Notes on matrices with diagonally dominant properties

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ABSTRACT

In this paper, we analyze the relation between some classes of matrices with variants of the diagonal dominance property. We establish a sufficient condition for a generalized doubly diagonally dominant matrix to be invertible. Sufficient conditions for a matrix to be strictly generalized diagonally dominant are also presented. We provide a sufficient condition for the invertibility of a cyclically diagonally dominant matrix. These sufficient conditions do not assume the irreducibility of the matrix.

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1. Introduction and notation

The equivalence between the Geršgorin eigenvalue inclusion theorem [10] and the Desplanques Theorem [4], which asserts the invertibility of any strictly diagonally dominant matrix, was first observed by Rohrbach [15]. Since then, new inclusion regions for the eigenvalues of a matrix have been established, and new variants of the diagonal dominance property with sufficient conditions for the invertibility of the matrix were introduced; see [1,2,17].

Graph theory plays an important role in advancing the theory of matrices with a diagonal dominance property. We denote by $\mathcal{M}_n$ the set of all $n \times n$ complex matrices. Let $A = (a_{ij}) \in \mathcal{M}_n$. The directed graph $\mathcal{G}(A)$ of $A$ is the directed graph on $n$ distinct points, known as vertices, $v_1, \ldots, v_n$ such that there is a directed arc $v_iv_j$ if and only if $a_{ij} \neq 0$. We denote the set of positive integers by $\mathbb{N}$, and for every $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ is denoted by $(n)$. If $v_i$ and $v_j$ are distinct vertices in $\mathcal{G}(A)$ and $m \in \mathbb{N}$, we say that there is a directed path $\Gamma$ of length $m$ from $v_i$ to $v_j$ if there exist $m + 1$ distinct vertices $v_{i_1}, \ldots, v_{i_{m+1}}$ in $\mathcal{G}(A)$ such that $v_{i_1} = v_i$, $v_{i_{m+1}} = v_j$ and $v_{i_k}v_{i_{k+1}}$ is a directed arc in $\mathcal{G}(A)$ from $v_{i_k}$ to $v_{i_{k+1}}$ for all $k \in (m)$.

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We write \( \Gamma = v_i, v_2, \ldots, v_m, v_{m+1} \) or simply, as \( \Gamma : v_i \rightarrow v_j \). The length of \( \Gamma \) is denoted by \( \ell(\Gamma) \). The matrix \( A \) is called irreducible if and only if \( \mathcal{G}(A) \) is strongly connected; see Chapter 6 of [11]. Several results in the literature that provide sufficient conditions for the invertibility of matrices with a diagonal dominance property require the irreducibility of the matrix; see [2,5–9,13,18]. If \( p \in \mathbb{N} \setminus \{1\} \) and there are \( p + 1 \) vertices \( v_{i_1}, \ldots, v_{i_p} \) such that \( v_{i_1}, ..., v_{i_p} \) are distinct vertices, \( i_{p+1} = i_1 \) and \( v_{i_k}v_{i_{k+1}} \) is a directed arc in \( \mathcal{G}(A) \) for all \( k \in \langle p \rangle \), we say that \( \Gamma = v_i, v_2, \ldots, v_p, v_{p+1} \) is a cycle in \( \mathcal{G}(A) \) of length \( p \). We do not consider cycles of length 1, known as trivial cycles or loops. The set of all cycles in \( \mathcal{G}(A) \) is denoted by \( \mathcal{C}(A) \).

**Definition 1.1.** Let \( A = (a_{ij}) \in \mathcal{M}_n \), and let \( k \in \langle n \rangle \). Define \( r_k(A) \) by

\[
    r_k(A) = \sum_{j=1, j \neq k}^{n} |a_{kj}|	ag{1.1}
\]

In (1.1), it is understood that \( r_1(A) = 0 \) if \( A \) is an \( 1 \times 1 \) matrix. We also define the set \( J(A) \) by

\[
    J(A) = \{i \in \langle n \rangle : |a_{ii}| > r_i(A)\}.	ag{1.2}
\]

(i) We say that \( A \) is diagonally dominant if \( |a_{jj}| \geq r_j(A) \) for all \( j \in \langle n \rangle \). If \( J(A) = \langle n \rangle \), we call \( A \) strictly diagonally dominant.

(ii) We say that \( A \) is strictly generalized diagonally dominant (or invertible H-matrix); see [7], if there exists a diagonal matrix \( Y \) such that \( AY \) is strictly diagonally dominant.

(iii) We call \( A \) diagonally dominant with nonzero elements chain; see Definition 2 of [7], if \( A \) is diagonally dominant, \( J(A) \) is nonempty and for every \( p \notin J(A) \), there is \( q \in J(A) \) such that a directed path \( \Gamma : v_p \rightarrow v_q \) exists in \( \mathcal{G}(A) \).

In the following terms, we assume that \( n \in \mathbb{N} \setminus \{1\} \):

(iv) Let \( S_1 \) be a nonempty proper subset of \( \langle n \rangle \). For each \( k \in \langle n \rangle \), define \( r_k^{S_1}(A) \) by

\[
    r_k^{S_1}(A) = \sum_{j \in S_1, j \neq k} |a_{kj}|.	ag{1.3}
\]

In (1.3), if \( S_1 = \{k\} \) then \( r_k^{\{k\}}(A) = 0 \).

(v) If \( S_1 \) and \( S_2 \) are nonempty proper disjoint subsets of \( \langle n \rangle \) with \( S_1 \cup S_2 = \langle n \rangle \), we say that \( (S_1, S_2) \) is a separation of \( \langle n \rangle \). If \( (S_1, S_2) \) is a separation of \( \langle n \rangle \), we define the real-valued function \( f_A^{S_1} \) with domain of definition \( S_1 \times S_2 \) by

\[
    f_A^{S_1}(i, j) = (|a_{ii}| - r_i^{S_1}(A)) (|a_{jj}| - r_j^{S_1}(A)) - r_i^{S_2}(A) r_j^{S_2}(A)\tag{1.4}
\]

for all \( i \in S_1 \) and \( j \in S_2 \). We will use the function \( f_A^{S_1} \) frequently in Sections 2 and 3 of this paper.

(vi) The matrix \( A \) is called generalized doubly diagonally dominant; see [14], if \( J(A) \) is nonempty and there exists a separation \( (S_1, S_2) \) of \( \langle n \rangle \) such that \( f_A^{S_1} \geq 0 \), where \( f_A^{S_1} \) is the function defined by (1.4). If \( J(A) \) is nonempty and there exists a separation \( (S_1, S_2) \) of \( \langle n \rangle \) such that \( f_A^{S_1} > 0 \), we say that \( A \) strictly generalized doubly diagonally dominant. (We would use the obvious convention that a nonzero \( 1 \times 1 \) matrix is strictly generalized doubly diagonally dominant.)

(vii) We call \( A \) cyclically diagonally dominant if for every cycle \( \Gamma \in \mathcal{C}(A) \), we have \( \Pi_{v_i \in \Gamma} |a_{ii}| \geq \Pi_{v_i \in \Gamma} r_i(A) \). The notation means that if \( \Gamma = v_i, v_2, \ldots, v_p, v_{p+1} \) is a cycle in \( \mathcal{G}(A) \) with \( i_{p+1} = i_1 \) then each of the two products in the inequality contains exactly \( p \) terms, and the index \( i \) takes on the values \( i_1, \ldots, i_p \).

To simplify the terminology, we adopt the following abbreviated notations:

\[
    \mathbf{D} = \{ A \in \mathcal{M}_n : A \text{ is diagonally dominant} \},
\]

\[
    \mathbf{SD} = \{ A \in \mathcal{M}_n : A \text{ is strictly diagonally dominant} \}.
\]
SGD = \{A \in \mathcal{M}_n : A is strictly generalized diagonally dominant\},

DC = \{A \in \mathcal{M}_n : A is diagonally dominant with nonzero elements chain\},

GDD = \{A \in \mathcal{M}_n : A is generalized doubly diagonally dominant\}

and

SGDD = \{A \in \mathcal{M}_n : A is strictly generalized doubly diagonally dominant\}.

The identity matrix in \(\mathcal{M}_n\) is denoted by \(I_n\). For a matrix \(A = (a_{ij}) \in \mathcal{M}_n\), we will sometimes use the notation \((A)_{kk}\) to denote the entry \(a_{kk}\). We denote by \(\mathbb{C}\) the set of complex numbers. The set of all complex eigenvalues of \(A \in \mathcal{M}_n\), also known as the spectrum of \(A\), is denoted by \(\sigma(A)\). The elements of the linear space \(\mathbb{C}^n\) are represented by \(n \times 1\) column vectors. The zero vector in \(\mathbb{C}^n\) is denoted by \(\mathbf{0}\).

The transpose of any \(n \times 1\) matrix \(X\) is denoted by \(X^t\). Let \(n \in \mathbb{N} \setminus \{1\}\) and \(x = (x_1, \ldots, x_n)^t \in \mathbb{C}^n\). For every nonempty proper subset \(S = \{\tau_1, \ldots, \tau_m\}\) of \(\langle n \rangle\), where \(\tau_1 < \cdots < \tau_m\), we denote the vector \((x_{\tau_1}, \ldots, x_{\tau_m})^t\) by \(x(S)\).

The cardinality of a nonempty finite set \(S\) is denoted by \(\text{card} S\). We denote the empty set by \(\emptyset\).

The paper is organized as follows: In Section 2, we discuss the relation between the matrices defined in terms (i)–(iii) and (vi) of Definition 1.1. We also use the fact about the invertibility of every strictly generalized doubly diagonally dominant matrix (see Corollary 2.1) to provide an inclusion region for the eigenvalues of any \(A \in \mathcal{M}_n\), \(n \geq 2\). We show that this eigenvalues inclusion region is the same as the one given by Huang et al. (Theorem 2.1 in [12]). We present in Section 3 a sufficient condition for a generalized doubly diagonally matrix to be invertible and establish sufficient conditions for a matrix to be strictly generalized diagonally dominant. We also provide several examples that demonstrate our results. Two of the examples compare the results of Theorem 3.1 with some of the earlier results in the literature. In Section 4, we discuss some properties of cyclically diagonally dominant matrices and establish a sufficient condition for the invertibility of a cyclically diagonally dominant matrix. Unlike some of the earlier results in the literature, our sufficient conditions in Sections 3 and 4 do not require the irreducibility of the matrix.

2. Preliminary facts

The sets \(\text{DC}\) and \(\text{SGDD}\) play important roles in the development of the theory of matrices that have variants of the strict diagonal dominance property; see, for example, Theorem 3 of [7] and Theorem 2 of [14]. The two sets are clearly linked to the other two sets \(\text{D}\) and \(\text{GDD}\). Theorem 2.1 analyzes in depth the relation between the four sets. The following remark outlines some known facts about the sets defined in terms (i)–(iii) and (vi) of Definition 1.1.

Remark 2.1

1. It is clear from terms (i), (iii) and (vi) of Definition 1.1 that

\[
\text{SD} \subset \text{D} \cap \text{DC} \cap \text{SGD}, \quad \text{DC} \subset \text{D} \cap \text{GDD} \quad \text{and} \quad \text{SGDD} \subset \text{GDD}.
\]

2. It is known that \(\text{DC} \subset \text{SGD}\); see [7,16], and \(\text{SGDD} \subset \text{SGD}\); see Theorem 1 of [8]. It then follows from \(\text{SD} \subset \text{DC} \cap \text{SGDD}\) in (2.1) that

\[
\text{SD} \subset \text{DC} \subset \text{SGD} \quad \text{and} \quad \text{SD} \subset \text{SGDD} \subset \text{SGD}.
\]

Theorem 2.1. The sets \(\text{D}, \text{DC}, \text{GDD}\) and \(\text{SGDD}\) satisfy the following properties:

1. \(\text{DC}\) is a proper subset of \(\text{D} \cap \text{GDD}\).
2. \(\text{D} \not\subset \text{GDD}\) and \(\text{D} \not\subset \text{SGDD}\).
3. \(\text{SGDD} \not\subset \text{D}, \text{SGDD} \not\subset \text{DC}\) and \(\text{GDD} \not\subset \text{D}\).
4. \(\text{DC} \not\subset \text{SGDD}\) and \(\text{SGDD}\) is a proper subset of \(\text{GDD}\).
5. \(\text{D} \cap \text{SGD}\) is a proper subset of \(\text{DC}\).
Proof. (1) From $DC \subset D \cap GDD$ (in (2.1)), it suffices to show that there exists a matrix $A_1 \in D \cap GDD$ such that $A_1 \notin DC$. Let $A_1 = (a_{ij}) \in M_3$ be such that $|a_{11}| = |a_{12}|, |a_{22}| = |a_{21}|, |a_{33}| > r_3(A_1)$ and $a_{13} = a_{23} = 0$. Then $A_1 \in D$ and $f(A_1) \neq \emptyset$. Thus $A_1 \notin D \cap GDD$. Denote the vertices in the directed graph $g(A_1)$ of $A_1$ by $v_1, v_2$ and $v_3$. Since $a_{i3} = 0$ for $i = 1, 2$, we deduce that there is no directed path from $v_1$ to $v_3$. Hence from $f(A_1) = \{3\}$, we see that $A_1 \notin DC$.

(2) It is clear from term (vi) in Definition 1.1 that any matrix $A \in D$ with $J(A) = \emptyset$ satisfies $A \notin GDD$. This shows that $D \not\subseteq GDD$. It then follows from $SGDD \subset GDD$ (in (2.1)) that $D \not\subseteq SGDD$.

(3) We prove $SGDD \not\subseteq D$ by a counter example. Let $A_2 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $A_2 \notin D$. Let $S_1 = \{1, 2\}$. It can be easily seen that $f_{A_2}^{S_1}(1, 3) = 2$ and $f_{A_2}^{S_1}(2, 3) = 1$. Thus from the fact that $f(A_2) \neq \emptyset$, we see that $A_2 \in SGDD$. This proves $SGDD \not\subseteq D$. It then follows from $DC \subset D$ (in (2.1)) that $SGDD \not\subseteq DC$. Also, from $SGDD \not\subseteq D$ and $SGDD \subset GDD$ (in (2.1)), we get $GDD \not\subseteq D$.

(4) We prove $DC \not\subseteq SGDD$ by a counter example. Let $A_3 = (a_{ij}) \in M_3$ be such that

\[
|a_{ii}| = |a_{i,i+1}| > 0 \quad \text{for} \quad i = 1, 2, \tag{2.3}
\]

\[
a_{13} = a_{21} = 0 \tag{2.4}
\]

and

\[
|a_{33}| > r_3(A_3). \tag{2.5}
\]

It is clear from (2.3)-(2.5) and term (iii) of Definition 1.1 that $A_3 \in DC$.\hspace{1cm} (2.6)

Now, we prove that $A_3 \notin SGDD$. Let $(S_1, S_2)$ be a separation of (3). Then either

\[
\{1, 2\} \cap S_j \neq \emptyset \quad \text{for} \quad j = 1, 2 \tag{2.7}
\]

or

\[
\text{there exists } k \in \{1, 2\} \text{ such that } S_k = \{1, 2\}. \tag{2.8}
\]

First, suppose that (2.7) holds. Some calculations reveal that

\[
1 \in S_1 \implies f_{A_3}^{S_1}(1, 2) = 0
\]

and

\[
2 \in S_1 \implies f_{A_3}^{S_1}(2, 1) = 0.
\]

So,

\[
\{1, 2\} \cap S_j \neq \emptyset \quad \text{for} \quad j = 1, 2 \implies f_{A_3}^{S_1} \neq 0. \tag{2.9}
\]

Now, suppose that (2.8) holds. It then follows that

\[
S_1 = \{1, 2\} \implies f_{A_3}^{S_1}(1, 3) = 0
\]

and

\[
S_2 = \{1, 2\} \implies S_1 = \{3\} \quad \text{and} \quad f_{A_3}^{S_1}(3, 1) = 0.
\]

So, $f_{A_3}^{S_1} \neq 0$ if (2.8) holds. Thus from (2.7)-(2.9), we see that $A_3 \notin SGDD$. Hence from (2.6), we get $DC \not\subseteq SGDD$. It then follows from $SGDD \subset GDD$ (in (2.1)) and term (1) that $SGDD$ is a proper subset of $GDD$.\hspace{1cm}
We observe that there exist a diagonally dominant matrix $A$ for all $j$. The proposition is established through the following five steps:

**Proof.**

1. If $q \in S_2$ and $|a_{qq}| = r_q(A)$, then $r_{q}^{S_1}(A) > 0$ and $p \in J(A)$. Since $q \in S_2$, we deduce from (2.11) that $f_{A}^{S_1}(p, q) > 0$. Thus from $|a_{qq}| = r_q(A)$, we see that $|a_{qq}| - r_{q}^{S_1}(A) = r_{q}^{S_1}(A) > 0$ and $p \in J(A)$.

2. If $|a_{pp}| = r_p(A)$, then $r_{p}^{S_2}(A) > 0$ and $S_2 \subseteq J(A)$. The step is proven in a similar way to step 1.

3. If $J(A) \neq \emptyset$. This clearly follows if $|a_{pp}| > r_p(A)$. So, assume that $|a_{pp}| = r_p(A)$. Then from step 2, the result follows.

4. If $A \in \text{GDD}$. This follows from $A \in D$ and step 3.

5. If condition (1) holds, then $A \in \text{DC}$. Since $A \in D$, we have either $|a_{pp}| > r_p(A)$ or $|a_{pp}| = r_p(A)$. We consider each case separately.

   **Case 1:** $|a_{pp}| > r_p(A)$. It then follows from $A \in D$ and condition (1) that to prove $A \in \text{DC}$, it suffices to show that for every $q \in S_2$ with $|a_{qq}| = r_q(A)$ there is a directed path in $\mathcal{G}(A)$ from $v_q$ to $v_p$. Let $q \in S_2$ be such that $|a_{qq}| = r_q(A)$. Thus from step 1, we see that $r_{q}^{S_1}(A) > 0$. Hence there exists $i \in S_1$ such that $a_{qi} \neq 0$. Then from condition (1), the result follows.

   **Case 2:** $|a_{pp}| = r_p(A)$. It then follows from step 2 that $S_2 \subseteq J(A)$ and there exists $j_0 \in S_2$ such that $a_{p_{j_0}} \neq 0$. So, from $A \in D$ and condition (1), we deduce that $A \in \text{DC}$. □

The following proposition provides sufficient conditions for a matrix $A \in D$ to be in GDD and in the smaller set DC; see terms (1) and (2) of Theorem 2.1.

**Proposition 2.1.** Let $A = (a_{ij}) \in \mathcal{M}_n$ be such that $A \in D$. Assume that there exist a separation $(S_1, S_2)$ of $(n)$ and $p \in S_1$ such that

\[ f_{A}^{S_1}(p, j) > 0 \]  

for all $j \in S_2$. Then $A \in \text{GDD}$. Also,

1. If $A$ satisfies the following additional condition:

   **Condition (1):** For every $i \in S_1 \setminus \{p\}$ there exists a directed path $\Gamma_i : v_i \rightarrow v_p$ in $\mathcal{G}(A)$, where $v_i, i \in \langle n \rangle$, denote the vertices in the directed graph $\mathcal{G}(A)$ of $A$, then $A \in \text{DC}$.

   **Proof.** The proposition is established through the following five steps:

   **Step 1.** If $q \in S_2$ and $|a_{qq}| = r_q(A)$, then $r_{q}^{S_1}(A) > 0$ and $p \in J(A)$. Since $q \in S_2$, we deduce from (2.11) that $f_{A}^{S_1}(p, q) > 0$. Thus from $|a_{qq}| = r_q(A)$, we see that $|a_{qq}| - r_{q}^{S_1}(A) = r_{q}^{S_1}(A) > 0$ and $p \in J(A)$.

   **Step 2.** If $|a_{pp}| = r_p(A)$, then $r_{p}^{S_2}(A) > 0$ and $S_2 \subseteq J(A)$. The step is proven in a similar way to step 1.

   **Step 3.** If $J(A) \neq \emptyset$. This clearly follows if $|a_{pp}| > r_p(A)$. So, assume that $|a_{pp}| = r_p(A)$. Then from step 2, the result follows.

   **Step 4.** If $A \in \text{GDD}$. This follows from $A \in D$ and step 3.

   **Step 5.** If condition (1) holds, then $A \in \text{DC}$. Since $A \in D$, we have either $|a_{pp}| > r_p(A)$ or $|a_{pp}| = r_p(A)$. We consider each case separately.

   **Case 1:** $|a_{pp}| > r_p(A)$. It then follows from $A \in D$ and condition (1) that to prove $A \in \text{DC}$, it suffices to show that for every $q \in S_2$ with $|a_{qq}| = r_q(A)$ there is a directed path in $\mathcal{G}(A)$ from $v_q$ to $v_p$. Let $q \in S_2$ be such that $|a_{qq}| = r_q(A)$. Thus from step 1, we see that $r_{q}^{S_1}(A) > 0$. Hence there exists $i \in S_1$ such that $a_{qi} \neq 0$. Then from condition (1), the result follows.

   **Case 2:** $|a_{pp}| = r_p(A)$. It then follows from step 2 that $S_2 \subseteq J(A)$ and there exists $j_0 \in S_2$ such that $a_{p_{j_0}} \neq 0$. So, from $A \in D$ and condition (1), we deduce that $A \in \text{DC}$. □

**Remark 2.2.** We observe that there exist a diagonally dominant matrix $A \in \mathcal{M}_n, n \geq 2$, and a separation $(S_1, S_2)$ of $(n)$ with which Eq. (2.11) and condition (1) of Proposition 2.1 are satisfied, but $A$ is not strictly generalized doubly diagonally dominant. To see this, let $A = (a_{ij}) \in \mathcal{M}_3$ be such that (2.3)–(2.5) are all satisfied. It follows from term (4) of Theorem 2.1 that $A \notin \text{SGDD}$. It is clear that $A \in D$. Also, with the separation $(S_1, S_2) = ([1, 3], \{2\})$ and $p = 3$, it can be shown that Eq. (2.11) and condition (1) of Proposition 2.1 are satisfied.

From term (5) of Theorem 2.1, we know that $D \cap \text{SGDD}$ is a proper subset of DC. The following proposition provides a sufficient condition for a matrix $A \in \text{DC}$ to be in $D \cap \text{SGDD}$. 
**Proposition 2.2.** Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$, be such that $A \in \text{DC}$. In addition, suppose that $A$ satisfies the following condition:

Condition (1): If $(n) \setminus J(A)$ is nonempty and $k \in (n) \setminus J(A)$, then $r_k^{(A)}(A) > 0$.

Then $A \in \text{D} \cap \text{SGDD}$.

**Proof.** Since $A \in \text{DC}$, we see from term (iii) of Definition 1.1 that

$$A \in \text{D} \quad \text{and} \quad J(A) \neq \emptyset. \quad (2.12)$$

If $J(A) = (n)$ then, from $\text{SD} \subset \text{D} \cap \text{SGDD}$ (in (2.1)), the result follows. So, suppose that $J(A)$ is a proper subset of $(n)$. Thus from $J(A) \neq \emptyset$ (in (2.12)), we deduce that $(J(A), (n) \setminus J(A))$ is a separation of $(n)$. For simplicity, write $(J(A), (n) \setminus J(A))$ as $(S_1, S_2)$. Let $j \in S_2$. Then from $A \in \text{D}$ (in (2.12)) and condition (1), we infer that $|a_{ij}| - r_j^{S_2}(A) = r_j^{S_1}(A) > 0$. Thus $f_j^{S_1}(i, j) > 0$ for all $i \in S_1 = J(A)$. Hence $f_j^{S_1} > 0$. Then from (2.12), we see that $A \in \text{D} \cap \text{SGDD}$. □

We now present a few known facts about the invertibility of some special families of matrices introduced in Definition 1.1. Unlike the set $\text{SD}$, the sets $\text{D}$ and $\text{GDD}$ both contain singular matrices. In regard to the set $\text{SGD}$, we have:

**Lemma 2.1.** Every strictly generalized diagonally dominant matrix is invertible.

**Proof.** Let $A \in \mathcal{M}_n$ be strictly generalized diagonally dominant. Then there exists a diagonal matrix $Y \in \mathcal{M}_n$ such that $AY \in \text{SD}$. Thus from the Desplanques Theorem [4], we deduce that $AY$ is invertible. Hence $A$ is invertible. □

From Lemma 2.1 and $\text{SGDD} \subset \text{SGD}$ (in (2.2)), we have:

**Corollary 2.1.** Every strictly generalized doubly diagonally dominant matrix is invertible.

Corollary 2.1 has an interesting connection with the formation of an inclusion region for the spectrum of a square matrix. To see this, we first introduce the following useful characterization for a matrix $A \in \mathcal{M}_n$ to be in $\text{SGD}$.

**Remark 2.3.** Let $A \in \mathcal{M}_n$, $n \geq 2$. Then $A \in \text{SGD}$ if and only if there is a nonempty proper subset $S_1$ of $(n)$ such that $f_A^{S_1} > 0$ and there exists $k \in S_1$ with $|a_{kk}| - r_k^{S_1}(A) > 0$.

We now introduce the following definition:

**Definition 2.1.** Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$. Suppose that $(S_1, S_2)$ is a separation of $(n)$. For all $i \in S_1$ and $j \in S_2$, define the sets $\mathcal{D}_i^{S_1}(A)$ and $\mathcal{V}_j^{S_1}(A)$ by

$$\mathcal{D}_i^{S_1}(A) = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq r_i^{S_1}(A) \right\} \quad (2.13)$$

and

$$\mathcal{V}_j^{S_1}(A) = \left\{ z \in \mathbb{C} : \left( |z - a_{jj}| - r_j^{S_1}(A) \right) \left( |z - a_{jj}| - r_j^{S_2}(A) \right) \leq r_i^{S_1}(A) r_j^{S_1}(A) \right\}. \quad (2.14)$$

(Using (1.4), we have $\mathcal{V}_j^{S_1}(A) = \{ z \in \mathbb{C} : f_{i,j}^{S_1}(i, j) \leq 0 \}$.)

From Remark 2.3 and Definition 2.1, it is clear that Corollary 2.1 is equivalent to the following corollary:
Corollary 2.2. Let $A \in \mathcal{M}_n$, $n \geq 2$. Then for every separation $(S_1, S_2)$ of $(n)$, we have

$$\sigma(A) \supset \mathbb{V}^{S_1}(A) := \left( \bigcap_{i \in S_1} \mathbb{D}_i^{S_1}(A) \right) \cup \left( \bigcup_{i \in S_1, j \in S_2} \mathbb{V}_i^{S_1}(A) \right). \quad (2.15)$$

Theorem 3.12 in [17] provides an inclusion region for $\sigma(A)$ that contains the region $\mathbb{V}^{S_1}(A)$ given in (2.15). The result in [17] utilizes the equivalence of Corollaries 2.1 and 2.2, and follows from Theorem 3.11 of [17]. The latter result shows (using different terminology) that $\text{SGDD} \subset \text{SGD}$, and is obtained through the same technique used in Theorem 1 of [8].

Remark 2.4. Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$. Suppose that $(S_1, S_2)$ is a separation of $(n)$. We make the following two observations:

1. For $i \in S_1$ and $j \in S_2$, define the set $\mathbb{W}_i^{S_1}(A)$ by

$$\mathbb{W}_i^{S_1}(A) = \left\{ z \in \mathbb{C} : z \not\in \mathbb{D}_i^{S_1}(A) \cup \mathbb{D}_j^{S_2}(A) \text{ and } z \in \mathbb{V}_i^{S_1}(A) \right\}. \quad (2.16)$$

Theorem 2.1 in [12] then shows that

$$\sigma(A) \supset \mathbb{W}^{S_1}(A) := \left( \bigcup_{i \in S_1} \mathbb{D}_i^{S_1}(A) \right) \cup \left( \bigcup_{j \in S_2} \mathbb{D}_j^{S_2}(A) \right) \cup \left( \bigcup_{i \in S_1, j \in S_2} \mathbb{W}_i^{S_1}(A) \right). \quad (2.17)$$

2. We now show that the sets $\mathbb{V}^{S_1}(A)$ and $\mathbb{W}^{S_1}(A)$ defined in (2.15) and (2.17), respectively, are the same.

(i) First, suppose that $z \in \mathbb{V}^{S_1}(A)$. It is clear that

$$z \in \bigcap_{i \in S_1} \mathbb{D}_i^{S_1}(A) \implies z \in \mathbb{W}^{S_1}(A).$$

So, assume that $z \in \mathbb{V}^{S_1}(A) \setminus \left( \bigcap_{i \in S_1} \mathbb{D}_i^{S_1}(A) \right)$. We have two cases:

Case (a) Either there exists $i_0 \in S_1$ with $z \in \mathbb{D}_{i_0}^{S_1}(A)$ or there exists $j_0 \in S_2$ with $z \in \mathbb{D}_{j_0}^{S_2}(A)$. It then follows from the definition of $\mathbb{W}_i^{S_1}(A)$ (in (2.17)) that $z \not\in \mathbb{W}^{S_1}(A)$.

Case (b) For all $i \in S_1$ and $j \in S_2$, we have

$$z \not\in \mathbb{D}_i^{S_1}(A) \quad \text{and} \quad z \not\in \mathbb{D}_j^{S_2}(A). \quad (2.18)$$

It then follows from $z \in \mathbb{V}^{S_1}(A)$ that there exist $i_1 \in S_1$ and $j_1 \in S_2$ such that $z \in \mathbb{V}_{i_1j_1}^{S_1}(A)$. Thus from (2.16) and (2.18), we deduce that $z \not\in \mathbb{W}_{i_1j_1}^{S_1}(A)$. Hence from the definition of $\mathbb{W}^{S_1}(A)$ (in (2.17)), we infer that $z \not\in \mathbb{W}^{S_1}(A)$.

(ii) Now, suppose that $z \not\in \mathbb{V}^{S_1}(A)$. Then from (2.15), we have

$$z \not\in \left( \bigcap_{i \in S_1} \mathbb{D}_i^{S_1}(A) \right) \cup \left( \bigcup_{i \in S_1, j \in S_2} \mathbb{V}_i^{S_1}(A) \right). \quad (2.19)$$

Thus from (2.14) and (2.16), we deduce that

$$z \not\in \bigcup_{i \in S_1, j \in S_2} \mathbb{W}_i^{S_1}(A). \quad (2.20)$$
Also, it follows from (2.13) and $z \not\in \bigcap_{i \in S_1} \mathcal{D}^S_{i,j}(A)$ (in (2.19)) that there exists $k \in S_1$ such that $|z - a_{kk}| > r^S_k(A)$. Hence from (2.14) and $z \not\in \bigcup_{i \in S_1, j \in S_2} \mathcal{V}^S_j(A)$ (in (2.19)), we infer that

$$|z - a_{ij}| > r^S_j(A)$$  \hspace{1cm} (2.21)

for all $j \in S_2$. Then from $z \not\in \bigcup_{i \in S_1, j \in S_2} \mathcal{V}^S_j(A)$ (in (2.19)), we get $|z - a_{ii}| > r^S_i(A)$ for all $i \in S_1$. Thus from (2.13) and (2.21) we see that

$$z \not\in \left( \bigcup_{i \in S_1} \mathcal{D}^S_{i,j}(A) \right) \cup \left( \bigcup_{j \in S_2} \mathcal{D}^S_{j,k}(A) \right).$$

Hence from the definition of $\mathcal{V}^S_i(A)$ (in (2.17)) and (2.20), we get $z \not\in \mathcal{W}^S_i(A)$. This completes the proof that $\mathcal{W}^S_i(A) = \mathcal{W}^{-S}_i(A)$.

3. Matrices with a generalized type of diagonal dominance

For a matrix $A = (a_{ij}) \in \mathcal{M}_n$, we denote, unless otherwise stated, the vertices of $A$ in its directed graph $\mathcal{G}(A)$ by $v_1, \ldots, v_n$. We introduce in the following definition directed graphs in $\mathcal{G}(A)$ that are characterized by a separation of $\langle n \rangle$.

**Definition 3.1.** Let $A \in \mathcal{M}_n$, $n \geq 3$, and let $(S_1, S_2)$ be a separation of $\langle n \rangle$ such that $\text{card } S_1 \geq 2$. Suppose that $\Gamma$ is a directed path in $\mathcal{G}(A)$.

1. We say that $\Gamma$ is an $S_1$ — directed path if for every vertex $v_k$ in $\Gamma$ we have $k \in S_1$.
2. The directed path $\Gamma$ is called reducible $S_1$ — directed path if $\Gamma$ is an $S_1$ — directed path and for every vertex $v_k$ in $\Gamma$ we have $r^S_k(A) = 0$.
3. If $\Gamma$ is an $S_1$ — directed path that is not reducible, we call it an irreducible $S_1$ — directed path.

The following proposition provides a sufficient condition for a generalized doubly diagonally dominant matrix to be invertible. If $A \in \mathcal{M}_n$, $n \geq 2$, and $S$ is a nonempty proper subset of $\langle n \rangle$, we denote the principal submatrix of $A$ that lies in the rows and columns of $A$ indexed by $S$ as $A(S)$; see p. 17 of [11].

**Proposition 3.1.** Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$. Assume that there exists a separation $(S_1, S_2)$ of $\langle n \rangle$ such that the following conditions hold

Condition (1): $A(S_1) \in \mathcal{D}$.
Condition (2): $f^S_{A,S_1} \geq 0$.
Condition (3): $\max f^S_{A,S_1} > 0$.

Then

(i) $\mathcal{J}(A) \neq \emptyset$.
(ii) $\mathcal{J}(A(S_i)) \neq \emptyset$ for $i = 1, 2$.
(iii) $A(S_2) \in \mathcal{D}.$
(iv) If, in addition to conditions (1)–(3), $A$ satisfies the following two conditions:

Condition (4): If $i \in \{1, 2\}$, $z_i \in S_i$ and $|a_{z_i}| = r^S_{z_i}(A)$, then there exist $\vartheta_i \in \mathcal{J}(A(S_i))$ and an irreducible $S_1$ — directed path in $\mathcal{G}(A)$ from the vertex $v_{z_i}$ to the vertex $v_{\vartheta_i}$.
Condition (5): If $K(A) = \{(i_1, i_2) \in S_1 \times S_2 : f^S_{A,i_1,i_2} = 0, r^S_{i_1}(A) r^S_{i_2}(A) > 0\}$ is nonempty and
(ε₁, ε₂) ∈ K(A), then there exist (δ₁, δ₂) ∈ L(A) = \{(k₁, k₂) ∈ S₁ × S₂ : f_A^{S₁}(k₁, k₂) > 0\} and ε₁ ∈ (ε₁, ε₂) \setminus [δ₁, δ₂] such that there are directed paths \(\Gamma_i : v_{ε_i} \rightarrow v_{δ_i}\) and \(\Lambda_i : v_{ε_i} \rightarrow v_{δ_i}\) in \(G(A)\), then \(A\) is invertible.

**Remarks**

(a) Notice that if \(A \in GDD\), then there exists a separation (\(S₁, S₂\)) of \(\langle n \rangle\) such that \(A\) satisfies conditions (1) and (2) of Proposition 3.1. Also, if \(A \in M_{n}\) satisfies conditions (1)–(3) of Proposition 3.1, then \(A \in GDD\).

(b) It follows from condition (3) that the set \(L(A) = \{(i, j) ∈ S₁ × S₂ : f_A^{S₁}(i, j) > 0\}\) defined in condition (5) is nonempty.

**Proof of Proposition 3.1.** It follows from condition (3) that there exist \(α₁ \in S₁\) and \(α₂ \in S₂\) such that

\[
|a_{α₁α₁}| − r_A^{S₁}(A) \left( |a_{α₁α₂}| − r_A^{S₂}(A) \right) > r_A^{S₂}(A) r_A^{S₁}(A).
\]

Then from condition (1), we deduce that

\[
|a_{α₁α₁}| − r_A^{S₁}(A) > 0
\]

for \(i = 1, 2\), and

either \( |a_{α₁α₁}| − r_A^{S₁}(A) > r_A^{S₂}(A) \) or \( |a_{α₁α₂}| − r_A^{S₂}(A) > r_A^{S₁}(A) \).

Thus from \( r_A(i) = r_A^{S₁}(A) + r_A^{S₂}(A) \) for \(i = 1, 2\), term (i) follows. From (3.1), term (ii) follows. Also, from condition (2) and (3.1) (with \(i = 1\)), we infer that \( |a_{ij} − r_A^{S_j}(A) | ≤ 0 \) for all \(j \in S₂\). This proves term (iii). The conclusion on the invertibility will be proved by contradiction. Assume that \(A\) is singular and \(x = (x₁, \ldots, xₙ) ∈ \mathbb{C}^n\) is a nonzero vector in the null space of \(A\). If conditions (1)–(4) hold, we will show that \(A\) does not satisfy condition (5) by the following seven steps:

**Step 1.** \(A(S₁)\) and \(A(S₂)\) are both invertible. We prove that \(A(S₂)\) is invertible. The other statement is proven similarly. If card \(S₂ = 1\) then, from term (ii), we see that \(A(S₂)\) is invertible. So, assume that card \(S₂ = m > 1\). Denote the matrix \(A(S₂)\) by \(B = (b_{αβ})\), and the vertices in the directed graph \(G(B)\) by \(v₁, \ldots, v_m\). It follows from term (ii) that \(B\) has a strictly diagonally dominant row. Let \(p \in \langle m \rangle\) be such that \( |b_{pq} | > r_B(B) \). Then from condition (4), there exists \(q \in \langle m \rangle\) with \( |b_{qp} | > r_B(B) \) such that there is a directed path in \(G(B)\) from \(v_p\) to \(v_q\). Thus from term (iii), we deduce that \(B \in DC\). Hence from \(DC \subset SGD\) (in (2.2)) and Lemma 2.1, we see that \(A(S₂) = B\) is invertible.

**Step 2.** \( \min \{||x(S₁)||_{∞}, ||x(S₂)||_{∞}\} > 0\). This follows from step 1, \(Ax = 0\) and \(x \neq 0\).

**Step 3.** Let \((ω₁, ω₂) ∈ \{(α₁, α₂) ∈ S₁ × S₂ : |x_{α₁}| = ||x(S₁)||_{∞}, j = 1, 2\). Then

\[
|a_{ω₁ω₂}||x(S₁)| ≤ \sum_{τ = 1, τ \neq ω₁}^{n} |a_{ω₁τ}||x_{τ}| ≤ ||x(S₁)||_{∞} r_A^{S₁}(A) + ||x(S₂)||_{∞} r_A^{S₂}(A),
\]

(3.2)

where \(i, k = 1, 2\). Also, \(f_A^{S₁}(ω₁, ω₂) = 0\). Eq. (3.2) follows from \(Ax = 0\) and the definitions of \(ω₁\) and \(ω₂\). It follows from condition (1), step 2 and (3.2) that \(f_A^{S₁}(ω₁, ω₂) ≤ 0\). Then from condition (2), we get \(f_A^{S₁}(ω₁, ω₂) = 0\).

**Step 4.** If \(i, k = 1, 2\) and \(s_i ∈ S₁\) is such that \(|x_{s_i}| = ||x(S₁)||_{∞}\) and \(r_A^{S₁}(A) = 0\), then \(a_{s_iS₁} = r_A^{S₁}(A)\) and \( \sum_{τ ∈ S₁, τ \neq s_i} a_{τS₁} ||x(S₁)||_{∞} − |x_{τ}| = 0 \). Since \(s_i ∈ S₁\) and \(|x_{s_i}| = ||x(S₁)||_{∞}\), we deduce from step 3 that (3.3) holds. Then from \(r_A^{S₁}(A) = 0\) and step 2, we get \(a_{s_iS₁} ≤ r_A^{S₁}(A)\). Thus from condition (1) or term (iii) (depending on the value of \(i\)), we obtain \(a_{s_iS₁} = r_A^{S₁}(A)\). Hence from \(|x_{s_i}| = ||x(S₁)||_{∞}\), \(r_A^{S₁}(A) = 0\) and (3.2), we obtain \(\sum_{τ ∈ S₁, τ \neq s_i} a_{τS₁} ||x(S₁)||_{∞} − |x_{τ}| = 0\).

**Step 5.** Let \(i = 1, 2\). Then there exists \(ω_i ∈ S₁\) such that \(|x_{ω_i}| = ||x(S₁)||_{∞}\) and \(r_A(ω_i) > r_A^{S₁}(A)\). Let \(k = 1, 2 \setminus \{i\}\). Choose \(s_i ∈ S₁\) such that \(|x_{s_i}| = ||x(S₁)||_{∞}\). If \(r_A^{S₁}(A) = 0\), then from \(r_A(ω_i) = r_A^{S₁}(A) + r_A^{S₂}(A)\), step 5 follows. So, assume that \(r_A^{S₁}(A) = 0\). Thus from \(s_i ∈ S₁\), \(|x_{s_i}| = ||x(S₁)||_{∞}\) and
step 4, we see that $|a_{\delta_1\omega_1}| = r^S_{\omega_1}(A)$. Hence from condition (4), there is $\delta_1 \in S_i$ with $|a_{\delta_1\omega_1}| > r^S_{\omega_1}(A)$ such that there exists an irreducible $\delta_i$-directed path $\Gamma : v_{\delta_i} \rightarrow v_{\omega_1}$. Choose the first vertex $v_\varsigma$ in the directed path $\Gamma$ that satisfies $r^S_{\omega_1}(A) > 0$, and denote it by $v_{\omega_1}$. By induction, we deduce from $|x_\varsigma| = \|x(S_i)\|_\infty$ and step 4 that all vertices $v_j$ in the $\delta_i$-directed path $\Gamma$ from $v_{\delta_i}$ to $v_{\omega_1}$, including $v_{\omega_1}$, satisfy $j \in S_i$ and $|x_j| = \|x(S_i)\|_\infty$. Then from $r^S_{\omega_1}(A) > 0$, step 5 follows.

Step 6. Let $\{i, k\} = \{1, 2\}$, and let $\omega_1 \in S_i$ be such that $|x_{\omega_1}| = \|x(S_i)\|_\infty$. If $\delta \in (n) \setminus \{\omega_1\}$ and $\Gamma$ is a directed path in $G(A)$ from $v_{\omega_1}$ to $v_\delta$, then

$$|x_\delta| = \left\{ \begin{array}{ll} \|x(S_i)\|_\infty & \text{if } \delta \in S_i, \\ \|x(S_k)\|_\infty & \text{if } \delta \in S_k. \end{array} \right. \quad (3.3)$$

We first prove

$$\sum_{\tau \in S_i, \tau \neq \omega_1} |a_{\omega_1\tau}| (\|x(S_i)\|_\infty - |x_\tau|) + \sum_{\tau \in S_k} |a_{\omega_k\tau}| (\|x(S_k)\|_\infty - |x_\tau|) = 0. \quad (3.4)$$

If $r^S_{\omega_1}(A) = 0$, then from the definition of $\omega_1$ and step 4, Eq. (3.4) follows. So, assume that $r^S_{\omega_1}(A) > 0$. From step 5, choose $\omega_k \in S_k$ such that $|x_{\omega_k}| = \|x(S_k)\|_\infty$ and $r^S_{\omega_1}(A) > 0$. It follows from condition (1) and (3.2) that

$$0 \leq (|a_{\omega_1\omega_1}| - r^S_{\omega_1}(A)) \|x(S_1)\|_\infty \leq \|x(S_2)\|_\infty r^S_{\omega_1}(A). \quad (3.5)$$

Also, from term (iii) and (3.2), we have

$$0 \leq (|a_{\omega_2\omega_2}| - r^S_{\omega_2}(A)) \|x(S_2)\|_\infty \leq \|x(S_1)\|_\infty r^S_{\omega_2}(A). \quad (3.6)$$

Since $r^S_{\omega_1}(A) = 0$ and $\|x(S_1)\|_\infty \|x(S_2)\|_\infty > 0$ (see step 2), we see from $f^S_A(\omega_1, \omega_2) = 0$ (in step 3), (3.5) and (3.6) that $0 \leq (|a_{\omega_1\omega_1}| - r^S_{\omega_1}(A)) \|x(S_1)\|_\infty = \|x(S_k)\|_\infty r^S_{\omega_1}(A)$. Thus from (3.2), Eq. (3.4) follows.

Now, let $\delta \in (n) \setminus \{\omega_1\}$ be such that there exists a directed path $\Gamma$ in $G(A)$ from $v_{\omega_1}$ to $v_\delta$. By induction, we see that (3.3) follows from $|x_{\omega_1}| = \|x(S_i)\|_\infty$ and (3.4).

Step 7. There exists $(\omega_1, \omega_2) \in K(A) = \{(i_1, i_2) \in S_1 \times S_2 : f^S_A(i_1, i_2) = 0, r^S_{i_1}(A) r^S_{i_2}(A) > 0\}$ with the following property: For any $(\delta_1, \delta_2) \in L(A) = \{(k_1, k_2) \in S_1 \times S_2 : f^S_A(k_1, k_2) > 0\}$ and any $\omega_1 \in \{\omega_1, \omega_2\} \setminus \{\delta_1, \delta_2\}$, there exists $\delta_i \in \{\delta_1, \delta_2\}$ such that there is no directed path in $G(A)$ from $v_{\omega_i}$ to $v_{\delta_i}$. From step 5, there exists $(\omega_1, \omega_2) \in S_1 \times S_2$ such that $|x_{\omega_i}| = \|x(S_i)\|_\infty$ for $i = 1, 2$ and $r^S_{\omega_1}(A) r^S_{\omega_2}(A) > 0$. Then from $f^S_A(\omega_1, \omega_2) = 0$ (in step 3), we deduce that $(\omega_1, \omega_2) \in K(A)$. Let $(\delta_1, \delta_2) \in L(A)$ and $\omega_i \in \{\omega_1, \omega_2\} \setminus \{\delta_1, \delta_2\}$. If there were directed paths $\Gamma_1 : v_{\omega_i} \rightarrow v_{\delta_i}$ and $\Gamma_2 : v_{\omega_i} \rightarrow v_{\delta_i}$ in $G(A)$ then, from steps 3 and 6, we would have $f^S_A(\delta_1, \delta_2) = 0$, and this contradicts the fact that $(\delta_1, \delta_2) \in L(A)$. \(\square\)

The following example shows that conditions (1)–(4) of Proposition 3.1 are not sufficient conditions for the invertibility of a matrix.

**Example 3.1.** Let $A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then $A$ is singular. Let $S_1 = \{1\}$ and $S_2 = \{2, 3, 4\}$. \(\square\)
It is clear that $A$ satisfies conditions (1) and (2) of Proposition 3.1. Since $f^S_A (1, 3) > 0$, we see that $A$ satisfies condition (3) of Proposition 3.1. It is clear that

$$\{(2, 4) : i \in \{1, 2\}, \alpha \in S_i \text{ and } |a_{\alpha \alpha}| = r^S_{rS}(A)\}.$$ 

Then from $a_{33} = 2 > 1 = r^S_{rS}(A)$, $a_{43}a_{31} \neq 0$ and $S_1 = \{1\}$, we deduce that condition (4) of Proposition 3.1 is also satisfied.

We observe that conditions (1)–(3) and condition (5) of Proposition 3.1 are not sufficient conditions for the invertibility of the matrix.

**Example 3.2.** Let $A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then $A$ is singular. Let $S_1 = \{5\}$ and $S_2 = \{4\}$. It is clear that $A$ satisfies conditions (1) and (2) of Proposition 3.1. Since $f^S_A (5, 3) = 2$, we see that $A$ satisfies condition (3) of Proposition 3.1. It is clear that

$$\{(5, 4) : (i_1, i_2) \in S_1 \times S_2 : f^S_A(i_1, i_2) = 0, r^S_{i_1}(A) r^S_{i_2}(A) > 0\}.$$

Thus from $f^S_A (5, 3) > 0$ and $a_{45}a_{43} \neq 0$, we see that condition (5) of Proposition 3.1 is also satisfied.

We remark that neither condition (2) nor condition (3) of Proposition 3.1 is a necessary condition for a matrix $A$ to be invertible. Also, there are invertible matrices that satisfy conditions (1)–(3) of Proposition 3.1, but they do not satisfy either condition (4) or condition (5) of Proposition 3.1. The following examples explain these facts.

**Example 3.3.** (i) Let $A_1 = \begin{pmatrix} 1 & 1.9 & 1.1 \\ 0.1 & 1 & 0 \\ 0.1 & 1.2 & 1 \end{pmatrix}$. Then $A_1$ is invertible (in fact, $A_1 \in \text{SGD}$ as, with $Y = \text{diag}\{1, 0.2, 0.4\}$, we have $A_1Y \in \text{SD}$). It can be shown that $A_1$ does not satisfy condition (2) of Proposition 3.1 for any separation $(S_1, S_2)$ of (3).

(ii) Let $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $A_2$ is invertible. It is clear that $A_2$ does not satisfy condition (3) of Proposition 3.1 for any separation $(S_1, S_2)$ of (3).

(iii) Let $A_3 = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$. Then $A_3$ is invertible. Let $(S_1, S_2)$ be a separation of (3). It can be shown that $A_3$ satisfies conditions (1)–(3) of Proposition 3.1 if and only if $\{S_1, S_2\} = \{\{1, 2\}, \{3\}\}$. Suppose that $S_1 = \{1, 2\}$. The other case is dealt with similarly. Since $J(A(S_1)) = \{2\}$, $|a_{11}| = r^S_{rS}(A_3)$ and $a_{13} = a_{23} = 0$, we see that there is no irreducible $S_1$-directed path from the vertex $v_1$ to the vertex $v_2$ in the directed graph $G(A_3)$ of $A_3$. So, $A_3$ does not satisfy condition (4) of Proposition 3.1.

(iv) Let $A_4 = (a_{ij}) = \begin{pmatrix} 1 & 1/2 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 3/2 \end{pmatrix}$, where $\iota = i^{1/2}$. Then $A_4$ is invertible. Let $(S_1, S_2)$ be a
separation of (3). It can be shown that $A_4$ satisfies conditions (1)–(3) of Proposition 3.1 if and only if $\{S_1, S_2\} = \{(1), \{2, 3\}\}$. Suppose that $S_1 = \{1\}$. The other case is dealt with similarly. It is clear that the sets $K(A_4)$ and $L(A_4)$ defined in condition (5) of Proposition 3.1 (with $A$ replaced by $A_4$) are given by

$$K(A_4) = \{(i_1, i_2) \in S_1 \times S_2 : r^{S_1}_{A_4}(i_1, i_2) = 0, r^{S_2}_{r_1}(A_4) > 0\} = \{(1, 2)\}$$

and

$$L(A_4) = \{(k_1, k_2) \in S_1 \times S_2 : r^{S_1}_{A_4}(k_1, k_2) > 0\} = \{(1, 3)\}.$$  

Since $a_{i3} = 0$ for $i = 1, 2$, we see that there is no directed path in $G(A_4)$ from the vertex $v_2$ to the vertex $v_3$. So, condition (5) of Proposition 3.1 does not hold for the matrix $A_4$ with respect the separation $((1), \{2, 3\})$.

The following theorem provides a sufficient condition for a matrix $A \in M_n$, $n \geq 2$, to be strictly diagonally dominant.

**Theorem 3.1.** Let $A = (a_{ij}) \in M_n$, $n \geq 2$. Assume that there exists a separation $(S_1, S_2)$ of $(n)$ such that

$$\{i, j\} \in S_1 \times S_2 : r^S_i(A) r^S_j(A) > 0 \neq \emptyset$$

and conditions (1)–(5) of Proposition 3.1 hold. Then $A \in \text{SGD}$.

**Proof.** Since $A$ satisfies condition (2) of Proposition 3.1, we deduce from (3.7) that

$$\{p, q\} = \{(1, 2), \text{ and } |a_{ii}| - r_{i}^{S_1}(A) = 0 \implies r_{i}^{S_1}(A) = 0. \quad (3.8)$$

Also, since $A$ satisfies conditions (1) and (3) of Proposition 3.1, we see that

$$S_1' := \{i \in S_1 : |a_{ii}| > r_{i}^{S_1}(A)\} \neq \emptyset. \quad (3.9)$$

From (3.7), we have

$$S_2' := \{j \in S_2 : r_{j}^{S_1}(A) > 0\} \neq \emptyset. \quad (3.10)$$

Then from condition (2) and (3.9), we get

$$z_2 = \min_{j \in S_2} \frac{|a_{ij}| - r_{j}^{S_2}(A)}{r_{j}^{S_1}(A)} \geq \max_{i \in S_1'} \frac{r_{i}^{S_2}(A)}{|a_{ii}| - r_{i}^{S_1}(A)} = z_1. \quad (3.11)$$

From term (iii) of Proposition 3.1, (3.8) and (3.10), we see that $z_2 > 0$. Choose $y \in [z_1, z_2]$ such that $y > 0$, and define the diagonal matrix $Y = \text{diag}(y_1, \ldots, y_n)$ by $y_i = y$ if $i \in S_1$ and $y_i = 1$ if $i \in S_2$. Denote the vertices in the directed graph $G(AY)$ by $v'_1, \ldots, v'_n$. Since $Y$ is a diagonal matrix with nonzero diagonal entries, we see that $A \in \text{SGD}$ if and only if $AY \in \text{SGD}$. Then from $\text{DC} \subset \text{SGD}$ (in (2.2)), it suffices to prove that $AY \in \text{DC}$. This is being established through the following seven steps:

**Step 1.** If $m \in (n) \setminus \{1\}$ and $1 \leq i_1, \ldots, i_m \leq n$, then $\Gamma = v_{i_1} v_{i_2}, \ldots, v'_{i_m-1} v'_{i_m}$ is a directed path in $G(A)$. If and only if $G' = v'_{i_1} v'_{i_2}, \ldots, v'_{i_m-1} v'_{i_m}$ is a directed path in $G(AY)$. The step clearly follows from the definition of $Y$.

**Step 2.** Let $(\tau_1, \tau_2) \in S_1 \times S_2$ be such that $r^S_{A}(\tau_1, \tau_2) > 0$. Then $(\tau_1, \tau_2) \cap J(AY) \neq \emptyset$. Since $A$ satisfies condition (1) of Proposition 3.1, we deduce from $r^S_{A}(\tau_1, \tau_2) > 0$ and $y > 0$ that

$$y \left(\left|a_{\tau_1, \tau_1} - r_{\tau_1}^{S_1}(A)\right| > r_{\tau_1}^{S_2}(A)\right) \text{ or } |a_{\tau_2, \tau_2} - r_{\tau_2}^{S_1}(A)| > y r_{\tau_2}^{S_1}(A).$$

Thus from the definition of $Y$, we see that $(\tau_1, \tau_2) \cap J(AY) \neq \emptyset$. 

Step 3. \( J(AY) \neq \emptyset \). Since \( A \) satisfies condition (3) of Proposition 3.1, we see from step 2 that \( J(AY) \neq \emptyset \).

Step 4. \( AY \in D \). Since \( A \) satisfies condition (1) of Proposition 3.1, we see from (3.9) that if \( i \in S_1 \setminus S'_1 \) then \( |a_{ii}| = r^1_i(A) \). Thus from (3.8) and the definition of \( Y \), we deduce that

\[
i \in S_1 \setminus S'_1 \implies |(AY)_{ii}| = r_i(AY).
\]

(3.12)

Also, it follows from (3.10), term (iii) of Proposition 3.1 and the definition of \( Y \) that

\[
j \in S_2 \setminus S'_2 \implies |(AY)_{jj}| \geq r_j(AY).
\]

(3.13)

From (3.9)–(3.11) and the definition of \( Y \), we see that

\[
k \in S'_1 \cap S'_2 \implies |(AY)_{kk}| \geq r_k(AY).
\]

Thus from (3.12) and (3.13), step 4 follows.

Step 5. If \( \{i, k\} = \{1, 2\} \), \( \zeta \in S_1 \) and \( |(AY)_{\zeta,\zeta}| = r^1_\zeta(AY) \), then there exists \( \eta \in S_1 \setminus \{\zeta\} \) with \( r^1_\eta(A) > 0 \) such that there is a directed path in \( \mathcal{G}(AY) \) from the vertex \( v'_\zeta \) to the vertex \( v'_\eta \). Assume without loss of generality that \( i = 1 \). It follows from \( |(AY)_{\zeta,\zeta}| = r^1_\zeta(AY) \), the definition of \( Y \) and \( y \neq 0 \) that

\[
|a_{\zeta,\zeta} - r^1_\zeta(A)| = 0.
\]

(3.14)

Then from (3.8), we deduce that

\[
r^2_\zeta(A) = 0.
\]

(3.15)

From \( \zeta \in S_1 \), (3.14) and condition (4), there exists \( \vartheta \in S_1 \setminus \{\zeta\} \) with \( |a_{\vartheta,\vartheta}| > r^1_\vartheta(A) \) such that there is an irreducible \( S_1 \)–directed path \( \Gamma \) in \( \mathcal{G}(A) \) from the vertex \( v_\zeta \) to the vertex \( v_\vartheta \). Thus from (3.15), there exists \( \eta \in S_1 \setminus \{\zeta\} \) with \( r^2_\eta(A) > 0 \) such that the vertex \( v_\eta \) lies in the directed path \( \Gamma : v_\zeta \longrightarrow v_\vartheta \). Hence from step 1, step 5 follows.

Step 6. If \( i \in \{1, 2\} \) and \( \zeta \in S_1 \) such that \( |(AY)_{\zeta,\zeta}| = r^1_\zeta(AY) \), then there exists \( \delta \in J(AY) \) such that there is a directed path in \( \mathcal{G}(AY) \) from \( v'_\zeta \) to \( v'_\delta \). Assume without loss of generality that \( i = 1 \). Then from \( |(AY)_{\zeta,\zeta}| = r^1_\zeta(AY) \) and the definition of \( Y \), we deduce that

\[
y \left( |a_{\zeta,\zeta} - r^1_\zeta(A)\right) = r^2_\zeta(A).
\]

(3.16)

We have either \( r^2_\zeta(A) > 0 \) or \( r^2_\zeta(A) = 0 \). We consider each case separately.

Case 1: \( r^2_\zeta(A) > 0 \). If there exists \( \tau \in S_2 \) such that \( a_{\zeta,\tau} \neq 0 \) and \( |(AY)_{\tau,\tau}| > r^1_{\tau}(AY) \), then step 6 follows. So, from step 4, we may assume that

\[
|(AY)_{jj}| = r_j(AY)
\]

(3.17)

for all \( j \in S_2 \) with \( a_{\zeta,j} \neq 0 \). We prove:

\[
P_\zeta := \{ j \in S_2 : r^1_j(A) > 0, \exists \text{ directed path in } \mathcal{G}(AY) \text{ from } v'_\zeta \text{ to } v'_j \} \neq \emptyset.
\]

(3.18)

From step 1, it is clear that (3.18) follows if \( \{j \in S_2 : a_{\zeta,j} \neq 0, r^1_j(A) > 0\} \neq \emptyset \). So, we may assume that \( r^1_j(A) = 0 \) for all \( j \in S_2 \) with \( a_{\zeta,j} \neq 0 \). It then follows from (3.17) and the definition of \( Y \) that

\[
|(AY)_{jj}| = r^1_j(AY)
\]

(3.19)

for all \( j \in S_2 \) with \( a_{\zeta,j} \neq 0 \). Choose \( j_0 \in S_2 \) with \( a_{\zeta,j_0} \neq 0 \). Thus from (3.19) and steps 1 and 5, we see that there exists \( \varepsilon \in S_2 \) with \( r^1_\varepsilon(A) > 0 \) such that there is a directed path in \( \mathcal{G}(AY) \) from \( v'_\zeta \) to \( v'_\varepsilon \). This completes the proof of (3.18).
From (3.18), choose $\varepsilon \in \mathcal{P}_\zeta$. It is clear from step 4 that $|(AY)_{kk}| > r_k(AY)$. If $|(AY)_{kk}| > r_k(AY)$, then step 6 follows. So, assume that $|(AY)_{kk}| = r_k(AY)$. Thus from $\varepsilon \in S_2$ and the definition of $Y$, we get $|a_{kk}| - r^S_{i_2}(A) = y r^S_{i_2}(A)$. Hence from $y > 0$, (3.16) and $r^S_{i_2}(A) r^S_{i_2}(A) > 0$, we deduce that

(3.20)


thus from step 5, the fact that there exists a directed path in $\mathcal{G}(AY)$ from the vertex $v'_\zeta$ to $v'_\delta$, we see that there exists $(\delta_1, \delta_2) \in S_1 \times S_2$ with $\delta_1, \delta_2 > 0$ and $k \in \{\zeta, \varepsilon\} \setminus \{\delta_1, \delta_2\}$ such that there are directed paths $\Gamma_k : v_1 \rightarrow v_{\delta_1}$ and $\Lambda_k : v_1 \rightarrow v_{\delta_2}$ in $\mathcal{G}(A\varepsilon)$. Thus from steps 1 and 2 and the fact that there exists a directed path in $\mathcal{G}(AY)$ from the vertex $v'_\zeta$ to $v'_\delta$, we see that there exists $(\delta_1, \delta_2) \in S_1 \times S_2$ with $\delta_1, \delta_2 > 0$ such that there are directed paths in $\mathcal{G}(AY)$ from $v'_\zeta$ to each of the vertices $v'_\delta$ and $v'_\bar{\delta}$. This completes the proof of step 6 when $r^S_{i_2}(A) > 0$.

Case 2: $r^S_{i_2}(A) = 0$. It then follows from $|(AY)_{kk}| = r_k(AY)$ and the definition of $Y$ that $|(AY)_{kk}| = r^S_{i_2}(A)$. Thus from step 5, there exists $\eta \in S_1 \setminus \{\varsigma\}$ with $r^S_{i_2}(A) > 0$ such that there is a directed path in $\mathcal{G}(AY)$ from $v'_\zeta$ to $v'_\eta$. From step 3, we have $|(AY)_{\eta\eta}| > r_\eta(AY)$. If $|(AY)_{\eta\eta}| > r_\eta(AY)$, step 6 follows. If $|(AY)_{\eta\eta}| = r_\eta(AY)$ then, from $r^S_{i_2}(A) > 0$ and case 1, step 6 follows in this case too.

Step 7. We have $AY \in \mathbf{DC}$. This follows from steps (3), (4) and (6).

**Example 3.4.** Let $A = (a_{ij}) = \begin{pmatrix} 1 & 0.75 & 0.25 & 0 & 0 \\ 0.1 & 1 & 0.1 & 1.2 & 1.2 \\ 0 & 0 & 1 & 0 & 1.2 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. It is clear that $A$ is reducible. We make the following three observations:

1. We use Theorem 3.1 to show that $A \in \mathbf{SGD}$. Let $S_1 = \langle 3 \rangle$ and $S_2 = \{4, 5\}$. Then
   (i) $A$ satisfies condition (1) of Proposition 3.1.
   (ii) Since
   
   \begin{equation}
   f^S_{i_2}(1, j) = f^S_{i_2}(2, 4) = 0
   \end{equation}
   
   for $j = 4, 5$, and
   
   \begin{equation}
   f^S_{i_2}(2, 5) = 0.8, f^S_{i_2}(3, 4) = 1.8 \quad \text{and} \quad f^S_{i_2}(3, 5) = 1,
   \end{equation}
   
   we deduce that $A$ satisfies conditions (2) and (3) of Proposition 3.1.
   (iii) It is clear that
   \begin{equation}
   \{i, \alpha : i \in S_1, \alpha \in S_2 \setminus \alpha \in S_1 \} \quad \text{and} \quad |a_{\alpha\alpha}| = r^S_{i_2}(A) = \{(1, 1)\}.
   \end{equation}
   
   Thus from $3 \in S_1, a_{13} \neq 0$ and $\min\{a_{33} - r^S_{i_3}(A), r^S_{i_3}(A)\} > 0$, we infer that $A$ satisfies condition (4) of Proposition 3.1.
   (iv) We have
   \begin{equation}
   \{i, j \in S_1 \times S_2 : f^S_{i_2}(i, j) = 0, r^S_{i_2}(A) r^S_{i_2}(A) > 0\} = \{(2, 4), (3, 4)\}.
   \end{equation}
   
   So, Eq. (3.7) in Theorem 3.1 holds.
   (v) Since $f^S_{i_2}(2, 4) = 0$ (in (3.20)) and $f^S_{i_2}(3, 4) > 0$ (in (3.21)), we see from (3.22) that
   \begin{equation}
   \{i, j \in S_1 \times S_2 : f^S_{i_2}(i, j) = 0, r^S_{i_2}(A) r^S_{i_2}(A) > 0\} = \{(2, 4)\}.
   \end{equation}
   
   Then from $f^S_{i_2}(3, 5) = 1$ and $a_{23} a_{25} \neq 0$, we see that condition (5) of Proposition 3.1 is also satisfied.
From (i)–(v) and Theorem 3.1, we deduce that $A \in \textbf{SGD}$.

(2) Let $(S_1, S_2)$ be a separation of (5) such that conditions (1)–(3) of Proposition 3.1 are satisfied for the matrix $A$. We prove $\{S_1, S_2\} = \{(3), \{4, 5\}\}$. We have either $2 \in S_1$ or $2 \in S_2$. Assume without loss of generality that

$$2 \in S_1. \quad (3.23)$$

We prove $S_1 = \{3\}$. Since $A$ satisfies condition (1) of Proposition 3.1, we see from $|a_{22}| < |a_{24}| = |a_{25}|$ and (3.23) that

$$4, 5 \in S_2. \quad (3.24)$$

Thus from $|a_{33}| < |a_{35}|$ and term (iii) of Proposition 3.1, we deduce that

$$3 \in S_1. \quad (3.25)$$

It follows from (3.24) and $a_{24} = a_{25} = 1.2$ that $f^S_2(A) \geq 2.4$. So, if 1 were in $S_2$ we would deduce from (3.23), (3.25) and the entries of $A$ along the first two rows that $f^S_1(A) < 0$, and this contradicts condition (2) of Proposition 3.1. This proves that $1 \in S_1$. It then follows from (3.23), (3.24) and (3.25) that $S_1 = \{3\}$.

(3) Using a different terminology, Theorem 8 of [3] provides sufficient conditions for a matrix in $\mathcal{M}_n$ to be strictly generalized diagonally dominant. With a separation $(S_1, S_2)$ of $\langle n \rangle$, the theorem contains conditions (1)–(3) of Proposition 3.1 and the following condition, which we call the “Auxiliary Condition”:

The “Auxiliary Condition”: If $(\varepsilon_1, \varepsilon_2) \in S_1 \times S_2$ is such that $f^S_{A}(\varepsilon_1, \varepsilon_2) = 0$, then there is $(k, l) \in S_1 \times S_2$ with $f^S_{A}(k, l) > 0$ such that there exist directed paths $\Lambda_1 : v_{\varepsilon_1} \rightarrow v_l$ and $\Lambda_2 : v_{\varepsilon_2} \rightarrow v_k$ in $\mathcal{G}(A)$.

We show that Theorem 8 in [3] cannot be applied to the matrix $A$. Since Theorem 8 in [3] satisfies conditions (1)–(3) of Proposition 3.1, we see from term (2) that $\{S_1, S_2\} = \{(3), \{4, 5\}\}$. Suppose that $S_1 = \{3\}$. The other case is treated similarly. Since $f^S_{A}(1, 5) = 0$ (in (3.20)), we see from $a_{5j} = 0$ for all $j \in S_1 = \{3\}$ that there is no $k \in S_1$ such that there is a directed path in $\mathcal{G}(A)$ from the vertex $v_5$ to the vertex $v_k$. So, the “Auxiliary Condition” does not hold for the matrix $A$.

**Example 3.5.** Let $A = (a_{ij}) = \begin{pmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3/4 & 2 & 1/8 \\ 5/12 & 3/4 & 0 & 2 \end{pmatrix}$.

(1) We use Theorem 3.1 to show that $A \in \textbf{SGD}$. Let $S_1 = \{1, 3\}$ and $S_2 = \{2, 4\}$. Then:

(i) $A$ satisfies condition (1) of Proposition 3.1.

(ii) It is clear that

$$f^S_{A}(i, 2) \geq 0 \quad (3.26)$$

for $i = 1, 3$, and

$$f^S_{A}(1, 4) = 0 \quad \text{and} \quad f^S_{A}(3, 4) = 85/96. \quad (3.27)$$

Thus $A$ satisfies conditions (2) and (3) of Proposition 3.1.

(iii) It is clear that for every $i \in \{1, 2\}$ and $\alpha \in S_i$, we have $|a_{\alpha \alpha}| > r^S_{i}(A)$. So, condition (4) of Proposition 3.1 is satisfied.

(iv) From the entries of $A$ and the definitions of $S_1$ and $S_2$, we see that

$$\{(i, j) \in S_1 \times S_2 : r^S_{i}(A) r^S_{j}(A) > 0\} = \{(1, 4), (3, 4)\}. \quad (3.28)$$

Then Eq. (3.7) in Theorem 3.1 holds.
(v) It follows from (3.26)–(3.28) that
\[
 \{(i, j) \in S_1 \times S_2 : f^{S_1}_A (i, j) = 0, \ r^{S_1}_i (A) r^{S_1}_j (A) > 0 \} = \{(1, 4)\}.
\]
Then from \(a_{13} a_{34} \neq 0\) and \(f^{S_1}_A (3, 4) > 0\) in (3.27)), we deduce that condition (5) of Proposition 3.1 is satisfied for the matrix \(A\).

From (i)–(v) and Theorem 3.1, we deduce that \(A \in \text{SGD}\).

(2) Theorems 1–4 in [7] provide sufficient conditions for a matrix to be strictly generalized diagonally dominant. We show that the four theorems in [7] cannot be applied to the matrix \(A\) by Eq. (3.7) in Theorem 3.1 is not satisfied. One of the conditions of Theorem 4 in [7] is not satisfied.

We remark that for a matrix \(A\) satisfying conditions (1)–(5) of Proposition 3.1, the condition defined by Eq. (3.7) in Theorem 3.1 is not a necessary condition for \(A\) to be strictly generalized diagonally dominant. The following example explains this.

**Example 3.6.** Let \(A = (a_{ij}) = \begin{pmatrix} 1 & 3/2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1/2 & 2 & 1/4 \\ 1 & 0 & 0 & 2 \end{pmatrix}\). We make the following three observations:

1. It is clear that, with \(S_1 = \{1, 3, 4\}\) and \(S_2 = \{2\}\), conditions (1)–(3) of Proposition 3.1 are satisfied for the matrix \(A\).
2. We have either 1 \(\in S_1\) or 1 \(\in S_2\). Assume without loss of generality that \(1 \in S_1\).
3. We deduce from \(|a_{33}| > r^{S_1}_3 (A)\) and \(a_{13} a_{32} \neq 0\) that there exists an irreducible \(S_1\)-directed path from the vertex \(v_1\) to the vertex \(v_3\). Then \(A\) satisfies condition (4) of Proposition 3.1. Also, since \(r^{S_1}_2 (A) = 0\), we see that \(\{(i, j) \in S_1 \times S_2 : f^{S_1}_A (i, j) = 0, \ r^{S_1}_i (A) r^{S_1}_j (A) > 0 \} = \emptyset\). So, condition (5) of Proposition 3.1 is satisfied.
4. Let \((S_1, S_2)\) be a separation of \(\{4\}\) such that conditions (1)–(3) of Proposition 3.1 are satisfied for the matrix \(A\). We prove \(\{S_1, S_2\} = \{\{1, 3, 4\}, \{2\}\}\), and that Eq. (3.7) in Theorem 3.1 is not satisfied. We have either 1 \(\in S_1\) or 1 \(\in S_2\). Assume without loss of generality that

\[
1 \in S_1.
\]

We prove \(S_1 = \{1, 3, 4\}\). Since condition (1) of Proposition 3.1 holds for \(A\), we deduce from (3.29), \(a_{11} = 1\) and \(a_{12} = 3/2\) that

\[
2 \in S_2.
\]

It follows from (3.29), (3.30) and the entries of \(A\) along the first and third rows that if \(3 \in S_2\) then \(f^{S_1}_A (1, 3) < 0\), and this contradicts condition (2) of Proposition 3.1. Then we must have

\[
3 \in S_1.
\]

From (3.29)–(3.31) and the entries of \(A\) along the first and fourth rows, we see that if \(4 \in S_2\) then \(f^{S_1}_A (1, 4) < 0\), and this contradicts condition (2) of Proposition 3.1. This proves \(4 \in S_1\). Thus from (3.29)–(3.31), we infer that \(S_1 = \{1, 3, 4\}\). Finally, it follows from \(a_{2j} = 0\) for all \(j \in \{1, 3, 4\}\) and \(\{S_1, S_2\} = \{\{1, 3, 4\}, \{2\}\}\) that Eq. (3.7) does not hold.

(3) Let \(Y = \text{diag}(1, 0.1, 0.8, 1)\). Then \(AY \in \text{SD}\). So, \(A \in \text{SGD}\).
In some applications, it is easier to check if a matrix $A \in \mathcal{M}_n$ is strictly generalized doubly diagonally dominant rather than determining if $A \in \text{SGD}$. If $A \in \text{SGDD}$, then we could use the inclusion $\text{SGDD} \subset \text{SGD}$ in (2.2) to conclude that $A$ is strictly generalized diagonally dominant. This technique will be used in Theorem 3.2. We will also use in Theorem 3.2 the following notation: If $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$, and $y = (y_1, \ldots, y_n)^t \in \mathbb{C}^n$ with $y_k \neq 0$ for all $k \in \langle n \rangle$, we define for every nonempty proper subset $S$ of $\langle n \rangle$ and every $i \in \langle n \rangle$ the quantity $r_i^S(y, A)$ by

$$r_i^S(y, A) = \sum_{k \in S, k \neq i} |y_k||a_{ik}|.$$  

(3.32)

In (3.32), it is understood that $r_i^S(y, A) = 0$.

**Theorem 3.2.** Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$. Suppose that there exist a separation $(S_1, S_2)$ of $\langle n \rangle$ and a vector $y = (y_1, \ldots, y_n)^t \in \mathbb{C}^n$ with $y_k \neq 0$ for all $k \in \langle n \rangle$ such that the following two conditions are satisfied:

Condition (1): There exists $p \in S_1$ such that $|a_{pp}||y_p| > r_p^{S_1}(y, A)$.

Condition (2): For all $i \in S_1$ and $j \in S_2$, we have

$$\left(|a_{ii}||y_i| - r_i^{S_1}(y, A)\right)\left(|a_{jj}||y_j| - r_j^{S_2}(y, A)\right) > r_i^{S_2}(y, A) r_j^{S_1}(y, A).$$

Then $A \in \text{SGD}$.

**Proof.** Let $Y = \text{diag}(y_1, \ldots, y_n)$. Then from condition (1), we get

$$|(AY)_{pp}| > r_p^{S_1}(AY).$$

(3.33)

Also, since $A$ satisfies condition (2), we see from (1.4), (3.32) and the definition of $Y$ that $r_{AY}^{S_1} > 0$. Thus from (3.33) and Remark 2.3, we deduce that $AY \in \text{SGDD}$. Hence from $\text{SGDD} \subset \text{SGD}$ (in (2.2)), we infer that $AY \in \text{SGD}$. Then from $Y$ being a diagonal matrix, we see that $A \in \text{SGD}$. \qed

**Example 3.7.** Let $A = (a_{ij}) = $

$$
1 & 0.1 & 0.1 & 0.1 \\
1.1 & 1 & 0.1 & 0.1 \\
1.1 & 0.5 & 1 & 0.2 \\
1.1 & 1.1 & 0.2 & 1 \\
$$

Then, with the separation $(S_1, S_2) = (\{1\}, \{2, 3, 4\})$ of $\langle 4 \rangle$ and $y = (1, 2, 3, 4)^t$, we see from (3.32) that

$$r_1^{S_1}(y, A) = 0; \quad r_1^{S_2}(y, A) = 0.9,$$

$$r_2^{S_1}(y, A) = r_3^{S_1}(y, A) = r_4^{S_1}(y, A) = 1.1,$$

$$r_2^{S_2}(y, A) = 0.7, \quad r_3^{S_2}(y, A) = 1.8 \quad \text{and} \quad r_4^{S_2}(y, A) = 2.8.$$  

Thus conditions (1) and (2) of Theorem 3.2 are satisfied. Hence $A \in \text{SGD}$.

**4. Invertibility of cyclically diagonally dominant matrices**

In this section, we establish a sufficient condition for a cyclically diagonally dominant matrix to be invertible. As stated in Section 3, we denote the vertices in the directed graph $\mathcal{G}(A)$ of $A \in \mathcal{M}_n$ by $v_1, \ldots, v_n$.

**Definition 4.1.** Let $A \in \mathcal{M}_n$, $n \geq 2$. If $v_i$ is a vertex in $\mathcal{G}(A)$, we define $\mathcal{G}_\text{OUT}(v_i)$ to be the set of all vertices different from $v_i$ that can be reached from $v_i$ by a directed arc. Let $\lambda \in \sigma(A)$ and $x = (x_1, \ldots, x_n)^t$ be an eigenvector of $A$ corresponding to $\lambda$, and let $i \in \langle n \rangle$. We say that a vertex $v_k$ is a maximal vertex in $\mathcal{G}_\text{OUT}(v_i)$ with respect to the pair $(\lambda, x)$ if $v_k \in \mathcal{G}_\text{OUT}(v_i)$ and for every $v_l \in \mathcal{G}_\text{OUT}(v_i)$ we have $|x_k| \geq |x_l|$. A cycle $\Gamma = v_{i_1} v_{i_2}, \ldots, v_{i_p} v_{i_{p+1}}$ in $\mathcal{G}(A)$ is said to satisfy the maximal vertex property with respect to
the pair $(\lambda, x)$ if for every $j \in \langle p \rangle$, $x_{ij} \neq 0$ and the vertex $v_{ij+1}$ is a maximal vertex in $\mathcal{G}_{\text{out}}(v_j)$ with respect to the pair $(\lambda, x)$. The set of all cycles that satisfy the maximal vertex property with respect to the pair $(\lambda, x)$ is denoted by $\mathcal{C}_{\lambda, x}(A)$. Define $\mathcal{K}_{\lambda, x}(A)$ by

$$\mathcal{K}_{\lambda, x}(A) = \{ v_1 \in \mathcal{G}(A) : r_i(A) > 0 \text{ and } |x_k| = |x_l| > 0 \text{ for all } k, l \text{ such that } v_k, v_l \in \mathcal{G}_{\text{out}}(v_i) \}.$$  \hspace{1cm} (4.1)

**Lemma 4.1.** Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$, be singular. Suppose that $x = (x_1, \ldots, x_n)^t$ is an eigenvector of $A$ corresponding to the eigenvalue 0. Assume that $\Gamma = \{v_i, v_{i+1}, \ldots, v_{p+1} \}$ is a cycle in $\mathcal{G}(A)$. Then:

(a) If $\{v_{i_1}, \ldots, v_{p+1} \} \subset \mathcal{K}_{0,x}(A)$, then $\Gamma \in \mathcal{C}_{0,x}(A)$.

(b) If $A$ is cyclically diagonally dominant, $a_{jj} \neq 0$ for all $j \in \langle n \rangle$ and $\Gamma \in \mathcal{C}_{0,x}(A)$, then $\Pi^p_{j=1} |a_{jj}| = \Pi^p_{j=1} r_j(A)$ and $\{v_{i_1}, \ldots, v_{p+1} \} \subset \mathcal{K}_{0,x}(A)$, where $\mathcal{K}_{0,x}(A)$ is defined by (4.1) with $\lambda = 0$.

**Proof.** (a) Let $\{v_{i_1}, \ldots, v_{p+1} \} \subset \mathcal{K}_{0,x}(A)$ and $j \in \langle p \rangle$). Since $\Gamma$ is a cycle, we have $v_{j+1} \in \mathcal{G}_{\text{out}}(v_j)$ if $j > 1$, and $v_{i_1} \in \mathcal{G}_{\text{out}}(v_{p+1})$. Then from $\{v_{i_1}, \ldots, v_{p+1} \} \subset \mathcal{K}_{0,x}(A)$, we see that

$$|x_j| > 0 \text{ and } |x_{j+1}| = |x_k| \text{ for every vertex } v_k \in \mathcal{G}_{\text{out}}(v_j)$$

for all $j \in \langle p \rangle$. Thus $\Gamma \in \mathcal{C}_{0,x}(A)$.

(b) Suppose that $A$ is cyclically diagonally dominant, $a_{jj} \neq 0$ for all $j \in \langle n \rangle$ and $\Gamma \in \mathcal{C}_{0,x}(A)$. Then from $Ax = 0$ and $v_{j+1}$ being a maximal vertex in $\mathcal{G}_{\text{out}}(v_j)$ for all $j \in \langle p \rangle$, we get

$$|a_{jj}| |x_j| \leq \sum_{m=1, m \neq j}^n |a_{jm}| |x_m| \leq |x_{j+1}| \sum_{m=1, m \neq j}^n |a_{jm}|$$ \hspace{1cm} (4.2)

for all $j \in \langle p \rangle$. From $\Gamma \in \mathcal{C}_{0,x}(A)$, we have $\Pi^p_{j=1} |x_j| = \Pi^p_{j=1} |x_{j+1}| > 0$. Thus from (4.2), $A$ being cyclically diagonally dominant and $a_{jj} \neq 0$ for every $j \in \langle n \rangle$, we deduce that $\Pi^p_{j=1} |a_{jj}| = \Pi^p_{j=1} r_j(A) > 0$. Hence from $a_{jj}, x_j \neq 0$ for every $j \in \langle p \rangle$, we infer that the inequalities in (4.2) are equalities. Then from $x_{j+1} \neq 0$ and $v_{j+1}$ being a maximal vertex in $\mathcal{G}_{\text{out}}(v_j)$ for all $j \in \langle p \rangle$, we see that $|x_m| = |x_{j+1}| > 0$ for all $j \in \langle p \rangle$ and $m \in \langle n \rangle$ that satisfy $v_m \in \mathcal{G}_{\text{out}}(v_j)$. Thus from $r_j(A) > 0$ for all $j \in \langle p \rangle$, we get $\{v_{i_1}, \ldots, v_{p+1} \} \subset \mathcal{K}_{0,x}(A)$. \hspace{1cm} \Box

The following proposition follows from Theorem 2.5 of [17].

**Proposition 4.1.** Let $A = (a_{jk}) \in \mathcal{M}_n$, $n \geq 2$, be a singular cyclically diagonally dominant matrix such that $a_{jj} \neq 0$ for all $j \in \langle n \rangle$. Suppose that $x = (x_1, \ldots, x_n)^t$ is an eigenvector of $A$ corresponding to the eigenvalue 0. If $x_i \neq 0$ for some $i \in \langle n \rangle$, then there exists a cycle $\Gamma \in \mathcal{C}_{0,x}(A)$ such that one of the following two conditions holds:

**Condition (1):** $v_i$ is a vertex in $\Gamma$.

**Condition (2):** There exist a vertex $v_m$ distinct from $v_i$ and a directed path $\Lambda : v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_s} \rightarrow v_{\alpha_{s-1}}$, $s \geq 2$, from $v_i$ to $v_m$ with the following properties: $v_m$ is a vertex in $\Gamma$, $\Lambda \cap \Gamma = \{v_m\}$ and for every $j \in \langle s-1 \rangle$, $v_{\alpha_{j+1}}$ is a maximal vertex in $\mathcal{G}_{\text{out}}(v_{\alpha_j})$ and $|x_{\alpha_{j+1}}| > 0$.

**Theorem 4.1.** Suppose that $A = (a_{jk}) \in \mathcal{M}_n$, $n \geq 2$, satisfies the following conditions:

**Condition (I):** $A$ is cyclically diagonally dominant matrix and $a_{jj} \neq 0$ for all $j \in \langle n \rangle$.

**Condition (II):** There exists a cycle $\Gamma_1$ in $\mathcal{C}(A)$ such that $\Pi_{v_j \in \Gamma_1} |a_{jj}| > \Pi_{v_j \in \Gamma_1} r_j(A)$.

**Condition (III):** If $\Gamma \in \mathcal{C}(A)$ satisfies $\Pi_{v_j \in \Gamma} |a_{jj}| = \Pi_{v_j \in \Gamma} r_j(A)$, then for every vertex $v_i \in \Gamma_1$ there exists a vertex $v_k \in \Gamma$ such that $a_{jk} \neq 0$.

Then $A$ is invertible.
Lemma 4.1 that

(b) of Lemma 4.1, we deduce that

\[
\frac{V}{\pi_1} \Gamma_1
\]

Since \( \Pi_{\gamma_1} \gamma_1 \) \( \gamma_1 \), we deduce from condition (I), the assumption that \( A \) is singular and Lemma 4.1 that there exists a vertex \( v_i \in \Gamma_1 \) such that

\[
v_i \notin K_{0,A}(x).
\]

Since \( v_i \in \Gamma_1 \), we see from \( \Pi_{\gamma_1} \gamma_1 \) \( \gamma_1 \) and condition (III) that there exists a vertex \( v_k \in \Gamma_0 \) such that \( a_{ik} \neq 0 \). Thus from \( v_k \in K_{0,A}(x) \) and (4.1), we infer that \( |x_i| > 0 \).

Hence from condition (I), the assumption that \( A \) is singular and Proposition 4.1, there exists a cycle \( \Gamma = v_{t_1} v_{t_2}, \ldots, v_{t_p} v_{t_{p+1}} \in C_{0,A}(x) \) such that \( v_i \) and \( \Gamma \) satisfy one of conditions (1) and (2) of Proposition 4.1. Since \( \Gamma \in C_{0,A}(x) \), we see from condition (I), the assumption that \( A \) is singular and term (b) of Lemma 4.1 that

\[
\Pi_{j=1}^p a_{t_j t_j} = \Pi_{j=1}^p r_j(A) \text{ and } \{v_{r_1, \ldots, v_{r_p}} \subset K_{0,A}(x).}
\]

So, from (4.4), we deduce that \( v_i \) and \( \Gamma \) cannot satisfy condition (1) of Proposition 4.1. Suppose that there exist a vertex \( v_m \) distinct from \( v_i \) and a directed path \( \Lambda : v_i \rightarrow v_m \) such that \( v_m \) is a vertex in \( \Gamma, \Lambda \cap \Gamma = \{v_m\} \) and for every directed arc \( v_{r_1} v_{r_{p+1}} \) in \( \Lambda, v_{r_j} v_{r_{j+1}} \) is a maximal vertex in \( G_{\text{out}}(v_{r_j}) \) and \( |x_{r_{j+1}}| > 0 \). It follows from \( \Pi_{j=1}^p a_{r_j r_j} = \Pi_{j=1}^p r_j(A) \) (in (4.5)), \( v_i \in \Gamma_1 \), and condition (III) that there exists \( g \in \{1, \ldots, p\} \) such that \( a_{r_g} \neq 0 \). Thus from \( v_{r_g} \in K_{0,A}(x) \) (in (4.5)) and (4.1), we see that

\[
|x_i| = |x_{r_{g+1}}| = |x_i| > 0 \text{ for all vertices } v_i \in G_{\text{out}}(v_{r_g}).
\]

Let \( m = t_j \), and let \( \hat{\Gamma} \) be the part of \( \Gamma \) defined as follows:

(i) If \( f \geq g \) or \( f = t_j \) then \( \hat{\Gamma} \) is the path of \( \Gamma \) from \( v_{t_j} \) to \( v_{t_g} \).

(ii) If \( f > g \) and \( t_j \neq t_g \) then \( \hat{\Gamma} : v_{t_1} v_{t_1}, \ldots, v_{t_p} v_{t_{p+1}}, \ldots, v_{t_g} v_{t_{g+1}} \). Let \( \Gamma_2 \) be the cycle consisting of \( \Lambda, \hat{\Gamma} \) and the directed arc \( v_{t_p} v_{t_1} \). Write \( \Gamma_2 \) as \( \gamma_2 = v_{t_1} v_{t_2}, \ldots, v_{t_p} v_{t_{g+1}} \). It follows from the construction of \( \Lambda, \Gamma \in C_{0,A}(x) \) and (4.6) that every vertex \( v_{t_{g+1}} \in \Gamma_2 \) is a maximal vertex in \( G_{\text{out}}(v_{t_g}) \) and \( x_{t_j} \neq 0 \) for all \( j \in (g) \). Then \( \Gamma_2 \in C_{0,A}(x) \). Thus from condition (I), the assumption that \( A \) is singular and term (b) of Lemma 4.1, we deduce that \( v_j \in K_{0,A}(x) \) for every vertex \( v_j \in \Gamma_2 \). In particular, \( v_i \in K_{0,A}(x) \). This contradicts (4.4).

Example 4.1. Let \( A = (a_{ij}) = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 8 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 5 & 1 \\ 1 & 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \). It is clear that \( A \) is reducible. The cycles in \( C(A) \) are

\[
\Gamma_1 = v_1 v_2, v_2 v_1 \text{ with } 16 = \Pi_{\gamma_1} |a_{ij}| > \Pi_{\gamma_1} r_1(A) = 9; \Gamma_2 = v_3 v_4, v_4 v_3 \text{ with } 15 = \Pi_{\gamma_2} |a_{ij}| = \Pi_{\gamma_2} r_1(A); \Gamma_3 = v_5 v_6, v_6 v_5 \text{ with } 16 = \Pi_{\gamma_3} |a_{ij}| > \Pi_{\gamma_3} r_1(A) = 2; \Gamma_4 = v_1 v_5, v_5 v_1 \text{ with } 16 = \Pi_{\gamma_4} |a_{ij}| > \Pi_{\gamma_4} r_1(A) = 6; \Gamma_5 = v_1 v_2, v_2 v_5, v_5 v_1 \text{ with } 128 = \Pi_{\gamma_5} |a_{ij}| > \Pi_{\gamma_5} r_1(A) = 18; \Gamma_6 = v_1 v_6, v_6 v_5, v_5 v_1 \text{ with } 32 = \Pi_{\gamma_6} |a_{ij}| > \Pi_{\gamma_6} r_1(A) = 6 \text{ and } \Gamma_7 = v_1 v_2, v_2 v_6, v_6 v_5, v_5 v_1 \text{ with } 256 = \Pi_{\gamma_7} |a_{ij}| > \Pi_{\gamma_7} r_1(A) = 18. \text{ Then from } a_{ij} \neq 0 \text{ for all } j \in \{6\}, \text{ we deduce that } A \text{ satisfies conditions (I) and (II) of Theorem 4.1. Since } \Gamma_2 = v_3 v_4, v_4 v_3 \text{ and }
\]

\[
\{\Gamma \in C(A) : \Pi_{\gamma_3} |a_{ij}| = \Pi_{\gamma_3} r_1(A) = \{\Gamma_2\}.
\]
we see from $\Gamma_1 = \mathbf{v}_1 \mathbf{v}_2, \mathbf{v}_2 \mathbf{v}_1, \prod_{i \in \Gamma_1} [a_{ii}] > \prod_{i \in \Gamma_1} r_i(A)$ and $a_{31} a_{32} \neq 0$ that $A$ also satisfies condition (III) of Theorem 4.1. So, $A$ is invertible.

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References