Enhancement of Stability of Linear Parabolic Systems by Static Feedback

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Abstract—We study enhancement of stability or stabilization of a class of linear parabolic systems via static feedback. Static feedback scheme is most difficult when both actuators and observation weights admit spillovers. We propose a simple static feedback law of enhancing stability property or achieving stabilization. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In the last two decades, the study of feedback stabilization for parabolic systems has gathered much attention from both mathematical and practical viewpoints. Let $H$ be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\| \cdot \|_H$. The control system with state $u$ is a differential equation in $H$ described by

$$\frac{du}{dt} + Lu = \sum_{k=1}^{N} f_k(t) h_k, \quad t > 0, \quad u(0) = u_0.$$  \hspace{1cm} (1)

Here $f_k(t)$ denote inputs, $h_k$ actuators, and $L$ a closed operator with dense domain $\mathcal{D}(L)$ such that the resolvent $(\lambda - L)^{-1}$ satisfies the decay estimate

$$\| (\lambda - L)^{-1} \| \leq \frac{c}{1 + |\lambda|}, \quad \lambda \in \hat{\Sigma},$$

where $\hat{\Sigma}$ denotes some sector described by $\hat{\Sigma} = \{ \lambda - b; \theta_0 \leq |\arg \lambda| \leq \pi \}, \ 0 < \theta_0 < \pi/2, \ b \in \mathbb{R}^1.$

The output of the system is a finite number of observations with weights $w_k \in H$

$$\langle u, w_k \rangle_H, \quad 1 \leq k \leq N.$$  \hspace{1cm} (2)

Setting $f_k(t) = \langle u, w_k \rangle_H, \ 1 \leq k \leq N$, we have the closed-loop feedback control system

$$\frac{du}{dt} + Lu = \sum_{k=1}^{N} \langle u, w_k \rangle_H h_k, \quad t > 0, \quad u(0) = u_0.$$  \hspace{1cm} (3)
Let us briefly review the stabilization scheme. Given a \( \mu > 0 \), the problem is to construct \( w_k \)s and \( h_k \)s in order that

\[
\left\| \exp \left( -t \left( L - \sum_{k=1}^{N} \langle , w_k \rangle H h_k \right) \right) \right\|_{L(H)} \leq \text{const} e^{-\mu t}, \quad t \geq 0.
\]

Now assume that \( \sigma(L) \cap \{ \lambda \in \mathbb{C}; \Re \lambda < \mu \} \) consists only of the eigenvalues. The projection operator associated with these eigenvalues is denoted as \( P \) with \( \dim PH < \infty \). Then the problem is successfully solved, e.g., [1,2], if

(i) \( (L|_{PH}, \{ Ph_1, \ldots, Ph_N \}) \) is a controllable pair and \( w_k \)s are freely constructed in the subspace \( PH \); or as the dual assumption,

(ii) \( (L|_{PH}, \{ Pw_1, \ldots, Pw_N \}) \) is an observable pair and \( h_k \)s are freely constructed in \( PH \),

where \( L|_{PH} \) denotes the restriction of \( L \) onto the invariant subspace \( PH \). It is not plausible, however, that \( w_k \) or \( h_k \) are freely constructed in the finite-dimensional subspace \( PH \) in applications such as boundary observation and control. In fact, what we could manipulate is almost limited to the finite-dimensional parameters in \( PH \). Based on such a construction, the spillovers \( (1 - P)w_k \) or \( (1 - P)h_k \) are not negligible terms in analysis of stability. Another demerit is that the constant in the above estimate of the semigroup generally increases as \( \mu \) is chosen large.

When \( w_k \) and \( h_k \) satisfy, respectively, the above observability conditions and the controllability conditions, and admit spillovers, a new dynamic feedback scheme containing a finite-dimensional compensator in the feedback loop is introduced to achieve the stabilization [3-5]. This scheme contains more parameters that we can manipulate, and has been so far extensively studied and applied to practical problems of flexible structures. Although the static feedback scheme in (3) is simple, the stabilization problem remains unsolved when both \( w_k \) and \( h_k \) admit spillovers.

In view of these facts, the stabilization study is valuable in the presence of the spillovers of \( w_k \) and \( h_k \). We study in this paper, the static feedback scheme in (1), that is, \( f_k(t) \) being the feedback of \( (u, w_k) \) \( H \), \( 1 \leq k \leq N \), and generalize the result in [1,2] to some extent. More precisely, this means that we can enhance the stability property a little when \( h_k \) satisfy the controllability conditions.

The more precise assumption on the spectrum is stated as follows. The spectrum \( \sigma(L) \) consists of two disjoint closed sets \( \sigma_1 \) and \( \sigma_2 \): \( \sigma(L) = \sigma_1 \cup \sigma_2 \), and \( \sigma_1 \cap \sigma_2 = \emptyset \). Here,

(i) \( \sigma_1 \) consists only of the eigenvalues \( \lambda_i, 1 \leq i \leq n \) on the vertical line: \( \Re \lambda = \omega \);

(ii) for each \( \lambda_i, 1 \leq i \leq n \), there is a set of the eigenvectors \( \varphi_{ij}, 1 \leq j \leq m_i (< \infty) \) such that the set \( \{ \varphi_{ij} \}_{j=1}^{m_i} \) forms a basis for the subspace \( (2\pi\sqrt{-1})^{-1} \int_{C_i}(\lambda - L)^{-1}H d\lambda \), where \( C_i \) denotes a small contour encircling \( \lambda_i \);

(iii) \( \min_{\lambda \in \sigma_2} \Re \lambda > \omega \).

By setting \( f_k(t) = -\gamma (u, w_k) H \) in (1), our control system is, instead of (3), described as

\[
\frac{du}{dt} + Lu = -\gamma \sum_{k=1}^{N} (u, w_k) H h_k, \quad t > 0, \quad u(0) = u_0,
\]

where \( \gamma > 0 \) denotes a small parameter. When there is no control, the semigroup of the unperturbed equation satisfies the estimate

\[
\| e^{-tL} \|_{L(H)} \leq ce^{-\omega t}, \quad t \geq 0.
\]

Henceforth, \( c \) with or without subscript will denote a various positive constant. We show that the power \( \omega \) is improved a little for the perturbed equation (4) in the presence of the spillovers of \( w_k \) and \( h_k \).
2. MAIN RESULT

According to the assumptions on \( \sigma(L) \), let \( P \) denote the projection operator associated with \( \lambda_1, \ldots, \lambda_n \)

\[
P = \frac{1}{2\pi \sqrt{-1}} \int_{C_i} (\lambda - L)^{-1} \, d\lambda.
\]

Set \( L_1 = L|_{P\mathcal{H}} \) and \( L_2 = L|_{Q\mathcal{H}} \) with \( \mathcal{D}(L_2) = \mathcal{D}(L) \cap Q\mathcal{H} \), where \( Q = 1 - P \). By setting \( u_1 = Pu \) and \( u_2 = Qu \), (4) is decomposed into two equations

\[
\frac{du_1}{dt} + L_1 u_1 = -\gamma \sum_{k=1}^{N} \langle u_1, P w_k \rangle_H P h_k - \gamma \sum_{k=1}^{N} \langle u_2 Q w_k \rangle_H P h_k
\]

and

\[
\frac{du_2}{dt} + L_2 u_2 = -\gamma \sum_{k=1}^{N} \langle u_1, P w_k \rangle_H Q h_k - \gamma \sum_{k=1}^{N} \langle u_2 Q w_k \rangle_H Q h_k.
\]

For the basis \( \{ \varphi_{ij}; 1 \leq i \leq n, 1 \leq j \leq m_i \} \), \( u_1, h_k, \) and \( L_1 \) are equivalent to

\[
u = ^t(u_{11} \ldots u_{1m_1} u_{21} \ldots u_{nm_n}), \quad h_k = ^t(h_{i1}^k \ldots h_{i1m_1}^k h_{21}^k \ldots h_{nm_n}^k),
\]

and

\[
A = \text{diag} (A_1 A_2 \ldots A_n), \quad A_i = \text{diag}(\lambda_1 \lambda_2 \ldots \lambda_i),
\]

respectively, where \( ^t(\cdots) \) denotes the transpose of a vector. Then (5) is equivalent to the equation in \( \mathbb{C}^s, s = m_1 + \cdots m_n \)

\[
\frac{d\nu}{dt} + (A + H W) \nu = -\gamma \sum_{k=1}^{N} \langle u_2 Q w_k \rangle_H h_k, \quad \nu(0) = \nu_0, \quad (5')
\]

where

\[
H = (h_1 h_2 \ldots h_N); \quad S \times N, \quad \text{and} \quad W = \left( \langle \varphi_{ij} w_k \rangle_H; \quad \begin{array}{c} k \downarrow 1, \ldots, N \vspace{1mm} \quad (i, j) \rightarrow (1,1), \ldots, (n,m_n) \end{array} \right).
\]

Setting

\[
H_i = \left( h_{ij}^k; \quad k \downarrow 1, \ldots, N \vspace{1mm} \quad j \downarrow 1, \ldots, m_i \right), \quad \text{and} \quad W_i = \left( \langle \varphi_{ij} w_k \rangle_H; \quad k \downarrow 1, \ldots, N \vspace{1mm} \quad j \downarrow 1, \ldots, m_i \right),
\]

we see that

\[
H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix}, \quad W = (W_1 W_2 \ldots W_n).
\]

The matrices \( H_i \) and \( W_i \) are the so called controllability and observability matrices, respectively. One of our main results is stated as follows.

**Theorem 1.** Consider the simplest case where \( m_1 = m_2 = \cdots = m_n \) and set \( N = m_i \). Suppose that

\[
\text{rank} H_i = m_i, \quad 1 \leq i \leq n.
\]
Choose $w_k$s so that $W_i = H_i^{-1}$, $1 \leq i \leq n$. Then, if $\gamma > 0$ is small enough, there is an $O(\gamma^2)$, such that

$$\left\| \exp \left( -t \left( L + \gamma \sum_{k=1}^{N} \langle \cdot, w_k \rangle_H h_k \right) \right) \right\|_{\mathcal{L}(H)} \leq c e^{-\left(\omega + \gamma + O(\gamma^2)\right)t}, \quad t \geq 0. \quad (9)$$

**Remark 1.** The essential difference between our result and the preceding ones lies in the construction of $w_k$ and $h_k$: the only requisite is that $w_k$ satisfy the finite-dimensional conditions: $W_i = H_i^{-1}$. The resultant spillovers $Qw_k$ and $Qh_k$ are the quantities that we cannot manipulate: consequently, they cannot generally remain in $PH$.

**Remark 2.** Theorem 1 is easily applied to a class of boundary control systems with no essential change. Problems caused by unboundedness on the boundary are merely of technical nature, and these difficulties are easily handled via standard arguments.

**Sketch of the proof.** Equations (5') and (6) are rewritten as integral equations by

$$u(t) = e^{-t(A+\gamma HW)}u(0) - \gamma \int_{0}^{t} e^{-(t-s)(A+\gamma HW)} \sum_{k=1}^{N} \langle u_2(s), Qw_k \rangle_H h_k ds,$$

and

$$u_2(t) = e^{-tF}u_2(0) - \gamma \int_{0}^{t} e^{-(t-s)F} (Qh_1 \ldots Qh_N)Wu(s) ds,$$

respectively, where

$$F = L_2 + \gamma \sum_{k=1}^{N} \langle \cdot, Qw_k \rangle_H Qh_k, \quad D(F) = D(L_2).$$

Combining these equations, we will derive an integral inequality for $|u(t)|$. Note that

$$\|e^{-tL_2}\|_{\mathcal{L}(H)} \leq M_1 e^{-\beta t}, \quad t \geq 0,$$

where $\min_{\lambda \in \sigma_2} \Re \lambda > \beta > \omega$. Thus, it is easily seen via the standard perturbation argument that

$$\|e^{-tL_2}\|_{\mathcal{L}(H)} \leq M_1 e^{-(\beta - M_1 c_1 \gamma)t}, \quad t \geq 0, \quad c_1 = \sum_{k=1}^{N} \|Qw_k\| \|Qh_k\| \quad (10).$$

The eigenvalues of $A + \gamma HW$ are nonlinear functions of $\gamma$. According to the choice of $w_k$, we have the following proposition which forms the key.

**Proposition 2.** There exist a constant $M > 0$ and $O(\gamma^2)$ such that

$$e^{-t(A+\gamma HW)} \leq M e^{-(\omega + \gamma + O(\gamma^2))t}, \quad t \geq 0, \quad (11)$$

where $M$ is independent of $\gamma$.

**Proof.** The idea is to obtain the concrete expression of $(\lambda - A - \gamma HW)^{-1}$. In calculating

$$e^{-t(A+\gamma HW)} = \frac{1}{2\pi \sqrt{-1}} \int e^{-t\lambda} (\lambda - A - \gamma HW)^{-1} d\lambda,$$
we need to estimate the residue of the integrand at each singularity. Setting $A_{ij} = H_i H_j^{-1}$, $1 \leq i, j \leq n$, we calculate as

$$(\lambda - A - \gamma HW)^{-1} = \left( \begin{array}{cccc}
I_N & 0 & \cdots & 0 \\
0 & A_{12} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{1n}
\end{array} \right)^{-1}$$

Each element of the second matrix of the above right-hand side is a rational function of $\lambda$ with the denominator $d_n$

$$d_n = \begin{vmatrix}
\lambda - \lambda_1 - \gamma & -\gamma & \cdots & -\gamma \\
-\gamma & \lambda - \lambda_2 - \gamma & \cdots & -\gamma \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma & -\gamma & \cdots & \lambda - \lambda_n - \gamma
\end{vmatrix}$$

Thus, each singularity is a simple pole as long as $\gamma > 0$ is small. Let $\lambda_i(\gamma)$, $1 \leq i \leq n$ be the solutions to the equation: $d_n = 0$, where $\lambda_i(\gamma) \to \lambda_i$ as $\gamma \to 0$. Differentiating the both sides of $d_n = 0$ with respect to $\gamma$ and setting $\gamma = 0$, we see that

$$\frac{d}{d\gamma} \lambda_i(\gamma) \Bigg|_{\gamma=0} = 1, \quad 1 \leq i \leq n.$$ 

Then, the assertion of the proposition immediately follows.

Set $\alpha(\gamma) = \gamma + O(\gamma^2)$. When $\gamma$ is small, we may assume that

$$\omega + \alpha(\gamma) < \beta - M_1 c_1 \gamma.$$ 

Based on estimates (10), (11), and the integral equations of $u$ and $u_2$, we can derive

$$|u(t)| \leq M e^{-(\omega + \alpha(\gamma))t} |u(0)| + \frac{M M_1 c_2 c_3}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} e^{-(\omega + \alpha(\gamma))t} \|u_2(0)\|$$

$$+ M M_1 c_2 c_3 \gamma^2 \int_0^t K(t - \sigma) |u(\sigma)| d\sigma,$$

where

$$c_2 = \sum_{k=1}^N \|P w_k\| \|Q h_k\|, \quad c_3 = \sum_{k=1}^N \|Q w_k\| \|P h_k\|,$$

and

$$K(t) = \int_0^t e^{-(\omega + \alpha(\gamma))(t - r)} e^{-(\beta - M_1 c_1 \gamma - \alpha(\gamma)) r} \frac{e^{-(\omega + \alpha(\gamma))t}}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} dt, \quad t \geq 0.$$ 

Thus, estimate (13) is rewritten as

$$|u(t)| \leq M_2 e^{-(\omega + \alpha(\gamma))t} \|u_0\| + \frac{M M_1 c_2 c_3}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} \gamma^2 \int_0^t e^{-(\omega + \alpha(\gamma))(t - \sigma)} |u(\sigma)| d\sigma.$$ 

Gronwall’s inequality implies that

$$|u(t)| \leq M_2 \|u_0\| \exp \left( - \left( \omega + \gamma + O(\gamma^2) - \frac{M M_1 c_2 c_3}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} \gamma^2 \right) t \right), \quad t \geq 0.$$ 

This leads to a similar estimate for $\|u_2(t)\|$. Thus, we have proven the desired estimate (9).
3. GENERALIZATION

In Theorem 1, we have assumed that the multiplicities $m_i$ are the same. We consider in this section, the general case where they are labelled as

$$ m_1 \geq m_2 \geq \cdots \geq m_n. \quad (15) $$

In order to generalize Theorem 1 under (15), the key is to obtain an estimate similar to (11).

**Theorem 3.** Take $N = m_1$. In (7), choose $w_k$ and $h_k$ such that

$$ H_i = (H_{i1}0), \quad H_{i1}; m_i \times m_i, $$

$$ \operatorname{rank} H_i = m_i, \quad \text{and} \quad W_i = \begin{pmatrix} H_{i1}^{-1} \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n. \quad (16) $$

Then, the assertion of Theorem 1 is correct.

As to Proposition 2, each element of $(\lambda - \Lambda - \gamma HW)^{-1}$ is shown to be a rational function of $\lambda$ with the denominator which is one of the following $d_1, \ldots, d_n$:

$$ d_i = \begin{vmatrix} \lambda - \lambda_1 - \gamma & -\gamma & \cdots & -\gamma \\ -\gamma & \lambda - \lambda_2 - \gamma & \cdots & -\gamma \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & \lambda - \lambda_i - \gamma \end{vmatrix}, \quad 1 \leq i \leq n. \quad (17) $$

A further generalization is possible. For positive integers $i$ and $j$ with $2 \leq i < j \leq n$ and $\lambda \in \mathbb{C}$, set

$$ \Xi_{i(i+1) \ldots j}(\lambda) = (\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_j) + \cdots + (\lambda - \lambda_i) \cdots (\lambda - \lambda_{j-1}) $$

$$ = \prod_{k=i}^{j} \left( \lambda - \lambda_k \right) \cdot \left( \frac{1}{\lambda - \lambda_i} + \cdots + \frac{1}{\lambda - \lambda_j} \right). \quad (18) $$

Here it is assumed that $n \geq 3$.

**Theorem 4.** Take $N = m_1$, and assume that

$$ H_i = (H_{i1} H_{i2}), \quad H_{i1}; m_i \times m_i, $$

$$ \det H_{i1} \neq 0, \quad \text{and} \quad W_i = \begin{pmatrix} H_{i1}^{-1} \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n. \quad (19) $$

Assume finally that

$$ \Xi_{i(i+1) \ldots j}(\lambda_h) \neq 0, \quad 1 \leq h < i < j \leq n, \quad 1 \leq i \leq n. \quad (20) $$

Then the assertion of Theorem 1 is correct.

Combination of $h$, $i$, and $j$ in (20) can be easily recognized by the following table, where $A_i = \lambda - \lambda_i$. 

| $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $\cdots$ | $A_{n-1}$ | $A_n$ |
|-------|-------|-------|-------|-------|-----------|-----------|
| $A_3$ | $A_4$ | $A_5$ | $A_6$ | $\cdots$ | $A_{n-1}$ | $A_n$ |
| $A_4$ | $A_5$ | $A_6$ | $\cdots$ | $A_{n-1}$ | $A_n$ |
| $A_5$ | $A_6$ | $\cdots$ | $A_{n-1}$ | $A_n$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_{n-2}$ | $A_{n-1}$ | $A_n$ |
| $A_{n-1}$ | $A_n$ |
When \( n = 3 \) and \( n = 4 \), for example, (20) means

\[ \lambda_1 \neq \frac{\lambda_2 + \lambda_3}{2} \]

and

\[ \lambda_1 \neq \frac{\lambda_2 + \lambda_3}{2}, \quad \lambda_1 \neq \frac{\lambda_3 + \lambda_4}{2}, \quad \lambda_2 \neq \frac{\lambda_3 + \lambda_4}{2}, \]

\[ A_3(\lambda_1)A_4(\lambda_1) + A_2(\lambda_1)A_4(\lambda_1) + A_2(\lambda_1)A_3(\lambda_1) = (\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4) + (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_4) + (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \neq 0, \]

respectively. In this case, the \((i,j)^{th}\) element of \((\lambda - A - \gamma HW)^{-1}\) is a rational function of \( \lambda \) with the denominator of the form

\[ \prod_{k=p_{i,j}}^{q_{i,j}} d_k, \quad \text{where} \ 1 \leq p_{i,j} \leq q_{i,j} \leq n, \]

and the analysis is more difficult.

The detailed proofs of Theorems 3 and 4 including the estimates of type (11) will appear elsewhere.

REFERENCES