Compressive buckling analysis of plates in unilateral contact

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Abstract

The unilateral contact buckling behaviour of delaminated plates in a composite member is studied in this paper, where the two-dimensional mechanical problem is simplified to a one-dimensional mathematical model following the assumption of a buckling mode function in terms of the lateral coordinates. After the governing differential equations for the plate parts in the contact and non-contact regions have been solved, the problem is reduced to just two nonlinear algebraic equations allowing the buckling coefficient to be obtained through an iteration method. A simplified method is also deduced based on an assumed zero length of contact. The numerical results of the buckling coefficients based on both methods show good agreement.

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1. Introduction

Composite members composed of several layers (especially where the layers vary significantly in thickness and material properties), may potentially exhibit compressive buckling under mechanical or thermal loadings, during which sections of the different layers become separated (delaminated), and then buckle away from each other. Most commonly, a thin layer or ‘skin’ will buckle away from a thicker layer, or ‘core’, typically of lower modulus. When the bonding action between the layers is ignorable, this type of buckling problem may be modelled by treating the distinct layers as elastic plates in a state of unilateral contact (Fig. 1). Buckling problems of this type are difficult to analyse due to the inherent nonlinearities and the complexity resulting from the unilateral contact effects.

Experiments conducted as part of a research project aimed at the development of composite components for the construction industry in New Zealand, CSA (2004) have shown the pivotal importance of initial skin buckling in the behaviour of composite wall panels, and also the failure initiation role played by post-buckling...
behaviour. Fig. 1 shows the cross-section of a number of composite members in which delamination-type plate buckling could play a significant role. Confidentiality conditions currently prevent the publication of specific details of certain specimens being developed, but do not constrain the dissemination of the governing mechanics.

Seide (1958) and Co (1977) were among the earliest researchers of the contact buckling problem, which was extended recently to a compressive plate on a tensionless rigid foundation by Shahwan and Waas (1994, 1998) and Smith et al. (1999a). As a practical application, Wright (1995), Uy and Bradford (1996) and Smith et al. (1999b,c) studied the local buckling problem of composite steel-concrete members. Chai (2001) conducted analytical and experimental studies of post-local buckling response of unilaterally constrained thin plates. In general, the contact buckling problem of a thin plate resting on a tensionless rigid medium may be considered as a simple non-contact plate model with only one half-wave. In a new approach to this type of problem, de Holanda and Goncalves (2003) and Shen and Li (2004) constructed a numerical elastic contact post-buckling model for simply supported plates on tensionless deformable foundations using perturbation and iteration techniques. Most of the existing literature is confined to the case of a plate on a rigid or elastic spring constraint.

In this paper, the unilateral contact buckling problem between two infinite plates with immovable ends is investigated. Assuming a buckling mode function in terms of lateral coordinates allows the formulation of a one-dimensional mathematical model, taking into account the coupling effect between the two plates. After solving the governing buckling equations of the plate parts in both the contact and non-contact regions, the problem is reduced to two nonlinear algebraic equations allowing the buckling coefficient to be obtained by means of an iteration method.

2. Mechanical model of two plates in contact

Consider a single wavelength part of the two interacting plates (Fig. 2).
For thin plates in compression, the governing equations are

\[ D_1 \left( w_{1e,xxxx} + 2w_{1e,xyy} + w_{1e,yyy} + \frac{\pi^2 K_1}{c^2} w_{1e,xx} \right) = q_r, \quad q_r \geq 0, \quad (1a) \]

\[ D_2 \left( w_{2e,xxxx} + 2w_{2e,xyy} + w_{2e,yyy} + \frac{\pi^2 K_2}{c^2} w_{2e,xx} \right) = -q_r, \quad q_r \geq 0, \quad (1b) \]

\[ D_i \left( w_{i,xxxx} + 2w_{i,xyy} + w_{i,yyy} + \frac{\pi^2 K_i}{c^2} w_{i,xx} \right) = 0, \quad i = 1, 2, \quad (1c) \]

\[ D_i = \frac{E_i t_i^3}{12(1 - \nu_i^2)}, \quad i = 1, 2 \quad \text{and} \]

\[ K_i = \frac{c^2 \sigma_{xi} t_i}{\pi^2 D_i}, \quad i = 1, 2, \quad (3) \]

where \( w_{ie}, w_i \) are the vertical displacements in the contact and non-contact regions respectively of plate \( i \); \( D_i \) is the well-known plate rigidity and \( K_i \) is a dimensionless coefficient; \( q_r \) is the contact force between the two plates; \( E_i, \nu_i, t_i \) are the elastic modulus, Poisson’s ratio and thickness of plate \( i \) and \( \sigma_{xi} \) is the normal stress in the \( x \)-direction. The subscripts \( \partial x, \partial y \) indicate partial differentiation, \( \partial / \partial x(\partial / \partial y) \), etc.

Assuming the two plates have the same compressive strain,

\[ \sigma_{e1}/E_1 = \sigma_{e2}/E_2, \quad (4) \]

(3) may be rewritten as

\[ K_2 = \frac{r_1^2(1 - v_1^2)}{r_2^2(1 - v_2^2)} K_1. \quad (5) \]

The inequality \( q_r \geq 0 \) in (1), which means that plate 1 is supported by plate 2, requires

\[ K_2 < K_1, \quad \text{or} \quad r_1^2(1 - v_1^2) < r_2^2(1 - v_2^2). \]

Considering that \( w_{1e} = w_{2e} = w_3 \) in the contact area, adding (1a) to (1b), one obtains

\[ w_{3,xxxx} + 2w_{3,xyy} + w_{3,yyy} + \frac{\pi^2 K_3}{c^2} w_{3,xx} = 0, \quad (6) \]

where \( K_3 = \frac{D_1 K_1 + D_2 K_2}{D_1 + D_2}. \quad (7) \)

Thus the mechanical model is reduced to three Eqs. (1c) and (6), governing buckling.

Assuming \( w(x,y) = f(x)g(y) \), Eqs. (1c) and (6) may be rewritten as

\[ f_i'''' g + 2f_i'' g'' + f_i g''' + \frac{\pi^2}{c^2} K_i f_i'' g = 0, \quad i = 1, 2, 3. \quad (8) \]

Assuming \( q(x, y) = q_s(x)g(y) \), (1a) may be rewritten as

\[ D_i \left( f_i'''' g + 2f_i'' g'' + f_i g''' + \frac{\pi^2}{c^2} K_i f_i'' g \right) = q_s g, \quad (9) \]

where primes denote differentiation with respect to \( x \) or \( y \).

The boundary conditions of the problem are

\[ g(\pm c/2) = g''(\pm c/2) = 0 \quad \text{for simply supported plates} \]

\[ g(\pm c/2) = g'(\pm c/2) = 0 \quad \text{for clamped plates}. \quad (10a, 10b) \]
The continuity condition between \( f_i, f_2, f_3 \) is defined by

\[
\begin{align*}
  f_i(-a/2) &= f_3(b/2), & i &= 1, 2, \\
  f_i'(-a/2) &= f_3'(b/2), & i &= 1, 2 \quad \text{(11a)} \\
  f_i''(-a/2) &= f_3''(b/2), & i &= 1, 2. \quad \text{(11b)}
\end{align*}
\]

To satisfy the boundary condition of (10), the lateral (buckling) mode function \( g(y) \) may be assumed as

\[
\begin{align*}
  g(y) &= \cos \frac{\pi y}{c} \quad \text{for a simply supported plate} \quad \text{(12a)} \\
  g(y) &= \left[ 1/4 - (y/c)^2 \right]^2 \quad \text{for a clamped plate}, \quad \text{(12b)}
\end{align*}
\]

where (12a) is the exact buckling mode function; while (12b) is approximate, but of good precision (Jones and Milne, 1976; Smith et al., 1999a).

Taking the integral of (8) and (9) after multiplying both sides by function \( g(y) \), and introducing \( f_i(x_i) = f_i(\xi_i) \), where \( \xi_{1,2} = x_{1,2}/a, \phi_{1,2} = a/c, \xi_3 = x_3/b, \phi_3 = b/c \), we get

\[
\begin{align*}
  \tilde{f}_i''(\xi_i) - \phi_i^2(\lambda_i - \pi^2 K_i)\tilde{f}_i''(\xi_i) + \lambda_2\phi_i^4\tilde{f}_i(\xi_i) &= 0, & i &= 1, 2, 3 \quad \text{(13)} \\
  \text{and} \quad D_1[\tilde{f}_i'''(\xi_3)] - \phi_3^2(\lambda_1 - \pi^2 K_1)\tilde{f}_i''(\xi_3) + \lambda_2\phi_3^4\tilde{f}_i(\xi_3)]/b^4 &= q_s, \quad \text{(14)}
\end{align*}
\]

where \( \lambda_1 = 2\pi^2, \lambda_2 = \pi^4 \) for a simply supported plate; or \( \lambda_1 = 24, \lambda_2 = 504 \) for a clamped plate.

The symmetric solution of (13) may be written as

\[
f_i(\xi_i) = A_i f_{i1} + A_2 f_{i2}, \quad \text{(15)}
\]

where the functions \( f_{i1} \) and \( f_{i2} \) depend on the value of the parameter \( \Lambda_i = \left( \frac{\pi^2 K_i - \lambda_1}{2} \right)^2 - \lambda_2 \) as follows:

**Case 1:** \( \Lambda_i > 0 \):

\[
\begin{align*}
  f_{i1}(\xi_i) &= \cos(x_i \xi_i), \quad \text{(16a)} \\
  f_{i2}(\xi_i) &= \sin(x_i \xi_i), \quad \text{(16b)} \\
  x_i, \beta_i &= \phi_i \left[ \frac{\pi^2 K_i - \lambda_1}{2} \pm \sqrt{\Lambda_i} \right]^{1/2}. \quad \text{(16c)}
\end{align*}
\]

**Case 2:** \( \Lambda_i = 0 \):

\[
\begin{align*}
  f_{i1}(\xi_i) &= \cos(x_i \xi_i), \quad \text{(17a)} \\
  f_{i2}(\xi_i) &= \xi_i \sin(x_i \xi_i), \quad \text{(17b)} \\
  x_i &= \phi_i \left[ \frac{\pi^2 K_i - \lambda_1}{2} \right]^{1/2}. \quad \text{(17c)}
\end{align*}
\]

**Case 3:** \( \Lambda_i < 0 \):

\[
\begin{align*}
  f_{i1}(\xi_i) &= (e^{x_i \xi_i} + e^{-x_i \xi_i}) \cos(\beta_i \xi_i), \quad \text{(18a)} \\
  f_{i2}(\xi_i) &= (e^{x_i \xi_i} - e^{-x_i \xi_i}) \sin(\beta_i \xi_i), \quad \text{(18b)} \\
  x_i, \beta_i &= \phi_i \left[ \sqrt{\left( \frac{\pi^2 K_i - \lambda_1}{2} \right)^2 - \Lambda_i} \mp \left( \frac{\pi^2 K_i - \lambda_1}{2} \right) \right]^{1/2}. \quad \text{(18c)}
\end{align*}
\]

The continuity condition, (11), may be rewritten as

\[
\tilde{f}_i(-1/2) = \tilde{f}_3(1/2), \quad i = 1, 2, \quad \text{(19a)}
\]
\[
\tilde{f}_i^\prime(-1/2)/a = \tilde{f}_i^\prime(1/2)/b, \quad i = 1, 2 \quad \text{and} \quad \tilde{f}_i''(-1/2)/a^2 = \tilde{f}_i''(1/2)/b^2, \quad i = 1, 2.
\]  
(19b)

(19) may be expressed in matrix form as

\[
[A][A] = 0,
\]  
(20)

where

\[
A = \begin{bmatrix}
  f_{11}(-1/2) & f_{21}(-1/2) & -f_{12}(-1/2) & -f_{22}(-1/2) & 0 & 0 \\
  f_{11}'(-1/2)/a & f_{21}'(-1/2)/a & -f_{12}'(-1/2)/a & -f_{22}'(-1/2)/a & 0 & 0 \\
  f_{11}''(-1/2)/a^2 & f_{21}''(-1/2)/a^2 & -f_{12}''(-1/2)/a^2 & -f_{22}''(-1/2)/a^2 & 0 & 0 \\
  f_{11}(-1/2) & f_{21}(-1/2) & 0 & 0 & -f_{13}(1/2) & -f_{23}(1/2) \\
  f_{11}'(-1/2)/a & f_{21}'(-1/2)/a & 0 & 0 & -f_{13}'(1/2)/b & -f_{23}'(1/2)/b \\
  f_{11}''(-1/2)/a^2 & f_{21}''(-1/2)/a^2 & 0 & 0 & -f_{13}''(1/2)/b^2 & -f_{23}''(1/2)/b^2
\end{bmatrix}
\]

and \([A] = [A_{11} \quad A_{21} \quad A_{12} \quad A_{22} \quad A_{13} \quad A_{23}]^T\).

For a non-trivial solution of (20) we require that the determinant of the coefficient matrix should vanish,

\[
|A| = 0.
\]  
(21)

Observing that the shearing force equilibrium equation on the borderline between the contact and non-contact areas is

\[
D_1[w_{3,xxx} + (2 - v_1)w_{3,xyy}]_{x = b/2} + D_2[w_{3,xxx} + (2 - v_2)w_{3,xyy}]_{x = b/2}
= D_1[w_{1,xxx} + (2 - v_1)w_{1,xyy}]_{x = a/2} + D_2[w_{2,xxx} + (2 - v_2)w_{2,xyy}]_{x = -a/2}
\]  
(22)

and recalling that \(w_i(x, y) = f_i(x)g_i(y)\), and using the relationships of (11b) and (15), (22) may be rewritten as

\[
D_1[A_{13}f_{13}''(1/2) + A_{23}f_{23}''(1/2)]/b^3 - D_1[A_{11}f_{11}''(-1/2) + A_{21}f_{21}''(-1/2)]/a^3
= D_2[A_{12}f_{12}''(-1/2) + A_{22}f_{22}''(-1/2)]/a^3 - D_2[A_{13}f_{13}''(1/2) + A_{23}f_{23}''(1/2)]/b^3.
\]  
(23)

Considering the contact pressure between the two plates, (14) may be rewritten as the following inequality:

\[
q_x = D_1[A_{13}f_{13}''(\xi_3) - \phi_3^f(\xi_3)] = A_{23}f_{23}''(\xi_3) + A_{23}f_{23}''(\xi_3) - \phi_3^f(\xi_3) + A_{23}f_{23}''(\xi_3) - \phi_3^f(\xi_3) = 0, \quad |\xi_3| \leq 1/2.
\]  
(24)

Enforcing the tensionless condition on the borderline contact force requires

\[
Q_x = D_1[A_{11}f_{11}''(-1/2) + A_{21}f_{21}''(-1/2)]/a^3 - D_1[A_{13}f_{13}''(1/2) + A_{23}f_{23}''(1/2)]/b^3 \geq 0.
\]  
(25)

In the non-contact area, the displacements of plate 1 cannot be less than those of plate 2, thus \(w_1 \geq w_2\), which means

\[
A_{11}\tilde{f}_{11}(\xi_1) + A_{21}\tilde{f}_{21}(\xi_1) - [A_{12}\tilde{f}_{12}(\xi_1) + A_{22}\tilde{f}_{22}(\xi_1)] \geq 0, \quad |\xi_1| \leq 1/2.
\]  
(26)

Considering Eqs. (5) and (7), and inequalities (24)–(26), leads to solutions of Eqs. (21) and (23), for values of \(K_1\) and \(\phi_2\) for varying \(\phi_1\). The true value of \(\phi_1\) is that which minimizes \(K_1\). The minimum \(K_1\) defines the buckling coefficient \(K_{1cr}\).

3. Simplified model of plates in contact

If the contact length is assumed negligible, \(b \to 0\), and the boundary condition for (19) becomes
\( \tilde{f}_1(1/2) = \tilde{f}_2(1/2) \) and 
\( \tilde{f}'_1(1/2) = \tilde{f}'_2(1/2) = 0. \)  

(27a)  

(27b)

The shearing force equilibrium equation at the boundary may be written as

\[ D_1 \tilde{f}''(1/2) + D_2 \tilde{f}''(1/2) = 0 \]  

(27c)

and thus 
\[ [\tilde{A}] \tilde{A} = 0, \]  

(28)

**Fig. 3.** Relationship between buckling coefficient and elastic modulus ratio.

**Fig. 4.** Relationship between buckling coefficient and thickness ratio.
where

\[
\bar{A} = \begin{bmatrix}
    f_{11}(1/2) & f_{21}(1/2) & -f_{12}(1/2) & -f_{22}(1/2) \\
    f'_{11}(1/2) & f'_{21}(1/2) & 0 & 0 \\
    0 & 0 & f'_{12}(1/2) & f'_{22}(1/2) \\
    D_1f''_{11}(1/2) & D_1f''_{21}(1/2) & D_2f''_{12}(1/2) & D_2f''_{22}(1/2)
\end{bmatrix}
\]

and

\[
[\bar{A}] = [A_{11} \ A_{21} \ A_{12} \ A_{22}]^T.
\]

For a non-trivial solution of (28) we require that the determinant of the coefficient matrix vanish,

\[
|\bar{A}| = 0.
\]
The inequality for tensionless contact force on the borderline is

\[ Q_\tau = -D_1[A_{11}f''_{11}(1/2) + A_{21}f''_{21}(1/2)]/a^3 \geq 0. \]  

(30)

Considering (5) and (30), and solving (29) yields values of \( \phi_1 \) for varying \( \phi_1 \). The true value of \( \phi_1 \) is that which minimizes \( K_1 \), with the minimum \( K_1 \) defining the buckling coefficient, \( K_{1cr} \).

4. Numerical results and verification

For two long, finite-width plates in contact, with Poisson’s ratio \( \nu_1 = \nu_2 \), the calculated buckling coefficients for plate 1 (\( K_{1cr} \)) for varying ratios of elastic modulus and varying thickness ratios are shown in Figs. 3 and 4.

![Fig. 7. Relationship between wavelength and modular ratio.](image)

![Fig. 8. Relationship between contact length and modular ratio.](image)
The calculated contact forces between the two plates, $Q_x$, are shown in Figs. 5 and 6, where model 1 refers to the precise method of (21) and (23), and model 2 denotes the simplified method of (29). The results show that the buckling coefficients and the contact forces increase when plate 2 has a higher elastic modulus or a greater thickness, tending to a constant value when the relative rigidity of plate 2 is sufficiently large to simulate a rigid constraint. The results also show that the simplified method compares well with model 1 for accuracy of calculated buckling coefficients, despite inevitable errors in wavelength due to the assumption of zero contact length. The wavelength of $\phi_1 + \phi_2$ (or $(a + b)/c$) and contact length of $\phi_2$ (or $b/c$) are shown in Figs. 7 and 8. The buckling mode displacements of plates with lateral edges simply supported are given in Figs. 9 and 10, which show that the assumption of zero contact length in the simplified method leads to displacement...
incompatibility between the two plates. The pressure distributions between the two plates in the contact area are shown in Fig. 11.

To verify the results above we consider two extreme conditions. ‘Plate a’ represents the conditions when the flexural stiffness of plate 2 is very small, and the behaviour of the skin sheet approaches that of a long, unconstrained plate. ‘Plate b’ corresponds to the case when the flexural stiffness of plate 2 is very large, and the behaviour of the skin sheet is similar to that of a long plate resting on a tensionless, rigid constraint. The comparison between the results of the methods in this paper and existing theory (Bloom and Coffin, 2001; Shahwan and Waas, 1998; Uy and Bradford, 1996) are shown in Table 1.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Method in the paper</th>
<th>Existing theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>E2/E1</td>
<td>K_{cr}</td>
<td>\phi_1 + \phi_2</td>
</tr>
<tr>
<td>Simply supported</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10^{-5} (t_2/t_1 = 2)</td>
<td>4.0</td>
<td>2.00 (Model 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.00 (Model 2)</td>
</tr>
<tr>
<td>10 (t_2/t_1 = 10)</td>
<td>5.33</td>
<td>1.74 (Model 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.73 (Model 2)</td>
</tr>
<tr>
<td>Clamped</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10^{-5} (t_2/t_1 = 2)</td>
<td>6.98</td>
<td>1.33 (Model 1)</td>
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<td></td>
<td></td>
<td>1.32 (Model 2)</td>
</tr>
<tr>
<td>10 (t_2/t_1 = 10)</td>
<td>10.01</td>
<td>1.15 (Model 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.14 (Model 2)</td>
</tr>
<tr>
<td></td>
<td>Plate b^a</td>
<td>9.80 (Bloom)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10.30 (Uy)</td>
</tr>
</tbody>
</table>

\( ^a \) The single wavelength part of an infinite plate with lateral edges clamped and resting on rigid constraints is similar to an unconstrained plate with four edges clamped.

Table 1
Comparison of results

5. Conclusion

Two unilateral contact models have been presented for the problem of local buckling of delaminated plates in a composite panel. Good agreement with existing solutions has been demonstrated for two limiting cases. Numerical results show that the simplified model, in which the contact length is assumed to be zero, generally
leads to satisfactory results for both the buckling coefficients and the interaction forces at the points of contact between the plates, although there are inevitable errors in the calculation of wavelength and buckling mode.

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References