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On a New Digraph Reconstruction Conjecture*

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Some classes of digraphs are reconstructed from the point-deleted subdigraphs for each of which the degree pair of the deleted point is also known. Several infinite families of known counterexamples to the Digraph Reconstruction Conjecture (DRC) turn out to be reconstructible in this sense. A new conjecture concerning reconstruction of digraphs in this sense is proposed and none of the known counterexample pairs to the DRC is a counterexample pair to this new conjecture.

1. INTRODUCTION

A *digraph* D consists of a finite set $V(D)$ of points and a set $X(D)$ of ordered pairs of distinct points. Any ordered pair (u, v) in $X(D)$ is called an *arc form* u *to* v. It is also called an *outarc incident with* u and an *inarc* incident with v. The *outdegree* (*indegree*) of a point v is the number of outarcs (inarcs) incident with v and the ordered pair (outdegree of v, indegree of v) is called the *degree pair* of v. If both (u, v) and (v, u) are in $X(D)$, they together are called a symmetric pair of arcs joining u and v. If (u, v) is in $X(D)$ and (v, u) is not in $X(D)$, (u, v) is called an *unpaired outarc incident with* u and an *unpaired inarc incident* with v. The ordered triple (r, s, k) is called the *degree triple* of the point v if v is incident with r unpaired outarcs, s unpaired inarcs and k symmetric pairs of arcs.

The pair (D, f) is called a *colored digraph* when D is a digraph and f is a mapping from $V(D)$ into a nonempty set (set of colors). For any $v \in V(D)$, $f(v)$ is called the color of v, and f is called a coloring of the digraph D. A colored digraph (D', ϕ') is called a *subdigraph* of the colored digraph (D, ϕ) if D' is a subdigraph of D and ϕ' is the restriction of ϕ to $V(D')$. When there is no ambiguity about the coloring, the colored digraph (D, ϕ) will be written as D itself. Two colored digraphs (D_1, f_1) and (D_2, f_2) are *isomorphic* if there exists a bijection ϕ form $V(D_1)$ onto $V(D_2)$ such that for $u, v \in V(D_1)$,

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$(u, v) \in X(D_1)$ iff $(\phi(u), \phi(v)) \in X(D_2)$ and $f_1(u) = f_2(\phi(u))$ for each $u \in V(D_1)$. Let D be a digraph or a colored digraph with points $v_1, v_2, ..., v_n$. Let $D_i = D - v_i$ and p_i and d_i be, respectively, the degree pair and degree triple of the point v_i for each i. The collection (D_i, p_i), $i = 1$ to n and (D_i, d_i), $i = 1$ to n are called the *degree pair associated deck* (DPA deck) and *degree triple associated deck* (DTA deck) of D respectively. A digraph (colored digraph) D is said to be *N-reconstructible* if every digraph (colored digraph) having the same DPA deck as D is isomorphic to D.

For $n \geqslant 2$, the number of symmetric pairs of arcs incident with the point missing from each subdigraph can be determined from the DPA deck and thus the DTA deck is known if the DPA deck is known.

The digraph reconstruction conjecture (DRC); first suggested by Harary in [3] took the following form in Manvel [4] because of the nonreconstructible tournaments on 5 points and 6 points discovered by Beineke and Parker [1].

DIGRAPH RECONSTRUCTION CONJECTURE (DRC). *A digraph D with at least seven points can be reconstructed from its subdigraphs $D_i = D - v_i$.*

Later, the conjecture was found to be true for tournaments of order 7, but counterexamples were found among 8-point tournaments in [5]. Finally for each integer $\geqslant 5$ of the form $2^m + 2^n$ with $0 \leqslant n < m$, Stockmeyer constructed [7] six related pairs of counterexamples to the DRC, including a pair of tournaments. Thus the hope that DRC will be true for digraphs of sufficiently high order is lost forever. In this paper we prove that some classes of digraphs including the infinite families D_n and E_n of counterexamples to the DRC in [6] are N-reconstructible. We revive the DRC in a slightly weaker form.

There is only one digraph with p points and degree pairs $(p - i, i - 1)$, $1 \leqslant i \leqslant p$. It is a tournament and will be denoted by T_p. In general, we take the definition as in [6].

2. N-RECONSTRUCTIBLE CLASSES

Any pair of nonisomorpic digraphs having the same DPA deck will be a counterexample pair to the DRC. The complete list of counterexample pairs to the DRC on $\leqslant$ six points and such a list for tournaments on $\leqslant$ eight points are known [8, 5]. But the digraphs in none of these pairs have the same DPA deck. Hence digraphs with at most six points and tournaments with at most eight points are N-reconstructible.

THEOREM 1. *If D is a digraph with n points, $(n \geqslant 5)$ such that a point deleted subdigraph of D is isomorphic to T_{n-1}, then D is N-reconstructible.*

Proof. Let the DPA deck of digraph D be given. Let (D_i, d_i), $i = 1$ to n be the DTA deck of D determined from its DPA deck. Let $D_1 = T_{n-1}$. We will prove that D can be uniquely reconstructed from (D_i, d_i), $i = 1$ to n.

Now $d_1, d_2, \ldots, d_n$ constitute the degree triple sequence of D while d_2, $d_3, \ldots, d_n$ gives the degree triples in D of the points of D that are in D_1. Write the first coordinates in d_i, $i = 2$ to n and the outdegree sequence of D_1 in nondecreasing order and take the vector difference. The entries in the outdegree sequence of D_1 in positions where 1 occurs in the resulting binary sequence will give the collection A of outdegrees of the points of D_1 that dominated the deleted point in D. Similarly the collection B of indegrees of the points of D_1 that were dominated by the deleted point in D can be found. Since T_{n-1} has no symmetric pair of arcs, the first coordinates of those triples among $d_2, \ldots, d_n$ that have third coordinate 1 give the collection C of outdegrees of the points of D_1 that were joined to the deleted point by symmetric pairs of arcs in D. Since points of D_1 have distinct indegrees and outdegrees, the points corresponding to A, B and C, respectively, can be uniquely determined in D_1. Hence adjoining a new point to D_1 and joining it accordingly with points of D_1, D is uniquely determined. ∎

The digraphs whose dominance matrices are

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

form a counterexample pair to the digraph reconstruction conjecture even though each of them has T_5 as a point deleted subdigraph.

THEOREM 2. *If (D, ϕ) is a colored digraph with n points, $(n \geqslant 5)$ such that a point deleted subdigraph of D is isomorphic to T_{n-1}, then (D, ϕ) is N-reconstructible.*

Proof. In the DPA deck of (D, ϕ), each point occurs in precisely $n - 1$ cards and hence the number of points of each color in (D, ϕ) can be determined and for any member of the deck, the color of the deleted point can be found. Now (D, ϕ) can be uniquely determined as in the proof of Theorem 1 as it uses only degree pair considerations.

THEOREM 3. *Colored digraphs with at most four points are N-reconstructible.*

Proof. The proof is routine and is omitted.

THEOREM 4. *Let D be a p-point digraph. If there exists a $(p-i)$-point digraph H such that*

(1) $i \geqslant 2$;

(2) *H has only the trivial automorphism*;

(3) *exactly one induced subdigraph (say H') of D is isomorphic to H*;

(4) *(D', f) is N-reconstructible for every coloring f of D', where D' is the subdigraph of D induced by $V(D) - V(H')$, then D is N-reconstructible.*

Proof. Under the hypothesis of the theorem, only i cards of the deck of D have the digraph H as an induced subdigraph. Call these cards $D_1, D_2,..., D_i$. Label the vertices of H with $i+1$, $i+2,..., p$. Since H has only the trivial automorphism, wherever H occurs as an induced subdigraph of D_j, $1 \leqslant j \leqslant i$, the points $i+1$, $i+2,..., p$ can be located and labeled accordingly. Let us assume that the points of H in D, $D_1,..., D_i$ are so labeled. For each j, $1 \leqslant j \leqslant i$, define a coloring f_j on the points of D_j other than $i+1$, $i+2,..., p$ as follows:

$$f_j(v) = (R(v), S(v), T(v)),$$

where

$$R(v) = \{x \mid i+1 \leqslant x \leqslant p \text{ and } \exists \text{ an unpaired arc from } v \text{ to } x \text{ in } D_j\},$$

$$S(v) = \{x \mid i+1 \leqslant x \leqslant p \text{ and } \exists \text{ an unpaired arc from } x \text{ to } v \text{ in } D_j\},$$

$$T(v) = \{x \mid i+1 \leqslant x \leqslant p \text{ and } \exists \text{ a symmetric pair of arcs between } x \text{ and } v \text{ in } D_j\}.$$

Let f be the coloring given to the vertices of D other than $i+1,..., p$ in the same way. Now let (D', f), $(D'_1, f_1),...,$ (D'_i, f_i) be the colored digraphs obtained from D, D_1, $D_2,..., D_i$ by deleting the points that induce H in each of them. $(D'_1, f_1),...,$ (D'_i, f_i) can be found out from the deck of D and they are clearly the point-deleted subdigraphs of (D', f). Hence the color of the point of D' whose deletion gives D'_j for any j, $1 \leqslant j \leqslant i$ can be determined from $(D'_1, f_1),...,$ (D'_i, f_i). If it is (R, S, T) and the degree triple associated with D_j as a subdigraph of D is (r, s, t) then the degree triple associated with (D'_j, f_j) as a subdigraph of (D', f) is $(r, s, t) - (|R|, |S|, |T|)$. Since (D', f) is N-reconstructible by (4), (D', f) can be uniquely obtained from $(D'_1, f_1),...,$ (D'_i, f_i). From (D', f), D can be uniquely obtained by adjoining a copy of H and arcs between D' and H as indicated by the coloring f. ∎

COROLLARY 4.1. *Let D be a p-point digraph. For $i = 2, 3,$ or 4, if there exists a $(p-i)$-point digraph H such that*

H has only the trivial automorphism and exactly one induced subdigraph of D is isomorphic to H,

then D is N-reconstructible.

Proof. Follows from Theorems 3 and 4.

Stockmeyer's counterexamples are based on a remarkable family $\{A_n \mid n > 0\}$ of tournaments. Let $p = 2^n$. Then A_n has vertex set $v_1,..., v_p$ and arc set $\{(v_i, v_j) \mid \text{odd}\,(j-i) \equiv 1 \pmod 4\}$, where, for any nonzero integer k, odd(k) is the odd integer obtained on dividing k by the appropriate power of 2 (Thus odd$(-6) = -3$ and odd$(8) = 1$). The first $p/2$ points of A_n have outdegree 2^{n-1} while the remaining ones have outdegree $2^{n-1} - 1$. The tournament D_n is obtained from A_n by adding two points v_0 and v_{p+1} such that v_0 dominates precisely the points $v_2, v_4,..., v_p$ and v_{p+1} while v_{p+1} dominates precisely the points $v_1, v_3,..., v_{p-1}$. The tournament E_n is obtained from A_n by adding two points v_0 and v_{p+1} such that v_0 dominates precisely the points $v_1, v_3,..., v_{p-1}$ and v_{p+1} while v_{p+1} dominates precisely the points $v_2, v_4,..., v_p$. We now proceed to prove that D_n and E_n are N-reconstructible for each $n \geqslant 3$.

LEMMA 5. *Let $n \geqslant 3$, $p = 2^n$, $1 \leqslant i, x \leqslant p/2$ and $0 \neq |x - i| \neq p/4$. Then in D_n, v_i dominates $v_x \Leftrightarrow v_i$ dominates $v_{p/2+x} \Leftrightarrow v_{p/2+i}$ dominates v_x.*

Proof. The proof is routine and is omitted.

COROLLARY 5.1. *Let $p = 2^n$, $n \geqslant 3$, $1 \leqslant k \leqslant p/4$ and $1 \leqslant j \leqslant p/2$. In D_n, if v_k does not dominate v_j then v_k does not dominate $v_{p/2+j}$ for $j \neq k$.*

Proof. Now $|j - k| = p/4 \Rightarrow j = k + p/4 \Rightarrow v_k$ dominates v_j, contradicting our hypothesis. Hence $|j - k| \neq p/4$ and the proof follows by Lemma 5. ∎

THEOREM 6. *A_n has only the trivial automorphism and for $n \geqslant 3$, exactly one induced subdigraph of D_n is isomorphic to A_n.*

Proof. Let $p = 2^n$. It is proved in [6] that A_n has the identity automorphism group and from the construction of D_n we obviously have $D_n - \{v_0, v_{p+1}\} \cong A_n$. Moreover, from the construction we easily see that $D_n - \{v_0, v_i(i \neq p+1)\}$ and $D_n - \{v_i(i \neq 0), v_{p+1}\}$ both have points of outdegree $2^{n-1} + 1$ and hence cannot be isomorphic to A_n. By actual inspection we see that for $n = 3$ and 4, $D_n - \{v_0, v_{p+1}\}$ is the only induced subdigraph isomorphic to A_n. To show this in general we show by induction that for $m \geqslant 5$, D_m satisfies the following hypothesis.

Hypothesis A. For all i, j with $1 \leqslant i, j \leqslant 2^m$, there exists a point v_k, $1 \leqslant k \leqslant 2^{m-1}$, which dominates neither v_i nor v_j in D_m (and hence $D_m - \{v_i, v_j\}$ has a point of outdegree $2^{m-1} + 1$).

Hypothesis A is easily verified for D_5 so assume that $n \geqslant 6$ and that hypothesis A holds for D_{n-1}. First we note that for $j = i + p/2$ we can take

$v_k = v_{i+1}$ if $1 \leqslant i < p/2$ and $v_k = v_1$ if $i = p/2$. The remaining possibilities are covered by the following three cases.

Case 1. $1 \leqslant i, j \leqslant p/2$.

By our assumption, there exists a k such that $1 \leqslant k \leqslant 2^{n-2} = p/4$ and v_k dominates neither v_i nor v_j in D_{n-1}. Hence v_k dominates neither v_i nor v_j in D_n.

Case 2. $p/2 < i, j \leqslant p$.

Let $i = p/2 + x$ and $j = p/2 + y$. Now $1 \leqslant x, y \leqslant p/2$. By case 1 there exists a k, $1 \leqslant k \leqslant p/4$ such that v_k dominates neither v_x nor v_y. Hence by Corollary 5.1, v_k dominates neither $v_{p/2+x}$ nor $v_{p/2+y}$. That is, v_k dominates neither v_i nor v_j and $1 \leqslant k \leqslant p/4$.

Case 3. $i \leqslant p/2, j > p/2$ and $j - i \neq p/2$.

Now $1 \leqslant i, j - p/2 \leqslant p/2$. Hence by case 1, there exists a k, $1 \leqslant k \leqslant p/4$ such that v_k dominates neither v_i nor $v_{j-p/2}$ in D_n. Hence by Corollary 5.1, v_k dominates neither v_i nor v_j.

Thus we see that D_n also satisfies hypothesis A if D_{n-1} does. Thus in D_n for $n \geqslant 5$, A_n occurs as an induced subdigraph only once. ▮

THEOREM 7. *D_n as well as E_n for $n = 3, 4, \ldots$ are N-reconstructible.*

Proof. D_n and E_n have isomorphic point-deleted subdigraphs (proved in [6]). Hence by Kelly's Lemma for digraphs, D_n and E_n will have the same number of induced subdigraphs isomorphic to A_n. Now the proof follows from Theorem 6 and Corollary 4.1. ▮

3. CONCLUDING OBSERVATIONS

D_n as well as E_n for $n = 3, 4, \ldots$ give an infinite family of tournaments that are N-reconstructible but not reconstructible under the DRC. Also the digraphs $A_p, A_p^*, B_p, B_p^*, C_p, C_p^*, E_p, E_p^*, F_p$ and F_p^* defined in [7] are N-reconstructible by Corollary 4.1 when p is of the form $2^n + 2$ or $2^n + 2^2$ $(n \geqslant 3)$, since each of them has exactly one 2^n-point induced subdigraph which is a tournament and this tournament has only the trivial automorphism (as it is isomorphic to A_n mentioned in Section 2). But none of these digraphs are reconstructible under the DRC (proved in [7]). Also oriented separable graphs without pendant points can be easily proved to be N-reconstructible and the proof is entirely analogous to that for separable graphs without pendant points in [2]. In view of these results, we revive the DRC in the following weaker form.

CONJECTURE. *All digraphs are N-reconstructible.*

Equivalently, the conjecture can be stated as follows, since one-point digraphs are certainly N-reconstructible.

If D and E are digraphs with points $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, respectively, such that $n > 1$, $D - u_i \cong E - v_i$ and u_i and v_i have the same degree pair for each i, then $D \cong E$.

Obviously digraphs that are reconstructible form their point-deleted subdigraphs are N-reconstructible and any counterexample to the above conjecture must be a counterexample to the DRC. But none of the known counterexamples, (even those with fewer than seven points) to the DRC is a counterexample to the above conjecture. If we consider a graph as a special type of digraph, then the above conjecture implies the standard reconstruction conjecture, since when the number of points is at least three, one can determine the degree of the missing point for each subgraph in the deck. Thus the above conjecture is weaker than the DRC but stronger than the graph reconstruction conjecture. We also believe that every colored digraph with at least two points is N-reconstructible.

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