Journal of Computational and Applied Mathematics 37 (1991) 57–69
North-Holland

On the use of a corresponding sequence algorithm for δ-fractions

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Received 15 July 1990
Revised 21 November 1990

Abstract

Baltus, C., S.C. Cooper, C. Craviotto and J.H. McCabe, On the use of a corresponding sequence algorithm for δ-fractions, Journal of Computational and Applied Mathematics 37 (1991) 57–69.

This paper discusses an algorithm for generating a new type of continued fraction, a δ-fraction, from a given power series. The δ-fraction corresponds to the given power series at $z = 0$. Included are convergence results and truncation error bounds.

Keywords: Continued fractions, algorithms, error bounds.

1. Introduction

In recent years there has been a revitalization of the analytic theory of continued fractions. With the aid of high-speed digital computers we are now in a position to take advantage of the algorithmic character of continued fractions. Continued fractions and the closely related Padé approximants are being applied to problems in theoretical physics, chemistry and engineering.

* This author's investigation was supported in part by funds provided by Washington State University.

There are several reasons that continued fractions are of such importance. One is that their approximants may converge in larger regions in the complex plane than the corresponding power series, which may not converge at all. Typically, the approximants are rational functions and as such often provide easily accessible information about zero and poles. Finally, the convergence is apt to be fast.

The purpose of this article is to discuss an algorithm for generating a new type of continued fraction, a δ-fraction, from a given power series. Before turning to the main discussion, we remind the reader of some basic facts and definitions.

Most of the analytic theory of continued fractions is based in the complex plane. A continued fraction is an ordered pair

$$\langle\langle\{a_n\}, \{b_n\}\rangle, \{f_n\}\rangle,$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of complex-valued functions, $a_n \not\equiv 0$ for $n \geqslant 1$, and $\{f_n\}$ is a sequence of complex-valued functions whose values may include the point at ∞. The sequence $\{f_n\}$ is defined as follows. Let

$$s_n(w) := \frac{a_n}{b_n + w}, \quad n \geqslant 1, \qquad s_0(w) := b_0 + w.$$

A second sequence $\{S_n\}$ is defined inductively by

$$S_0(w) := s_0(w), \qquad S_n(w) := S_{n-1}(s_n(w)), \quad n \geqslant 1.$$

The function f_n is then obtained by setting $f_n = S_n(0)$. The functions a_n and b_n are called the elements of the continued fraction or the partial numerators and partial denominators, respectively, and f_n is called the nth approximant. The continued fraction can be written in the more intuitive form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n + \ddots}}}}$$

which is usually abbreviated to

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots \tag{1.1}$$

or

$$b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}.$$

The continued fraction can also be defined in terms of the functions A_n and B_n, which satisfy the initial conditions

$$A_{-1} = 1, \qquad A_0 = b_0, \qquad B_{-1} = 0, \qquad B_0 = 1, \tag{1.2}$$

and the three-term recurrence relations

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \quad B_n = b_n B_{n-1} + a_n B_{n-2}, \qquad n \geqslant 1. \tag{1.3}$$

For the nth approximant of the continued fraction one then has

$$f_n = \frac{A_n}{B_n}, \quad n \geqslant 0. \tag{1.4}$$

The functions A_n and B_n are often referred to as the nth numerator and nth denominator, respectively.

A continued fraction $b_0 + \mathrm{K}(a_n/b_n)$ is said to correspond at $z = 0$ to a fixed power series

$$c_0 + c_1 z + c_2 z^2 + \cdots,$$

if the approximants f_n are functions of z that satisfy

$$\sum_{i=0}^{\infty} c_i z^i - f_n(z) = \mathrm{O}(z^{\sigma(n)}),$$

where $\sigma(n) \to \infty$ as $n \to \infty$. (Here the notation $\mathrm{O}(z^{\sigma(n)})$ indicates a power series in increasing powers of z starting with a power of z not less than $\sigma(n)$.)

Several forms of continued fraction have been introduced and studied extensively. Three particular continued fractions with well-known correspondence properties are the following.

(1) C-fractions [8,11,20] are of the form

$$\frac{p_1 z^{\alpha_1}|}{|1} + \frac{p_2 z^{\alpha_2}|}{|1} + \cdots + \frac{p_n z^{\alpha_n}|}{|1} + \cdots, \tag{1.5}$$

where each α_n is a positive integer and each p_n is a nonzero complex constant. If $\alpha_n = 1$ for all n, the continued fraction is a *regular* C-fraction and if, in addition, $p_n > 0$ for all n, it is a Stieltjes fraction.

(2) P-fractions [8,12] are of the form

$$b_0(z) + \frac{1|}{|b_1(z)} + \frac{1|}{|b_2(z)} + \cdots + \frac{1|}{|b_n(z)} + \cdots, \tag{1.6}$$

where each $b_n(z)$ is a polynomial in $1/z$.

(3) General T-fractions [7,8,13,14,18,19] are of the form

$$\frac{F_1 z|}{|1 + G_1 z} + \frac{F_2 z|}{|1 + G_2 z} + \cdots + \frac{F_n z|}{|1 + G_n z} + \cdots, \tag{1.7}$$

where each F_n is a nonzero complex constant and each G_n is a complex constant.

Of the other frequently studied continued fractions, some are obtained by contracting or expanding the above or by equivalence transformations. Other ones arise in connection with moment problems. Further ones are described in connection with their links with the Padé table of the given power series [2,3], and this approach can be extended to derive interpolating continued fractions.

The possible nonexistence of a regular C-fraction for a given power series, the failure of P-fractions to have "simple" elements and the failure of general T-fractions to terminate for rational functions are examples of the perceived deficiencies in the types of existing classes of continued fractions that inspired Lange to introduce a new class.

In his seminal paper on δ-fractions [10], Lange asked the following question. Is there a class of continued fraction that has the following properties?

(a) The elements a_n ands b_n are polynomials in z of degree at most one.

(b) Any regular C-fraction is a member of the class.

(c) Given a power series, there exists a unique member of the class that corresponds to it.

(d) If the series is the expansion about $z=0$ of a rational function, then the continued fraction terminates.

As Lange remarks, regular C-fractions as a class do not satisfy (c), C-fractions fail to meet (a), P-fractions fail to meet conditions (a) and (b), while general T-fractions fail to meet conditions (c) and (d). In introducing δ-fractions, Lange provides a class of continued fractions for which all four conditions are met.

A δ-fraction is a finite or infinite continued fraction of the form

$$b_0 - \delta_0 z + \frac{d_1 z}{1-\delta_1 z} + \frac{d_2 z}{1-\delta_2 z} + \cdots + \frac{d_n z}{1-\delta_n z} + \cdots, \tag{1.8}$$

where b_0 and d_n are complex constants and the δ_n are real constants whose values are either zero or one. The δ-fraction is said to be *regular* if $d_{k+1}=1$ whenever $\delta_k=1$. The above conditions (a) and (c) are met. The regular C-fractions occur when $\delta_k=0$ for all k. In [10] Lange proves that for every formal power series (fps) there exists a uniquely determined regular δ-fraction corresponding to the series (providing one chooses the terminating form whenever possible), and that for any finite or infinite δ-fraction there is a uniquely determined corresponding power series. He also proves that a power series is the Maclaurin series of a rational function

$$R(z)=\frac{\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n}{1+\beta_1 z+\beta_2 z^2+\cdots+\beta_m z^m}$$

if and only if there exists a finite regular δ-fraction

$$b_0-\delta_0 z+\frac{d_1 z}{1-\delta_1 z}+\frac{d_2 z}{1-\delta_2 z}+\cdots+\frac{d_{n-1} z}{1-\delta_{n-1} z}+\frac{d_n z}{1}$$

corresponding to it. Several convergence theorems are provided by Lange for functions that are analytic at the origin. These are then applied to many examples of δ-fraction expansions of classical analytic functions. An example of such an expansion is that for Dawson's integral

$$F(z)=\mathrm{e}^{-z^2}\int_0^z \mathrm{e}^{t^2}\,\mathrm{d}t.$$

Namely,

$$F(z)=\frac{z}{1-z}+\frac{z}{1}-\frac{\frac{2}{3}z}{1}+\frac{\frac{2}{3}z}{1}+\frac{(\frac{4}{2})(\frac{1}{5}z)}{1}-\frac{(\frac{4}{2})(\frac{1}{5}z)}{1}+\cdots,$$

and, in general,

$$d_{4n-1}=-d_{4n}=\frac{(-1)^n n\binom{2n}{n}}{(4n-1)4^{n-1}},\qquad d_{4n+2}=-d_{4n+1}=\frac{(-1)^n 4^n}{(4n+1)\binom{2n}{n}}.$$

This expansion was obtained *not* from the power series expansion of $F(z)$ to which the fraction corresponds, but by extending the well-known continued fraction

$$F(z)=\frac{z|}{|1}+\frac{\frac{2}{3}z^2|}{|1}-\frac{\frac{4}{3\times5}z^2|}{|1}+\frac{\frac{6}{5\times7}z^2|}{|1}-\frac{\frac{8}{7\times9}z^2|}{|1}+\cdots.$$

The δ-fraction expansion for each classical function considered by Lange is obtained in the same way, by extending an existing continued fraction, using the following procedure which can be found in [8,18,20]. Let A_n/B_n, $n=1, 2, 3,\ldots$, be the nth approximant of the continued fraction

$$\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\cdots+\frac{a_n|}{|b_n}+\cdots.$$

If the section

$$\frac{a_k|}{|b_k}+\frac{a_{k+1}|}{|b_{k+1}}$$

is replaced with

$$\frac{a_k|}{|b_k-\rho}+\frac{\rho|}{|1}-\frac{a_{k+1}/\rho|}{|b_{k+1}+a_{k+1}/\rho},$$

then the approximants of the extended fractions are

$$\frac{A_1}{B_1},\ldots,\frac{A_{k-1}}{B_{k-1}},\frac{A_k-\rho A_{k-1}}{B_k-\rho B_{k-1}},\frac{A_k}{B_k},\ldots.$$

In many cases this procedure can be applied repeatedly to extend a given continued fraction to the form of a δ-fraction. However, as the following example shows, this procedure is not always sufficient. The regular δ-fraction and C-fraction expansions, respectively, for the function $x^3+x^4+x^5$ are

$$-x+\frac{x|}{|1-x}+\frac{x|}{|1}+\frac{x|}{|1}-\frac{2x|}{|1}+\frac{\frac{1}{2}x|}{|1}-\frac{\frac{3}{2}x|}{|1}+\frac{\frac{4}{3}x|}{|1}+\frac{\frac{1}{6}x|}{|1}-\frac{\frac{3}{2}x|}{|1}+\frac{2x|}{|1}$$

and

$$\frac{x^3|}{|1}-\frac{x|}{|1}+\frac{x^2|}{|1}+\frac{x|}{|1}.$$

None of the approximants of the two expansions agree except for the final ones, which equal the original function. Thus, it is impossible to obtain the δ-fraction from the C-fraction from extensions.

In addition to the above extension technique used by Lange, the only other method of deriving δ-fractions that has been given is for those functions satisfying Riccati equations [5]. The method is analogous to those used for obtaining C-fraction solutions [4,15] and general T-fraction solutions [6] and is independent of the corresponding power series. Here we describe an algorithm for obtaining a δ-fraction expansion directly from the power series to which it corresponds. Many algorithms for transforming power series into continued fractions are available if the approximants, or some sequence of them, form a path in the Padé table for the series [2,3]. In some cases, as Lange states, at least some of the approximants of the δ-fraction will be in the Padé table, but there are examples for which at most one of the approximants appears. Thus, methods with close connections to Padé tables do not appear promising. The following very general algorithm, orginally developed by Viscovatoff in 1803–1806 [9], can be used to derive continued fractions whose elements have the form $a_n(z) = p_n z^{\nu_n}$, where p_n is a constant and ν_n is a positive integer and $b_n(z)$ is a polynomial in z. Thus, C-fractions, general T-fractions, and P-fractions (after applying equivalence transformations) can be derived by this algorithm. The details of these applications are given in [16,17]. One major advantage of the algorithm is that, unlike other algorithms (e.g., the qd-algorithm), it will only fail if the particular continued fraction that is sought does not exist. This is never the case with δ-fractions. In Section 2 the general algorithm is presented. Section 3 is devoted to the application of the corresponding sequence algorithm to δ-fractions. For the sake of completeness, some convergence results are summarized and relevant truncation error bounds are given in Section 4.

2. The corresponding sequence algorithm

This algorithm originated in work done by Viscovatoff between 1803 and 1806. In its primitive form, it was a clever way of producing regular C-fractions from power series without having to invert the series [9]. In papers [16,17], Murphy and O'Donohoe generalized the algorithm to apply to all continued fractions of the form

$$\frac{p_1 z}{|q_1(z)} + \frac{p_2 z^{\nu_1}|}{|q_2(z)} + \cdots + \frac{p_n z^{\nu_{n-1}}|}{|q_n(z)} + \cdots, \tag{2.1}$$

where p_n is a constant, ν_n is a positive integer and $q_n(z)$ is a polynomial of degree μ_n for each $n \in \mathbb{N}$. They make the further stipulation that $q_n(0) = 1$ for all n. Let $g_0(z)$ be the (possibly divergent) power series

$$g_0(z) = a_0 + a_1 z + a_2 z^2 + \cdots. \tag{2.2}$$

The algorithm is a method for generating the continued fraction (2.1), which has the property that the nth approximant satisfies the following correspondence criterion:

$$g_0(z) - \frac{A_n(z)}{B_n(z)} = \mathrm{O}(z^{\sigma(n)}),$$

where $\sigma(n) = \sum_{i=1}^{n} \nu_i$. There may be coefficients in the partial denominators $q_n(z)$ that are undetermined. In such a case, the number of undetermined coefficients is $\lambda_n = \mu_n - \nu_n + 1$. They can be chosen arbitrarily, but of course once chosen they will affect subsequent partial numerators and denominators.

If we set $g_{-1}(z) = 1$ and $\nu_0 = 0$, then, provided we can determine the coefficients and constants involved, the set of recurrence relations

$$g_n(z) = z^{\nu_{n-1}} p_n g_{n-2}(z) - q_n(z) g_{n-1}(z), \tag{2.3}$$

for $n = 1, 2, 3, \ldots$, determine the continued fraction (2.1). Consider $\{g_n(z)\}$ as a sequence of power series of the form

$$g_n(z) = z^{\sigma(n)}\{a_{n,0} + a_{n,1} z + \cdots + a_{n,k} z^k + \cdots\}, \tag{2.4}$$

for $n = 0, 1, 2, \ldots$, where $a_{0,k} = a_k$ for $k = 0, 1, \ldots$. Substituting the appropriate expressions from (2.4) into (2.3) and equating coefficients of like powers of z, we have

$$a_{k,r} = E^{\nu_k}\{p_k a_{k-2,r} - q_k(E^{-1}) a_{k-1,r}\}, \tag{2.5}$$

where E^m is the shift operator defined by

$$E^m a_{k,r} = a_{k,r+m}. \tag{2.6}$$

The relation (2.5) must hold for all $k \geqslant 1$ and for $r \geqslant -\nu_k$. For $r < 0$, $a_{k,r} = 0$. The equations for $r = -\nu_n, -\nu_n + 1, \ldots, -2, -1$ are then used to determine p_n and the coefficients in $q_n(z)$. The undetermined coefficients arise in the case where $\mu_n > \nu_n - 1$. The algorithm will fail only if $a_{k,0}$ is zero for some k. As Murphy and O'Donohoe proved,

$$a_{k,0} = p_1 \times p_2 \times \cdots \times p_{k+1},$$

and one of the p_k's is zero only if the fraction does not exist or, in certain cases, is terminating and represents a rational function.

In the next section, we modify this algorithm slightly and adapt it to the problem of finding a regular δ-fraction that corresponds to a given power series.

3. The corresponding sequence algorithm for regular δ-fractions

Let $g_0(z)$ be the (possibly divergent) power series

$$g_0(z) = a_0 + a_1 z + a_2 z^2 + \cdots.$$

The following algorithm will generate the regular δ-fraction that corresponds to $g_0(z)$. If $g_0(z)$ is the power series expansion of a rational function, we choose the terminating form of the corresponding regular δ-fraction.

Let the function $g_n(z)$ be a power series of the form

$$g_n(z) = z^n\left(a_{n,0} + a_{n,1}z + a_{n,2}z^2 + \cdots + a_{n,k}z^k + \cdots\right), \quad a_{n,0} \neq 0, \ n \geqslant 1, \tag{3.1}$$

where $a_{0,k} = a_k$ for $k = 0, 1, \ldots$. Define the recurrence relations for $\{g_n(z)\}$ as follows:

$$g_1(z) = g_0(z) - (b_0 - \delta_0 z), \qquad g_2(z) = d_1 z - (1 - \delta_1 z) g_1(z), \tag{3.2a}$$

$$g_n(z) = d_{n-1} z g_{n-2}(z) - (1 - \delta_{n-1} z) g_{n-1}(z), \quad n \geqslant 3. \tag{3.2b}$$

Provided we can determine b_0, δ_n and d_n for all n, the recurrence relations (3.2) will generate a δ-fraction.

By substituting the appropriate expressions for g_0 and g_1 from (3.1) into the first equation in (3.2a), and equating coefficients of like powers of z, we have

$$b_0 = a_{0,0}, \quad a_{1,0} = a_{0,1} + \delta_0 \quad \text{and} \quad a_{1,k} = a_{0,k+1}, \text{ for } k = 1, 2, 3, \ldots .$$

If $a_{0,k} = 0$ for $k = 1, 2, 3, \ldots$, then we choose $\delta_0 = 0$ and hence $g_1(z) = 0$, forcing the δ-fraction to terminate with $b_0 - \delta_0 z$. If $a_{1,k} \neq 0$ for some $k \geqslant 0$, choose

$$\delta_0 = \begin{cases} 1, & \text{if } a_{0,1} = 0, \\ 0, & \text{if } a_{0,1} \neq 0. \end{cases}$$

so that $a_{1,0} \neq 0$.

By substituting the appropriate expressions for $g_1(z)$ and $g_2(z)$ from (3.1) into the second equation in (3.2a), and equating coefficients of like powers of z, we have

$$a_{2,k} = \delta_1 a_{1,k} - a_{1,k+1}, \text{ for } k = 0, 1, 2, \ldots, \quad \text{and} \quad d_1 = a_{1,0}.$$

If $a_{1,k}=0$ for $k=1, 2, 3, \dots$, choose $\delta_1=0$, implying that $g_2(z)=0$, and hence the δ-fraction terminates with $d_1 z/1$. If $a_{1,k}\neq 0$ for some $k\geqslant 0$, choose

$$\delta_1=\begin{cases}1, & \text{if } a_{1,1}=0,\\ 0, & \text{if } a_{1,1}\neq 0,\end{cases}$$

so that $a_{2,0}\neq 0$. Note that if $\delta_0=1$, then $a_{0,1}=0$, implying that $a_{1,0}=1$, and hence $d_1=1$.

If $n\geqslant 3$, we substitute the appropriate expressions for $g_n(z)$, $g_{n-1}(z)$ and $g_{n-2}(z)$ from (3.1) into (3.2b) and equate coefficients of like powers of z to obtain

$$a_{n,k}=d_{n-1}a_{n-2,k+1}+\delta_{n-1}a_{n-1,k}-a_{n-1,k+1},\quad \text{for } k=0, 1, 2, \dots,$$

and

$$d_{n-1}=\frac{a_{n-1,0}}{a_{n-2,0}}.$$

If $d_{n-1}a_{n-2,k+1}=a_{n-1,k+1}$ for $k=0, 1, 2, \dots$, we choose $\delta_{n-1}=0$, which forces $g_n(z)=0$, and the δ-fraction to terminate with $d_{n-1}z/1$. If $d_{n-1}a_{n-2,k+1}\neq a_{n-1,k+1}$ for some $k\geqslant 0$, choose

$$\delta_{n-1}=\begin{cases}1, & \text{if } a_{n-1,1}=d_{n-1}a_{n-2,1},\\ 0, & \text{if } a_{n-1,1}\neq d_{n-1}a_{n-2,1},\end{cases}$$

so that $a_{n,0}\neq 0$. Note that if $\delta_{n-2}=1$ and $n=3$, then $d_{n-1}=1$. Similarly, if $\delta_{n-2}=1$ and $n\geqslant 4$, then $d_{n-1}=1$. Therefore, the δ-fraction is regular.

The above algorithm generates a regular δ-fraction from the power series expansion of $g_0(z)$. The following theorem states that the regular δ-fraction generated from $g_0(z)$ actually corresponds to $g_0(z)$.

Theorem 1. *The regular δ-fraction obtained from the power series expansion $g_0(z)$ by the corresponding sequence algorithm for regular δ-fractions corresponds to $g_0(z)$ at $z=0$.*

Proof. Let $\{g_n\}$ be the sequence used in the corresponding sequence algorithm described above. Let A_n/B_n be the nth approximant ((1.3) and (1.4)) of the δ-fraction generated by the algorithm. Using induction, it is easy to show that

$$\frac{(-1)^n g_{n+1}}{B_n}=g_0-\frac{A_n}{B_n},\quad \text{for } n=0, 1, 2, \dots . \tag{3.3}$$

From the three-term recurrence relation for B_n (1.3), it is clear that B_n is a polynomial with constant term 1. Since g_{n+1} is of the form (3.1), A_n/B_n is the nth approximant of the δ-fraction, and from (3.3) we have

$$\sum_{i=0}^{\infty} a_i z^i-\frac{A_n}{B_n}=g_0-\frac{A_n}{B_n}=\frac{(-1)^n g_{n+1}}{B_n}=O(z^{n+1}).$$

Therefore, the regular δ-fraction corresponds at $z=0$ to $g_0(z)$ and has the order of correspondence specified by [10, Theorem 2.2]. □

The following is a summary of the algorithm, using series coefficients up to a_m.

If $a_{0,k} = 0$ for $k = 1, 2, 3, \ldots, m$
  then $\delta_0 = 0$ and stop
  else if $a_{0,1} = 0$ then $\delta_0 = 1$
    else $\delta_0 = 0$
$b_0 = a_{0,0}$
$a_{1,0} = a_{0,1} + \delta_0$
for $k = 1, 2, 3, \ldots, m-1$
  $a_{1,k} = a_{0,k+1}$
$d_1 = a_{1,0}$
for $k = 0, 1, 2, \ldots, m-2$
  $a_{2,k} = -a_{1,k+1}$
If $a_{2,k} = 0$ for $k = 0, 1, 2, \ldots, m-2$
  then $\delta_1 = 0$ and stop
  else if $a_{1,1} = 0$ then $\delta_1 = 1$
    else $\delta_1 = 0$
for $k = 0, 1, 2, \ldots, m-2$
  $a_{2,k} = \delta_1 a_{1,k} + a_{1,k+1}$
for $n = 3, 4, 5, \ldots, m-n$

$$d_{n-1} = \frac{a_{n-1,0}}{a_{n-2,0}}$$

  for $k = 0, 1, 2, \ldots, m-n$
    $a_{n,k} = d_{n-1}a_{n-2,k+1} - a_{n-1,k+1}$
  if $a_{n,k} = 0$ for $k = 0, 1, 2, \ldots, m-n$
    then $\delta_{n-1} = 0$ and stop
    else if $a_{n-1,1} = d_{n-1}a_{n-2,1}$ then $\delta_{n-1} = 1$
      else $\delta_{n-1} = 0$
  for $k = 0, 1, 2, \ldots, m$
    $a_{n,k} = a_{n,k} + \delta_{n-1}a_{n-1,k}$

From the algorithm, it is easy to see that in order to accurately compute δ_n and d_n, one must use the first $n+1$ coefficients in g_0. Thus, if a stopping point is preassigned, one can approximate g_0 by the approximate Taylor polynomial without affecting the result.

4. Convergence results and truncation error bounds

This section includes convergence results and truncation error bounds which are important considerations when implementing the algorithm. Included are two theorems on convergence of δ-fractions and one theorem on truncation error bounds. The two convergence theorems represent a reorganization of results from Lange's original paper [10]. The truncation error bounds are achieved by modifying results obtained in 1985 [1].

The first convergence theorem incorporates results found in [10, Theorems 3.2, 3.4 and 3.5].

Theorem 2. *Let*

$$1-\delta_0 z+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{d_n z}{1-\delta_n z}\right) \tag{4.1}$$

be a δ-fraction satisfying

$$\lim_{n\to\infty} d_n=0.$$

Then the δ-fraction (4.1) *converges in the open disk* $D=\{z\colon |z|<1\}$ *to a function* $F(z)$ *analytic at* $z=0$ *and meromorphic in* D. *The convergence is uniform on compact subsets of* D *which contain no poles of* F. *Furthermore*, $L(F)=g_0$.

If, in addition, $\delta_n=0$, $n\geqslant N$ *for some positive integer* N, *then the disk in the above statement may be replaced by* $\mathbb{C}$, *the complex plane.*

The next theorem combines results found in [10, Theorems 3.2 and 3.3] as well as a simple application of Worpitzky's criterion (see [8]).

Theorem 3. *Let*

$$1-\delta_0 z+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{d_n z}{1-\delta_n z}\right) \tag{4.2}$$

be a δ-fraction satisfying

$$0<|d_n|\leqslant M,\quad n\geqslant N, \tag{4.3}$$

where N *is a positive integer and* M *is a positive real number. Then the δ-fraction* (4.2) *converges in the disk*

$$D=\left[z\colon |z|\leqslant\left(\sqrt{1+M}+\sqrt{M}\right)^{-2}\right] \tag{4.4}$$

to a function $F(z)$ *analytic at* $z=0$ *and meromorphic in* D. *The convergence is uniform on compact subsets of* D *which contain no poles of* F. *Furthermore*, $L(F)=g_0$.

If, in addition, $\delta_n=0$, $n\geqslant N$, *then the disk* D *may be replaced by the larger disk*

$$D=\left[z\colon |z|\leqslant(4M)^{-1}\right]. \tag{4.5}$$

We now provide truncation error bounds, i.e., bounds on the error

$$\left|F(z)-\frac{A_n(z)}{B_n(z)}\right|,$$

which may be applied when the conditions of Theorems 2 and 3 are satisfied. We note the following. The bounds were originally derived for continued fractions of the form

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n(z)}{1}\right).$$

Thus, we consider the continued fraction

$$1-\delta_0 z+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{d_n z}{1-\delta_n z}\right) \tag{4.6a}$$

in the equivalent form

$$1-\delta_0 z+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n(z)}{1}\right). \tag{4.6b}$$

where

$$a_1(z)=\frac{d_1 z}{1-\delta_1 z} \quad \text{and} \quad a_n(z)=\frac{d_n z}{(1-\delta_{n-1}z)(1-\delta_n z)}, \quad n\geqslant 2.$$

For a fixed z and N, define

$$\overline{M}:=\sup_{n\geqslant N+1}|a_n| \quad \text{and} \quad \rho:=\tfrac{1}{2}-\sqrt{\tfrac{1}{4}-\overline{M}}.$$

Let B_n denote the nth denominator of the original δ-fraction (4.6a) and let B_n^* denote the denominator of the equivalent continued fraction (4.6b). Then

$$B_n=(1-\delta_1 z)(1-\delta_2 z)\cdots(1-\delta_n z)B_n^*.$$

Let h_n denote the ratio B_n/B_{n-1}. Then $h_n=(1-\delta_n z)h_n^*$ where $h_n^*=B_n^*/B_{n-1}^*$. Also

$$\frac{\prod_{j=1}^{n}|a_j(z)|}{|B_{n-1}^*|^2}=\frac{\prod_{j=1}^{n}|d_j z|}{|1-\delta_n z|\,|B_{n-1}|^2}.$$

Finally, define

$$\Gamma_n:=\frac{a_{n+1}(z)}{2-4|a_{n+2}(z)|}.$$

With this notation we have the following theorem adapted from [1, Theorems 3.2, 3.3 and 4.1].

Theorem 4. *Let a δ-fraction* (4.6a) *be given satisfying* (4.3) *for positive M and N and let D be defined by* (4.4) *or* (4.5) (*as the situation warrants*). *Let $a_n(z)$, $\overline{M}$, ρ and Γ_n, $n\geqslant N$, be defined as above. Let $z\in D$.*

(a) *Suppose*

$$|\Gamma_n|\leqslant\left|\frac{h_N}{1-\delta_N z}+\Gamma_N\right|. \tag{4.7}$$

Define $\rho_n:=|\Gamma_n|$. Then for $n\geqslant N+1$,

$$\left|F(z)-\frac{A_n(z)}{B_n(z)}\right|\leqslant\frac{2\rho_n\prod_{j=1}^{n}|d_j z|}{|1-\delta_n z|\,|B_{n-1}|^2\left(|h_n/(1-\delta_n z)+\Gamma_n|^2-\rho_n^2\right)}. \tag{4.8}$$

(b) *Suppose*

$$|h_N|>\rho|1-\delta_N z|. \tag{4.9}$$

Then for $n \geqslant N+1$,

$$\left|F(z)-\frac{A_n(z)}{B_n(z)}\right| \leqslant \frac{2\rho \prod_{j=1}^{n} |d_j z|}{|1-\delta_n z|\,|B_{n-1}|^2\left(|h_n/(1-\delta_n z)|^2-\rho^2\right)}. \tag{4.10}$$

(c) *If, in addition*,

$$|h_N| > \tfrac{1}{2}|1-\delta_N z|, \tag{4.11}$$

then for $n \geqslant 2$,

$$\left|F(z)-\frac{A_{N+n}(z)}{B_{N+n}(z)}\right| \leqslant 2R_{N+1}\left(\frac{1}{1+\rho^{-1}(1-2\rho)/(1+2\rho)}\right)^{n-1}, \tag{4.12a}$$

where

$$R_{N+1} := \frac{\rho \prod_{j=1}^{N+1} |d_j z|}{|1+\delta_{N+1}|\,|B_N|^2\left(|h_{N+1}/(1-\delta_{N+1})|^2-\rho^2\right)}. \tag{4.12b}$$

Proof. (a) If $z \in D$, where D is defined by (4.4), then

$$|a_n(z)| = \left|\frac{d_n z}{(1-\delta_{n-1}z)(1-\delta_n z)}\right| \leqslant \tfrac{1}{4},$$

as Lange shows in the proof of [10, Theorem 3.3]. Hence condition (3.18) of [1, Theorem 3.3] is satisfied. By hypothesis,

$$|\Gamma_n| \leqslant \left|\frac{h_n}{1-\delta_n z}+\Gamma_N\right|,$$

and thus condition (3.21) of [1, Theorem 3.3] holds, which gives us the bound (3.23) of [1, Theorem 3.3], and hence inequality (4.8) follows. If D is defined by (4.5), the result is trivial.

(b) We have shown $|a_n(z)| \leqslant \frac{1}{4}$ for $n=1, 2, 3, \ldots$, and hence $\overline{M} \leqslant \frac{1}{4}$. Define

$$\rho_n := \rho = \tfrac{1}{2} - \sqrt{\tfrac{1}{4}-\overline{M}} \leqslant \tfrac{1}{2},$$

which implies $0 < \rho < 1$. Also $|a_n| \leqslant \overline{M} = \rho(1-\rho)$, and thus conditions (3.12a) and (3.12b) of [1, Theorem 3.2] are satisfied. Our hypothesis

$$|h_N| > \rho|(1-\delta_N z)| \quad \text{implies} \quad |h_N^*| = \frac{|h_N|}{|1-\delta_N z|} > \rho,$$

and hence condition (3.13) of [1, Theorem 3.2] is satisfied, and bound (4.10) follows.

(c) Again define $\rho_n := \rho$ for $n=1, 2, 3, \ldots$. Then condition (4.1) of [1, Theorem 4.1] holds since $\rho(1-\rho) = \overline{M} \leqslant \frac{1}{4}$. Our condition (4.11) insures that condition (4.2), and hence bound (4.3) of [1, Theorem 4.1], holds. With the conditions of [1, Theorem 4.1] satisfied, our bound (4.12) follows. □

Note that condition (4.7) will be satisfied for all N sufficiently large if $F(z)$ is finite and $\lim_{n\to\infty} a_n = 0$.

Bounds (4.8) and (4.10) are a posteriori bounds since their computation requires knowledge of $A_n(z)$ and $B_n(z)$. The simpler bound (4.12) is an a priori bound since bounds on the error

$$\left| F(z) - \frac{A_{N+n}(z)}{B_{N+n}(z)} \right|$$

essentially require only knowledge of $A_N(z)$ and $B_N(z)$.

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