can exist for system (12) with \(0 < a < 1\) and \(b > 1\).

**Theorem 19.** System (12) with \(0 < a < 1\) and \(b > 1\) has only two invariant algebraic curves which correspond to the invariant straight lines \(x = 0\) and \(y = 0\).

**Proof.** We denote by \(k_x(x, y) := ax + by + 1\) the cofactor of the invariant straight line \(x = 0\) and by \(k_y(x, y) := x + y\) the cofactor of \(y = 0\). Let us consider the critical points of system (12) and its eigenvalues. The following table also contains for a critical point \((u_1, v_1)\) with \(u_1 = X/Y\) and \(v_1 = Z/Y\) for \(p_4\) and \(p_5\). We denote by \(k_{u_1}(u_1, v_1) := (a - 1)u_1 + v_1 + b - 1\) and by \(k_{v_1}(u_1, v_1) := -1 - u_1\) the cofactors of the invariant straight lines \(u_1 = 0\) and \(v_1 = 0\), respectively. For \(p_6\) we take local coordinates \((u_2, v_2)\) with \(u_2 = Y/X\) and \(v_2 = Z/X\) and we denote by \(k_{u_2}(u_2, v_2) := (1 - b)u_2 - v_2 + 1 - a\) and by \(k_{v_2}(u_2, v_2) := -bu_2 - v_2 - a\) the cofactors of the invariant straight lines \(u_2 = 0\) and \(v_2 = 0\), respectively.

- \(p_1 = [0, 0, 1]\) has eigenvalues \(\lambda_1 = 1\) and \(\mu_1 = 0\). Both \(x\) and \(y\) are null at this point and \(k_x(p_1) = \lambda_1\), \(k_y(p_1) = \mu_1\).
- \(p_2 = [-\frac{1}{a}, 0, 1]\) has eigenvalues \(\lambda_2 = -\frac{1}{a}\) and \(\mu_2 = -1\). The polynomial \(x\) does not vanish at this point but \(y\) does and \(k_y(p_2) = \lambda_2\).
- \(p_3 = [1, -1, b - a]\) has eigenvalues \(\lambda_3 = (1 - a + \sqrt{(1 + a)^2 - 4b})/(2(a - b))\) and \(\mu_3 = (1 - a - \sqrt{(1 + a)^2 - 4b})/(2(a - b))\). No \(x\) nor \(y\) is null at this point.
- \(p_4 = [0, 1, 0]\) has eigenvalues \(\lambda_4 = -1\) and \(\mu_4 = b - 1\). Both \(u_1\) and \(v_1\) are null at this point, \(k_{u_1}(p_4) = \mu_4\) and \(k_{v_1}(p_4) = \lambda_4\).
- \(p_5 = [1 - b, a - 1, 0]\) has eigenvalues \(\lambda_5 = (b - a)/(a - 1)\) and \(\mu_5 = 1 - b\). Here, \(u_1\) is not null at this point but \(v_1\) is, and \(k_{v_1}(p_5) = \lambda_5\).
- \(p_6 = [1, 0, 0]\) has eigenvalues \(\lambda_6 = 1 - a\) and \(\mu_6 = -a\). Both \(u_2\) and \(v_2\) are null at this point, \(k_{u_2}(p_6) = \lambda_6\) and \(k_{v_2}(p_6) = \mu_6\).
We notice that these six singular points are always different in the range of the parameters considered.

Assume that \( f(x, y) = 0 \) is an irreducible invariant algebraic curve of degree \( n \) for system (12), different from \( x = 0 \) and \( y = 0 \) and with cofactor \( k(x, y) = k_{00} + k_{01}x + k_{10}y \). In local coordinates \((u_1, v_1)\) this cofactor may be written \( k(u_1, v_1) = (k_{01} - n)u_1 + k_{00}v_1 + k_{10} - n \) and in local coordinates \((u_2, v_2)\) this cofactor becomes \( k(u_2, v_2) = (k_{01} - bn)u_2 + (k_{00} - n)v_2 + k_{10} - an \).

Now we apply Theorems 13, 14 and 15. It is clear that the curve \( f = 0 \) must satisfy that \( f(p_1) \neq 0 \), \( f(p_4) \neq 0 \) and \( f(p_6) \neq 0 \). Then \( k(p_1) = k(p_4) = k(p_6) = 0 \). The only polynomial of degree 1 which satisfies these three conditions is \( k(x, y) := n(ax + y) \).

At the focus point \( p_3 \) we may have \( f(p_3) \neq 0 \) or \( f(p_3) = 0 \). If \( f(p_3) \neq 0 \), then \( k(p_3) = 0 \) by Lemma 10. If \( f(p_3) = 0 \) then \( k(p_3) = \mu_3 + \lambda_3 = \frac{1}{n - b} \), by Lemma 16. This last dichotomy can be codified by \( k(p_3) = \varepsilon_3 \frac{n}{n - b} \) with \( \varepsilon_3 \in \{0, 1\} \). This last linear equation gives \( n = \varepsilon_3 \), from which we deduce that if such \( f(x, y) = 0 \) exists, then it is an invariant straight line \((n = 1)\). Easy calculations show that there is no invariant straight line with cofactor \( ax + y \). We conclude that no invariant algebraic curve \( f(x, y) = 0 \) different from \( x = 0 \) and \( y = 0 \) can exist.

The following theorem is related to the Liouvillian integrability of system (12).

**Theorem 20.** There is no Liouvillian first integral for system (12) with \( 0 < a < 1 \) and \( b > 1 \).

**Proof.** Assume that there is a Liouvillian first integral. By applying Theorems 4 and 19, we conclude that there exists an inverse integrating factor of the form

\[
V(x, y) := \exp \left( \frac{h(x, y)}{x^{n_1}y^{n_2}} \right) x^{c_1}y^{c_2},
\]

where \( h(x, y) \in \mathbb{R}[x, y], n_1, n_2 \in \mathbb{N} \) and \( c_1, c_2 \in \mathbb{R} \). We notice that, eventually, \( h, n_1, n_2, c_1 \) or \( c_2 \) can be null.

Since \( p_3 \) is a focus point for system (12) with \( 0 < a < 1 \) and \( b > 1 \), and neither \( x \) nor \( y \) is null at this point, we have that \( V(p_3) \) is a non-null well-defined real number and, by continuity, there is a neighborhood of \( p_3 \) in which \( V \) has no zeros. So, the first integral computed from \( V \) is continuous in a neighborhood of this focus point \( p_3 \), which gives a contradiction. Hence, no such first integral can exist.

### 3.2. Quadratic systems with an algebraic limit cycle

In this subsection we consider the families of quadratic systems with algebraic limit cycles known until the moment of composition of this paper. These families sweep all the algebraic limit cycles defined by polynomials of degrees 2 and 4 for a quadratic system, as it is proved in [10]. It is shown in [9] that there are no algebraic limit cycles of degree 3 for a quadratic system. In [13], two examples of quadratic systems with an algebraic limit cycle of degree 5 and 6 are described. We will show that none of these quadratic systems has a Liouvillian first integral.
The following result is due to Ch’in Yuan-shün [11] and characterizes the algebraic limit cycles of degree 2 for a quadratic system.

**Theorem 21** [11]. If a quadratic system has an algebraic limit cycle of degree 2, then after an affine change of variables, the limit cycle becomes the circle

\[ \Gamma := x^2 + y^2 - 1 = 0. \]  

Moreover, \( \Gamma \) is the unique limit cycle of the quadratic system which can be written in the form

\[ \begin{align*}
\dot{x} &= -y(ax + by + c) - (x^2 + y^2 - 1), \\
\dot{y} &= x(ax + by + c),
\end{align*} \]  

with \( a \neq 0, c^2 + 4(b + 1) > 0 \) and \( c^2 > a^2 + b^2 \).

In [15–17], Evdokimenko proves that there are no quadratic systems having limit cycles of degree 3. An easier proof of this fact can be found in a work by J. Chavarriga, J. Llibre and J. Moulin-Ollagnier [9].

The study of algebraic limit cycles of degree 4 was initiated by A.I. Yablonskii, who found the first family, see [30] and followed by V.F. Filipstov, see [18], who found another family affine-independent of the previous one. A third family was found by J. Chavarriga [6] and in the work by J. Chavarriga, J. Llibre and J. Sorolla [10] a fourth family is found and it is proved that any quadratic system with an algebraic limit cycle of degree 4 is affine-equivalent to one of the four encountered families. The fact that the algebraic limit cycle for these four families of quadratic systems does not coexist with any other limit cycle is proved by J. Chavarriga, H. Giacomini and J. Llibre in [7].

We summarize these four families of algebraic limit cycles for quadratic systems in the following result.

**Theorem 22** [10]. After an affine change of variables the only quadratic systems having an algebraic limit cycle of degree 4 are

(a) **Yablonskii’s system**

\[ \begin{align*}
\dot{x} &= -4abc x - (a + b)y + 3(a + b)cx^2 + 4xy, \\
\dot{y} &= (a + b)abx - 4abc y + \left(4abc^2 - \frac{3}{2}(a + b)^2 + 4ab\right)x^2 \\
&\quad + 8(a + b)cxy + 8y^2,
\end{align*} \]  

with \( abc \neq 0, a \neq b, ab > 0 \) and \( 4c^2(a - b)^2 + (3a - b)(a - 3b) < 0 \). This system has the invariant algebraic curve

\[ (y + cx^2)^2 + x^2(x - a)(x - b) = 0, \]

whose oval is a limit cycle for system (15).

(b) **Filipstov’s system**
\[
\begin{align*}
\dot{x} &= 6(1+a)x + 2y - 6(2+a)x^2 + 12xy, \\
\dot{y} &= 15(1+a)y + 3a(1+a)x^2 - 2(9+5a)xy + 16y^2,
\end{align*}
\] (17)

with \(0 < a < \frac{3}{17}\). This system has the invariant algebraic curve

\[
3(1+a)(ax^2+y)^2 + 2y^2(2y - 3(1+a)x) = 0,
\] (18)

whose oval is a limit cycle for system (17).

(c) Chavarriga’s system

\[
\begin{align*}
\dot{x} &= 5x + 6x^2 + 4(1+a)xy + ay^2, \\
\dot{y} &= x + 2y + 4xy + (2+3a)y^2,
\end{align*}
\] (19)

with \(-71+17\sqrt{17} \leq a < 0\) has the invariant algebraic curve

\[
x^2 + x^3 + x^2y + 2axy^2 + 2axy^3 + a^2y^4 = 0,
\] (20)

whose oval is a limit cycle for system (19).

(d) Chavarriga, Llibre and Sorolla’s system

\[
\begin{align*}
\dot{x} &= 2(1+2x-2ax^2 + 6xy), \\
\dot{y} &= 8 - 3a - 14ax - 2axy - 8y^2,
\end{align*}
\] (21)

with \(0 < a < \frac{1}{4}\) has the invariant algebraic curve

\[
\frac{1}{4} + x - x^2 + ax^3 + xy + x^2y^2 = 0,
\] (22)

whose oval is a limit cycle for system (21).

Furthermore, in [10] it is proved that the curve (16) has genus 0 and the curves (18), (20) and (22) have genus 1. We recall that the genus \(G\) of an algebraic curve is defined by

\[
G = \frac{1}{2}(n-1)(n-2) - (\delta + \kappa)
\]

where \(n\) is the degree of the irreducible polynomial which defines the curve, \(\delta\) is the number of nodes (ordinary double points) and \(\kappa\) is the number of cusps (double points such that the tangent vector reverses sign as the curve is transversed). See [19,29] for further information on planar algebraic curves.

In a work due to C. Christopher, J. Llibre and G. Świrszcz [13] two families of quadratic systems with an algebraic limit cycle of degrees five and six, respectively, are given. These two families are constructed by means of a birational transformation of system (21). A birational transformation is a rational change of variables such that its inverse is also rational. Moreover, they prove that there is also a birational transformation which converts Yablonskii’s system (15) into the system with a limit cycle of degree 2, (14). At the time of the composition of this paper, no other algebraic limit cycles for quadratic systems are known. Therefore, in order to prove that none of these quadratic systems with an algebraic limit cycle has a Liouvillian first integral, we only need to study the integrability of systems (14), (17), (19) and (21). All the other known cases are birationally equivalent to one of these ones and if one of them has a Liouvillian first integral, then the birational transformation of this first integral is a Liouvillian first integral for the transformed system. Therefore, since we will show that the systems (14), (17), (19) and (21) do not have a Liouvillian
Each one of the systems known until the moment of composition of this paper, has a Liouvillian first integral.

**Theorem 23.** Each one of the systems (14), (15), (17), (19) and (21) has only one invariant algebraic curve, when the limit cycle exists.

Theorem 23 is proved using analogous reasonings for each system (14), (17), (19) and (21). Since system (15) is birationally equivalent to system (14), then we do not need to study it.

The computation of the coordinates and nature of each singular point and the study of all the possibilities due to Theorems 13, 14 and 15 is easy but long. We only explicit these computations for systems (14) and (17), which exhaustively show all the encountered tricks.

**Proof for system (17).** We first list all the singular points (finite and infinite) and the type they belong to, depending on their ratio of eigenvalues. The singular point has associated eigenvalues \( \lambda_i \) and \( \mu_i \), \( i = 1, 2, \ldots, 7 \). We always assume that the parameter of the system belongs to the interval in which the limit cycle exists, that is, \( 0 < a < \frac{1}{17} \). We also point out that whether \( f_0(p_1) = 0 \) or not, where \( f_0 = 0 \) is the algebraic curve given in (18). In case \( f_0(p_1) = 0 \) we give the value of \( k_0(p_1) \), where \( k_0(x, y) \) is the cofactor of \( f_0(x, y) = 0 \). For the singular points at infinity \( (Z_0 = 0) \), we notice that all of them have the coordinate \( Y_0 \neq 0 \), so local coordinates \((u, v), \) where \( u = X/Y \) and \( v = Z/Y \), are taken at these points. The system in coordinates \((u, v)\) is:

\[
\begin{align*}
\dot{u} &= -4u + 2v + 2(2a + 3)u^2 - 9(1 + a)uv - 3a(1 + a)u^3, \\
\dot{v} &= v(-16 + 2(9 + 5a)u - 15(1 + a)v - 3a(1 + a)u^2). 
\end{align*}
\]  

We denote by \( k_v(u, v) = -16 + 2(9 + 5a)u - 15(1 + a)v - 3a(1 + a)u^2 \) the cofactor of the invariant straight line \( v = 0 \). The curve \( f_0 = 0 \) in these coordinates is given by \( f_0(u, v) = 3a^2(1 + a)u^4 + 6a(1 + a)u^2v + 3(1 + a)v^2 - 6(1 + a)uv + 4v \) and its cofactor is \( k_0(u, v) = -2(8 - 4(3 + 2a)u + 15(1 + a)v + 6a(1 + a)u^2) \).

We list all the singular points of Eq. (17):

- \( p_1 = [0, 0, 1] \) is a node point with \( \lambda_1 = 6(1 + a), \mu_1 = 15(1 + a), f_0(p_1) = 0 \) and \( k_0(p_1) = 2\mu_1 \).
- \( p_2 = [20(1 + a), 15(1 + a)^2, 8(1 - a)] \) is a focus point with \( f_0(p_2) \neq 0 \).
- \( p_3 = [2(a - w_1), -18 - 33a - 16a^2 - (3 + 2a)w_1, 24(a + 1)] \) is a node point with \( f_0(p_3) = 0 \) and \( k_0(p_3) = \lambda_3 \).
- \( p_4 = [2(a + w_1), -18 - 33a - 16a^2 + (3 + 2a)w_1, 24(a + 1)] \) is a saddle point with \( f_0(p_4) = 0 \) and \( k_0(p_4) = \lambda_4 \).
- \( p_5 = [0, 1, 0] \) is a node point with \( f_0(p_5) = 0 \) and \( k_0(p_5) = k_v(p_5) = -16 \).
- \( p_6 = [3 + 2a + w_2, 3a(a + 1), 0] \) is a saddle point with \( f_0(p_6) \neq 0 \) and \( k_v(p_6) = \lambda_6 \).
- \( p_7 = [3 + 2a - w_2, 3a(a + 1), 0] \) is a saddle point with \( f_0(p_7) \neq 0 \) and \( k_v(p_7) = \lambda_7 \).

We have used the notation \( w_1 = \sqrt{36 + 72a + 37a^2} \) and \( w_2 = \sqrt{9 - 8a^2} \).
Assume that there is another invariant algebraic curve \( f(x, y) = 0 \) with \( f(x, y) \in \mathbb{R}[x, y] \) which we suppose to be irreducible in \( \mathbb{R}[x, y] \) and the polynomials \( f_0(x, y) \) and \( f(x, y) \) relatively coprime. We denote by \( k(x, y) := k_{00} + k_{10}x + k_{01}y \) its cofactor. Let \( n \) be the degree of the polynomial \( f(x, y) \), then its corresponding cofactor for the system with coordinates \((u, v)\) is

\[
k(u, v) = k_{01} - 16a + (k_{10} + 2(9 + 5a)n)u + (k_{00} - 15(1 + a)n)v - 3a(1 + a)nu^2.
\]

Let us consider the singular points for which the ratio of the eigenvalues is either rationally independent or negative, that is, \( p_2, p_4, p_6 \) and \( p_7 \). Let us consider one of these points, say \( p_i \) with \( i = 2, 4, 6, 7 \). The invariant algebraic curve \( f = 0 \) either satisfies \( f(p_i) \neq 0 \) or \( f(p_i) = 0 \). If \( f(p_i) \neq 0 \) then \( k(p_i) = 0 \) by Lemma 10. On the other hand, if \( f(p_2) = 0 \) then \( k(p_2) = \lambda_2 + \mu_2 \) by Lemma 16. If \( f(p_4) = 0 \), then \( k(p_4) = \mu_4 \) because we have that \( f_0(p_4) = 0 \) and \( k_0(p_4) = \lambda_4 \), and we apply Theorems 13, 14 and 15. Analogously, if \( f(p_6) = 0 \) then \( k(p_6) = \mu_6 \) because \( k_0(p_6) = \lambda_6 \), and if \( f(p_7) = 0 \) then \( k(p_7) = \mu_7 \) because \( k_0(p_7) = \lambda_7 \).

We codify these conditions by the following equations \( k(p_2) = \varepsilon_2(\lambda_2 + \mu_2), k(p_4) = \varepsilon_4\mu_4, k(p_6) = \varepsilon_6\mu_6 \) and \( k(p_7) = \varepsilon_7\mu_7 \), where \( \varepsilon_i \in \{0, 1\}, i = 2, 4, 6, 7 \). These equations give a total of sixteen cases to study.

We solve these four linear equations for \( k_{00}, k_{10}, k_{01} \) and \( n \) and we get:

\[
n = \frac{1}{6(a^2 - 1)(9 + 11a)} \left[ (-18 + 48a + 118a^2 + 52a^3 + (26a^2 + 20a)
- 6)w_1\varepsilon_2 + (72 + 72a - 70a^2 - 74a^3 + (24 + 2a - 26a^2)w_1)\varepsilon_4
+ (-27a - 51a^2 - 22a^3 + (18 + 49a + 33a^2)w_2 - (9 + 11a)w_1
+ (9 + 11a)w_1w_2)\varepsilon_6 + (-27a - 51a^2 - 22a^3 - (18 + 49a + 33a^2)w_2
- (9 + 11a)w_1 - (9 + 11a)w_1w_2)\varepsilon_7 \right].
\]

For each one of the cases \( \varepsilon_i \in \{0, 1\}, i = 2, 4, 6, 7 \), we have that \( n \) is an algebraic function of the parameter \( a \). This algebraic function can be studied numerically without loss of precision, due to its simplicity. This study for each value of \( \varepsilon_i \) gives that there is no natural number in the range of the function \( n(a) \) when \( 0 < a < 3/13 \), (except in two cases which will be carefully remarked) and, hence, no invariant algebraic curve \( f(x, y) = 0 \), different from \( f_0(x, y) = 0 \), can exist. However, we give a rigorous algebraic proof of this fact, which can be obtained from the characteristics of the function \( n(a) \).

We differentiate the function \( n(a) \) with respect to \( a \) and we get a rational function of the form

\[
\frac{\partial n}{\partial a} = \frac{\alpha_0(a) + \alpha_1(a)w_1 + \alpha_2(a)w_2 + \alpha_3(a)w_1w_2}{(a^2 - 1)^2(9 + 11a)^2(8a^2 - 9)(37a^2 + 72a + 36)}
\]

where \( \alpha_i(a) \) are polynomials in \( a \). We notice that the denominator of this expression is strictly negative for \( a \in (0, 3/13) \). Let us consider the lowest degree polynomial in \( a \) which has the numerator as a factor. We denote it by \( N(a) \). We encounter that \( N(a) \) is a polynomial of degree 16 in \( a \). We compute it for any of the possible values \( \varepsilon_i \in \{0, 1\}, i = 2, 4, 6, 7 \),
\[ N(a) = (a - 1)^4(a + 1)^2(59049 + 389286a + 1048059a^2 + 1479654a^3 + 1158795a^4 + 478608a^5 + 81649a^6), \]

which is obviously strictly positive when \( a \in (0, 3/13) \).

Therefore, the function \( n(a) \) is a strictly increasing or decreasing function of \( a \). We compute its value in \( a = 0 \) and \( a = 3/13 \) for the fourteen cases and we deduce that there is no natural number in the range of this function except for two cases which correspond to \( \{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0, 1, 0, 1\} \) and \( \{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0, 0, 0, 1\} \). Let us particularly study these cases.

In the case \( \{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0, 1, 0, 1\} \), the function \( n(a) \) can be seen to be strictly increasing in the interval \( a \in (0, 3/13) \) by the described method. We have \( n(0) = 1 \) and \( n(3/13) = -73 - 13\sqrt{41} + 7\sqrt{161} + 3\sqrt{6601} \approx 2.754985 \). So, there is a value \( a^* \in (0, 3/13) \) such that \( n(a^*) = 2 \). We notice that we have computed the cofactor for this case. Straightforward computations show that system (17) has no invariant conic. These easy computations correspond to a linear system of equations on the coefficients of a conic.

In the case \( \{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0, 0, 0, 1\} \), the function \( n(a) \) can be seen to be strictly increasing in the interval \( a \in (0, 3/13) \) by the described method. We have \( n(0) = 1 \) and \( n(3/13) = 9 + \sqrt{161} + 7\sqrt{161} + 3\sqrt{6601} \approx 6.6375 \). So, there is a value \( a^* \in (0, 3/13) \) such that \( n(a^*) = 6 \). Let us consider the singular point \( p_3 \) which is a node point with \( \lambda_3 < \mu_3 < 0 \). We deduce the equation \( k(p_3) = \epsilon_3 \mu_3 + (s_3 - \epsilon_3)\lambda_3 \), where \( s_3 \in \{0, 1\} \) and \( s_3 \) is an integer number with \( s_3 \geq \epsilon_3 \). We notice that we have the cofactor \( k(x, y) \) and, once evaluated in \( p_3 \), we have a condition on \( \epsilon_3, s_3 \) and \( a \). When \( \epsilon_3 = 0 \), we can compute \( s_3(a) \) from this equation and we have that for \( a \in (0, 3/13) \) there is no integer number in the range on this function. When \( \epsilon_3 = 1 \), we compute \( s_3(a) \) from this equation and we have that there is an \( a^*_0 \in (0, 3/13) \) for which \( s(a^*_0) = 3 \). An easy computation using resultants shows that \( a^*_1 \neq a^*_2 \).

We consider the case when \( \epsilon_i = 1 \) for \( i = 2, 4, 6, 7 \), then \( N(a) \equiv 0 \) and \( n \equiv -1 \). So, no invariant algebraic curve \( f(x, y) = 0 \) can exist in this case.

In case \( \epsilon_i = 0 \) for \( i = 2, 4, 6, 7 \), we have that \( N(a) \equiv 0 \) and \( n \equiv 0 \). So, no invariant algebraic curve \( f(x, y) = 0 \) can exist in this case either.

The non-existence of the invariant algebraic curve \( f(x, y) = 0 \) has been shown in the sixteen possible cases given by \( \epsilon_i \in \{0, 1\}, i = 2, 4, 6, 7 \). \( \Box \)

**Proof for system (14).** We consider system (14) with the invariant algebraic curve \( f_0(x, y) = 0 \) defined by \( f_0(x, y) := x^2 + y^2 - 1 \) and with cofactor \( k_0(x, y) = -2x \). System (14) depends on three parameters \( a, b \) and \( c \) which satisfy \( a \neq 0 \), \( c^2 + 4(b + 1) > 0 \) and \( c^2 > a^2 + b^2 \) for the existence of the limit cycle. In order to show that there is no other invariant algebraic curve, different from \( f_0(x, y) = 0 \), we study two cases in which the nature of the singular points changes, that is, \( b < -1 \) and \( b \geq -1 \). We do not need to study all the singular points since only some of them are used. As before, the singular points at

\( i = 2, 4, 6, 7 \), and we prove that \( N(a) \) has a strictly defined sign except when \( \epsilon_i = 1 \) for all \( i \) or when \( \epsilon_i = 0 \) for all \( i \).
the line of infinity satisfy that $Y_0 \neq 0$ so we take local coordinates $u = X/Y$ and $v = Z/Y$ at these points. System (14) with local coordinates $(u, v)$ is

$$
\dot{u} = -(1 + b) - au - cv - (1 + b)u^2 + v^2 - au^3 - cu^2v, \\
\dot{v} = -uv(b + au + cv).
$$

(24)

The cofactor of the invariant straight line $v = 0$ is $k_v(u, v) = -u(b + au + cv)$.

Assume $b < -1$ and consider the following three singular points, which correspond to the three singular points at the line of infinity.

- $p_5 = \{-(1 + b), a, 0\}$, with eigenvalues $\lambda_5 = -(1 + b)/a$ and $\mu_5 = -(a^2 + (1 + b)^2)/a$, so it is a saddle point. At this point $f_0(p_5) \neq 0$ and $k_v(p_5) = \lambda_5$.
- $p_6 = \{1, 1, 0\}$ with eigenvalues $\lambda_6 = 2(a - (b + 1)i)$ and $\mu_6 = a - bk$, so the ratio $\lambda_6/\mu_6$ is not a real number. At this point $f_0(p_6) = 0$, $k_0(p_6) = \lambda_6$ and $k_v(p_6) = \mu_6$.
- $p_7 = \{-1, 1, 0\}$ with eigenvalues $\lambda_7 = 2(a + (b + 1)i)$ and $\mu_7 = a + bk$, so the ratio $\lambda_7/\mu_7$ is not a real number. At this point $f_0(p_7) = 0$, $k_0(p_7) = \lambda_7$ and $k_v(p_7) = \mu_7$.

Let $f(x, y) = 0$ be an invariant algebraic curve of degree $n$ of system (14) with cofactor $k(x, y) := k_{00} + k_{10}x + k_{01}y$ such that the polynomials $f(x, y)$ and $f_0(x, y)$ are relatively coprime, with $f(x, y) \in \mathbb{R}[x, y]$ and irreducible in $\mathbb{R}[x, y]$. The cofactor of this irreducible curve in local coordinates $(u, v)$ is $k(u, v) = k_{01} + (k_{10} - bn)u + k_{00}v - anx^2 - cnuv$. By Theorem 15 we have that the only possibility is $f(p_6) \neq 0$ and $f(p_7) \neq 0$. Therefore, by Lemma 10 we have that $k(p_6) = k(p_7) = 0$. In addition, the intersection of the curve $f = 0$ and the invariant straight line $v = 0$ must be one of the singular points, and the only possibility is that $f(p_5) = 0$. Since $p_5$ is a saddle point and $k_v(p_5) = \lambda_5$, by Theorems 13, 14 and 15, we get the equation $k(p_5) = \mu_5$. The combination of these three equations gives $n = 1$ and straightforward computations show that there is no invariant straight line for this system.

Assume that $b \geq -1$ and consider the same three singular points at infinity. But, $p_5$ is a node point in this case. Let us now consider the following two finite singular points.

- $p_3 = \{-ac + b\sqrt{a^2 + b^2 - c^2}, -bc - a\sqrt{a^2 + b^2 - c^2}, a^2 + b^2\}$ which is a complex point with eigenvalues $\lambda_3 = 2(ac - b\sqrt{a^2 + b^2 - c^2})/(a^2 + b^2)$ and $\mu_3 = \sqrt{a^2 + b^2 - c^2}$, whose ratio is never a real number. At this point $f_0(p_3) = 0$ and $k_0(p_3) = \lambda_3$.
- $p_4 = \{-ac - b\sqrt{a^2 + b^2 - c^2}, -bc + a\sqrt{a^2 + b^2 - c^2}, a^2 + b^2\}$ which is a complex point with eigenvalues $\lambda_4 = 2(ac + b\sqrt{a^2 + b^2 - c^2})/(a^2 + b^2)$ and $\mu_4 = -\sqrt{a^2 + b^2 - c^2}$, whose ratio is never a real number. At this point $f_0(p_4) = 0$ and $k_0(p_4) = \lambda_4$.

Assume that there is another invariant algebraic curve $f(x, y) = 0$ with cofactor $k(x, y)$ as in the previous case. As before, we have that $k(p_6) = k(p_7) = 0$. Since the polynomial $f(x, y)$ is assumed to be real, the behavior of the curve $f(x, y) = 0$ must coincide at the points $p_3$ and $p_4$. So, either $f(p_3) \neq 0$ and $f(p_4) \neq 0$ and then $k(p_3) = k(p_4) = 0$ or $f(p_3) = f(p_4) = 0$ and then $k(p_3) = \mu_3$ and $k(p_4) = \mu_4$. This condition can be codified by $k(p_3) = \varepsilon_3\mu_3$ and $k(p_4) = \varepsilon_3\mu_4$, with $\varepsilon_3 \in \{0, 1\}$. 


If $\varepsilon_3 = 0$, the resolution of these four linear equations on the coefficients of $k(x, y)$ and on $n$ give that $n = 0$, so no other invariant algebraic curve exists in this case.

If $\varepsilon_3 = 1$, the resolution of these four linear equations give $n = 1$ and easy computations show that there is no invariant straight lines for system (14).

Theorem 23 is crucial in order to prove the following result. In view of Theorem 4 the information relating a polynomial system with its invariant algebraic curves and exponential factors gives the Liouvillian or non Liouvillian integrability of the system.

**Theorem 24.** None of the systems (14), (15), (17), (19) and (21) has a Liouvillian first integral.

**Proof.** Theorem 23 states that each of these systems only has one invariant algebraic curve, namely $f_0(x, y) = 0$. Assume that there is a Liouvillian first integral, then, by Theorem 4, we have that it has an inverse integrating factor of the form

$$V(x, y) = \exp \left\{ \frac{h(x, y)}{f_0(x, y) n} \right\} f_0(x, y)^c,$$

where $h(x, y) \in \mathbb{R}[x, y]$, $n \in \mathbb{N}$ and $c \in \mathbb{R}$. We eventually may have that $h(x, y)$ is constant and/or $n = 0$ and/or $c = 0$. The form of this inverse integrating factor is given by Theorem 4 and the fact that the only invariant algebraic curve of the system is $f_0(x, y) = 0$. All these systems have a strong focus point in the region bounded by the limit cycle, namely $p$. For any of them it is easy to see that $f_0(p) \neq 0$, for the values of the parameters in which the limit cycle exists. Then, $V(p) \neq 0$ and, hence, the first integral constructed by this inverse integrating factor is continuous in a neighborhood of this focus point $p$. But, a first integral cannot be continuous in a neighborhood of a focus point without being constant on all the neighborhood. We have, then, a contradiction and we deduce that no such Liouvillian first integral can exist.

**References**


\[ \frac{dy}{dx} = \frac{\sum_{0\leq i+j\leq 2} a_{ij}x^i y^j}{\sum_{0\leq i+j\leq 2} b_{ij}x^i y^j}, \]


