The Monotone Convergence of the Two-Stage Iterative Method for Solving Large Sparse Systems of Linear Equations

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Abstract—This paper sets up the monotone convergence theory for the two-stage iterative method proposed by Frommer and Szyld in [1], and investigates the influence of the splitting matrices and the inner iteration number sequence on the monotone convergence rate of this method.

Keywords—Linear system of equations, Two-stage iterative method, Monotone convergence, Monotone convergence rate.

1. INTRODUCTION

For the large sparse system of linear equations

\[ Ax = b, \quad A \in L(R^n) \text{ nonsingular}, \quad x, b \in R^n. \quad (1.1) \]

the two-stage iterative methods are really much more efficient for getting its numerical solution. Many researchers, e.g., Nichols [2], Wachspress [3], Golub and Overton [4], Lanzkron, Rose and Szyld [5] and so on, have extensively discussed the iteration formulas of this type of methods, and deeply studied the corresponding convergence theories in the sense of asymptotic convergence. In particular, Frommer and Szyld [1] have investigated the convergence for the stationary case of this class of methods in detail when the coefficient matrix of the linear system (1.1) is an H-matrix, and they have also proved that the convergence result still suits to the nonstationary case. The problems then arise if this class of methods has the monotone convergence property, and how the splitting matrices as well as the inner iteration number sequence will influence the monotone convergence rates of these methods. This paper is going to give definite answers to these two problems.

The structure of this paper is as follows. After reviewing the two-stage iterative method and some related results on its convergence established in [1] in Section two, we further set up the monotone convergence theorem for this method, and the corresponding comparison theorem on
its monotone convergence rate, which describes the influence of the splitting matrices and the inner iteration number sequence on its monotone convergence rate, in Section three. Furthermore, several concrete and practical applications about this new comparison theorem are also discussed in this section.

2. PRELIMINARIES

In the subsequent discussions, we will closely follow the notations and concepts presented in [1] without explanation. The two-stage iterative method discussed in [1] has the form

\[ x_{k+1} = (F^{-1}G)^{s(k)}x_k + \sum_{j=0}^{s(k)-1} (F^{-1}G)^j F^{-1}(Nx_k + b), \quad k = 0, 1, 2, \ldots, \]  

(2.1)

where \( A = M - N \) is a splitting of the matrix \( A \), which implies that \( M \) is a nonsingular matrix, \( M = F - G \) is a splitting of the matrix \( M \), and \( s(k) \geq 1 \) is the inner iteration numbers at the kth outer iteration step. Clearly, the iteration matrix corresponding to (2.1) is

\[ T_s(k) = (F^{-1}G)^{s(k)} + \sum_{j=0}^{s(k)-1} (F^{-1}G)^j F^{-1}N \]

\[ = I - \left( I - (F^{-1}G)^{s(k)} \right) M^{-1}A, \]  

(2.2)

where \( I \) denotes the identity matrix. If we let

\[ R_s(k) = \sum_{j=0}^{s(k)-1} (F^{-1}G)^j F^{-1} = \left( I - (F^{-1}G)^{s(k)} \right) M^{-1}, \]  

(2.3)

then it holds

\[ T_s(k) = I - R_s(k)A. \]  

(2.4)

For this method, Frommer and Szyld established the following convergence theorems in [1].

**Theorem 2.1.** Let \( A \in L(R^n) \) be a monotone matrix, \( A = M - N \) be a regular splitting, and \( M = F - G \) be a weak regular splitting. Then, the two-stage iterative method (2.1) is convergent for any sequence \( s(k) \geq 1(k = 0, 1, 2, \ldots) \) of inner iterations.

**Theorem 2.2.** Let \( A = M - N \) and \( M = F - G \) be splittings of the matrices \( A \) and \( M \), respectively, such that \( \langle M \rangle - \langle N \rangle \) is an \( M \)-matrix and \( \langle M \rangle = (F) - |G| \). Then, the two-stage iterative method (2.1) is convergent for any sequence \( s(k) \geq 1(k = 0, 1, 2, \ldots) \) of inner iterations.

From these two theorems we can easily know that for any sequence \( \{s(k)\}_{k=0}^{\infty} \) satisfying \( s(k) \geq 1(k = 0, 1, 2, \ldots) \), the matrices \( R_s(k)(k = 0, 1, 2, \ldots) \) defined in (2.3) are nonsingular matrices under the conditions of these theorems, respectively.

3. MAIN RESULTS

This section will emphasize on the establishments of the monotone convergence theory of the two-stage iterative method (2.1), and the comparison theorem about its convergence rate in the sense of monotonicity.

**Theorem 3.1 (Monotone Convergence Theorem).** Let \( A \in L(R^n) \) be a monotone matrix, \( A = M - N \) be a regular splitting, \( M = F - G \) be a weak regular splitting, and \( s(k) \geq 1(k = 0, 1, 2, \ldots) \) be the inner iteration sequence. Assume that the initial values \( x_0 \), \( y_0 \) are taken to obey

\[ Ax_0 \leq b \leq Ay_0. \]
Then, the sequences \( \{x_k\} \) and \( \{y_k\} \) generated by

\[
x_{k+1} = T_s(k)x_k + R_s(k)b,
\]

\[
y_{k+1} = T_s(k)y_k + R_s(k)b,
\]

satisfy that

(i) \( x_k \leq x_{k+1} \leq y_{k+1} \leq y_k, \ k = 0, 1, 2, \ldots \); and

(ii) \( \lim_{k \to \infty} x_k = A^{-1}b = \lim_{k \to \infty} y_k \).

**Proof.** From the conditions of the theorem and (2.2)–(2.3), we know that the matrix \( A \) is monotone and there hold \( T_s(k) \geq 0, R_s(k) \geq 0(k = 0, 1, 2, \ldots) \). Now, the conclusion (i) can be directly demonstrated by induction, and (ii) is just a simple corollary of (i).

**Theorem 3.2.** Under the conditions of Theorem 3.1, we additionally suppose that \( R_s^{-1}(k)T_s(k) \geq 0(k = 0, 1, 2, \ldots) \). Then there hold \( Ax_k \leq b \leq Ay_k(k = 0, 1, 2, \ldots) \).

**Proof.** From (2.1)–(2.4), we can obtain

\[
Ax_{k+1} - b = A(T_s(k)x_k + R_s(k)b) - b
\]

\[
= A(I - R_s(k)A)x_k + AR_s(k)b - b
\]

\[
= Ax_k - b - AR_s(k)(Ax_k - b)
\]

\[
= (I - AR_s(k))(Ax_k - b)
\]

\[
= R_s^{-1}(k)(I - R_s(k)A)R_s(k)(Ax_k - b)
\]

\[
= R_s^{-1}(k)T_s(k)R_s(k)(Ax_k - b).
\]

Now, the relations \( Ax_k \leq b(k = 0, 1, 2, \ldots) \) follow directly from induction. Similarly, we can demonstrate the relations \( Ay_k \geq b(k = 0, 1, 2, \ldots) \).

The assumption \( R_s^{-1}(k)T_s(k) \geq 0(k = 0, 1, 2, \ldots) \) in Theorem 3.2 means that the splittings \( A = R_s^{-1}(k) - R_s^{-1}(k)T_s(k)(k = 0, 1, 2, \ldots) \) are all regular splittings of the matrix \( A \). Note that these splittings are just weak regular splittings under the conditions of Theorem 3.1.

Based upon Theorem 3.1 and Theorem 3.2, we can further compare the monotone convergence rates of two two-stage iterative methods resulted from two different splittings \( M = F - G = \tilde{F} - \tilde{G} \) of the matrix \( M \) as well as two different inner iteration number sequences \( s(k) \geq 1(k = 0, 1, 2, \ldots) \) and \( t(k) \geq 1(k = 0, 1, 2, \ldots) \).

For this purpose, we construct matrices

\[
\tilde{T}_{t(k)} = \left(\tilde{F}^{-1}\tilde{G}\right)^{t(k)} + \sum_{j=0}^{t(k)-1} \left(\tilde{F}^{-1}\tilde{G}\right)^{j} \tilde{F}^{-1}N,
\]

\( k = 0, 1, 2, \ldots \) \hspace{1cm} (3.1)

\[
\tilde{R}_{t(k)} = \sum_{j=0}^{t(k)-1} \left(\tilde{F}^{-1}\tilde{G}\right)^{j} \tilde{F}^{-1},
\]

Analogously to (2.4), we can verify that it also holds

\[
\tilde{T}_{t(k)} = I - \tilde{R}_{t(k)}A, \quad k = 0, 1, 2, \ldots \hspace{1cm} (3.2)
\]

**Theorem 3.3 (Comparison Theorem).** Let \( A \in L(R^n) \) be a monotone matrix, \( A = M - N \) be a regular splitting of the matrix \( A \), \( M = F - G = \tilde{F} - \tilde{G} \) be weak regular splittings of the matrix \( M \), and \( s(k), t(k) \geq 1(k = 0, 1, 2, \ldots) \) be the inner iteration number sequences with \( s(k) \geq 1 \) and \( t(k) \geq 1(k = 0, 1, 2, \ldots) \). Then, if either \( R_s^{-1}(k)T_s(k) \geq 0(k = 0, 1, 2, \ldots) \) or \( \tilde{R}_{t(k)}^{-1}\tilde{T}_{t(k)} \geq 0(k = 0, 1, 2, \ldots) \) hold, we have

(i) \( x_k \leq \tilde{x}_k \leq y_k \leq \tilde{y}_k, \) for \( Ax_0 \leq b \);

(ii) \( x_k \geq \tilde{x}_k \geq y_k \geq \tilde{y}_k, \) for \( Ax_0 \geq b \).
provided the following conditions are satisfied:

(a) \( GF^{-1} \geq 0 \) (or \( G\tilde{F}^{-1} \geq 0 \));

(b) \( \tilde{F}^{-1} \geq F^{-1} \) and \( t(k) \geq s(k) \) \((k = 0, 1, 2, \ldots)\).

Where the iterative sequence \( \{x_k\} \) is defined by (2.1), the iterative sequence \( \{\tilde{x}_k\} \) is defined by

\[
\tilde{x}_{k+1} = \tilde{T}_{t(k)}\tilde{x}_k + \tilde{R}_{t(k)}b, \quad k = 0, 1, 2, \ldots, \tag{3.3}
\]

and both of them are started from \( \tilde{x}_0 = x_0 \).

**Proof.** We only verify the conclusion (i), while (ii) can be proved similarly.

From (2.1)-(2.4) and (3.1)-(3.3) we can get

\[
\tilde{x}_{k+1} - x_{k+1} = \left( \tilde{T}_{t(k)}\tilde{x}_k + \tilde{R}_{t(k)}b \right) - \left( T_{s(k)}x_k + R_{s(k)}b \right) = \left( \tilde{x}_k + \tilde{R}_{t(k)}(b - Ax_k) \right) - \left( x_k + R_{s(k)}(b - Ax_k) \right) = (\tilde{x}_k - x_k) + \left( \tilde{R}_{t(k)} - R_{s(k)} \right)(b - Ax_k) = (I - \tilde{R}_{t(k)}A)(\tilde{x}_k - x_k) + \left( \tilde{R}_{t(k)} - R_{s(k)} \right)(b - Ax_k).
\]

Now, in accordance with Theorem 3.1 and Theorem 3.2 we easily know that to prove the validity of the conclusion (i) we only need to verify the correctness of the matrix inequalities

\[
\tilde{R}_{t(k)} \geq R_{s(k)}, \quad k = 0, 1, 2, \ldots,
\]

Noticing the definitions of \( R_{s(k)} \) and \( \tilde{R}_{t(k)} \) \((k = 0, 1, 2, \ldots)\), and the assumption \( t(k) \geq s(k) \) \((k = 0, 1, 2, \ldots)\), we see that it is sufficient to test \( \tilde{R}_{s(k)} \geq R_{s(k)} \) \((k = 0, 1, 2, \ldots)\), or

\[
\left( (F^{-1}G)^k \right) M^{-1} \leq \left( (F^{-1}G)^k \right) M^{-1}, \quad k = 0, 1, 2, \ldots, \tag{3.4}
\]

In fact, for \( k = 0 \) the inequality (3.4) is trivial. Suppose that for \( j = 0, 1, 2, \ldots, k - 1 \) the inequality (3.4) holds. Because the relations

\[
\tilde{F}^{-1}GM^{-1} = \tilde{F}^{-1}(F^{-1}G)M^{-1} = M^{-1} - \tilde{F}^{-1}
\]

and

\[
M (F^{-1}G)^{k-1} M^{-1} = F (I - F^{-1}G) (F^{-1}G)^{k-1} (I - F^{-1}G)^{-1} F^{-1} = F (F^{-1}G)^{k-1} F^{-1} = (GF^{-1})^{k-1} \geq 0
\]

can be obtained, we have

\[
\left( (F^{-1}G)^k \right) M^{-1} \leq \left[ (F^{-1}G) \right] \left[ M (F^{-1}G)^{k-1} M^{-1} \right] = (M^{-1} - \tilde{F}^{-1}) (GF^{-1})^{k-1} \leq (M^{-1} - F^{-1}) (GF^{-1})^{k-1} = [(F^{-1}G)^{-1}] \left[ M (F^{-1}G)^{k-1} M^{-1} \right] = (F^{-1}G)^k M^{-1},
\]

that is, (3.4) is true for \( k \). By induction (3.4) holds for all \( k = 0, 1, 2, \ldots \).

Up to now, we have completed the proof of this theorem.
At last, we use a concrete application about the afore-established comparison theorem to end this section. For the convenience of our statements and without loss of generality, we stipulate from now on that $\text{diag}^+(M) = I$. Also, we will tacitly approve that the matrix $A \in L(R^n)$ is monotone, and the splitting $A = M - N$ of the matrix $A$ is an $M$-splitting, i.e., $M$ is an $M$-matrix and $N \geq 0$.

Let $L$ and $U$ be, respectively, the strictly lower and upper triangular matrices of $(-M)$. Considering the following splittings of the matrix $M$: \[ M = I - (L + U) = (I - L) - U. \]

It can be easily seen that $I - L \leq I$.

If we take $F$ and $G$ to be one of the following:

(i) $F = I$, $G = L + U$;
(ii) $F = I - L$, $G = U$,

then we can correspondingly obtain two special two-stage iterative methods with the inner iterations to be

(a) the Jacobi iteration (see (i));
(b) the Gauss-Seidel iteration (see (ii)), respectively.

Let all the inner iteration number sequences be $s(k)(k = 0, 1, 2, \ldots)$, and represent the sequences generated by these two-stage iterative methods as $\{x_k^{(\xi)}\}$ with $x_0^{(\xi)} = x_0$, where $\xi = J, GS$. By applying Theorem 3.3 we can directly obtain the following conclusions:

(a) when $Ax_0^{(\xi)} \leq b(\xi = J, GS)$, there holds $x_k^{(GS)} \geq x_k^{(J)} (k = 0, 1, 2, \ldots)$;
(b) when $Ax_0^{(\xi)} \geq b(\xi = J, GS)$, there holds $x_k^{(GS)} \leq x_k^{(J)} (k = 0, 1, 2, \ldots)$,

that is to say, the two-stage iterative method with the Gauss-Seidel iteration as inner iteration converges faster than that with the Jacobi iteration as inner iteration in the sense of monotonicity.

REFERENCES