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On a proper acute triangulation of a polyhedral surface

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ABSTRACT

Let Σ be a polyhedral surface in $\mathbb{R}^3$ with n edges. Let L be the length of the longest edge in Σ, δ be the minimum value of the geodesic distance from a vertex to an edge that is not incident to the vertex, and θ be the measure of the smallest face angle in Σ. We prove that Σ can be triangulated into at most $CLn/(\delta\theta)$ planar and rectilinear acute triangles, where C is an absolute constant.

1. Introduction

A triangulation of a polygon (or a surface) is a dissection of the polygon (or the surface) into triangles in which any two triangles are either disjoint, have a single vertex in common, or have one entire edge in common. An acute (or non-obtuse) triangulation is a triangulation into all acute (or non-obtuse) triangles. It is known that any polygon has an acute triangulation [2]. Many authors are interested in the minimum number of triangles in an acute triangulation of an n-gon. Let us state some results concerning the number of triangles:

- An obtuse triangle can be triangulated into seven acute triangles, but not into fewer ones [13].
- A square can be triangulated into eight acute triangles, but not into fewer ones [3].
- Every quadrilateral can be triangulated into at most ten acute triangles, and there is a (concave) quadrilateral that requires ten acute triangles for acute triangulation [11].
- Every trapezoid other than a rectangle can be triangulated into at most seven acute triangles [16].
- Every n-gon can be triangulated into $O(n)$ acute triangles [12,17].

Acute triangulation of a two-dimensional surface means a triangulation into *geodesic* acute triangles. It follows from Colin de Verdiére and Marin [4] that every compact Riemannian surface X in $\mathbb{R}^3$ admits an acute triangulation. More precisely, if X is homeomorphic to a sphere, then X admits a triangulation with all angles in $[3\pi/10-\varepsilon, 2\pi/5+\varepsilon]$, if X is homeomorphic to a torus, then it admits a triangulation with all angles in $[\pi/3-\varepsilon, \pi/3+\varepsilon]$, and if X has genus ≥ 2, then X admits a triangulation with all angles in $[2\pi/7-\varepsilon, 5\pi/14+\varepsilon]$. Some results on the minimum numbers of triangles in acute triangulations of surfaces are as follows.

- A sphere can be triangulated into 20 geodesic acute triangles, but not into fewer ones [6].
- The surface of a cube can be triangulated into 24 geodesic acute triangles, but not into fewer ones [5].
- The surface of a regular icosahedron can be triangulated into 12 geodesic acute triangles, but not into fewer ones [9].
- The surface of a regular dodecahedron can be triangulated into 14 geodesic acute triangles, but not into fewer than 12 [10].
- Every flat torus can be triangulated into at most 16 geodesic acute triangles [8].

See also [18], for a short survey on acute triangulations.

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By a polyhedral surface, we mean a two-dimensional manifold in $\mathbb{R}^3$, with or without boundary, consisting of a finite number of polygons attached along their edges in such a way that no vertex comes to the interior of an edge. For example, the surface of a cube is a (closed) polyhedral surface. Each polygon of a polyhedral surface is called a *face* of the polyhedral surface and the edges of the polygons are called (*original*) *edges* of the polyhedral surface. It is known [2] that every polyhedral surface can be triangulated into *geodesic* acute triangles (with ignoring original edges). However, in a polyhedral surface, it would be natural to consider a triangulation in which all original edges of the polyhedral surface are used, possibly in subdivided forms. Let us call such a triangulation of a polyhedral surface Σ a *proper triangulation* of Σ. Thus, in a proper triangulation of Σ, no triangle can lie across an original edge of Σ, and every triangle lies in a face of Σ. Hence, a proper triangulation has only planar and rectilinear triangles. A proper acute triangulation of a polyhedral surface is a proper triangulation consisting of all acute triangles. To get a proper acute triangulation of a polyhedral surface, we have to triangulate all polygonal faces into acute triangles in such a way that they fit with each other on original edges. So, it is not obvious if every polyhedral surface admits a proper acute triangulation.

The surface of a cube has a proper acute triangulation into 56 triangles, 56 is the minimum number and such proper acute triangulation is combinatorially unique [7]. Saraf [15] presented a clever way to give a proper triangulation of a polyhedral surface into all non-obtuse triangles. It seems, however, that no proof on the existence of a proper acute triangulation for a general polyhedral surface is given so far. The purpose of this paper is to prove the existence of a proper acute triangulation for a general polyhedral surface, and to present a bound on the number of acute triangles in terms of some parameters of the polyhedral surface.

For a polyhedral surface Σ, the symbols n, L, δ, θ are used in the following sense throughout this paper:

n: the number of edges in Σ.
L: the length of the longest edge in Σ.
δ: the minimum value of the geodesic distance from a vertex to an edge that is not incident to the vertex.
θ: the smallest face angle of Σ.

Theorem 1. *A polyhedral surface Σ has a proper acute triangulation with at most $C(\frac{L}{\delta\theta})n$ triangles, where C is an absolute constant.*

Proof is done by combining the disk-packing method of Bern et al. [1] with the idea of Saraf [15] and Maehara [12]. The factor $\frac{L}{\delta\theta}$ in the upper bound is necessary in our proof. However, I do not know whether this factor is essential or not.

Problem. Is there a bound $B > 0$ such that every tetrahedral surface has a proper triangulation with at most B acute triangles?

2. Saraf-type triangulations

Let Σ be a polyhedral surface. Since each face of Σ is a polygon, each face can be triangulated using only non-obtuse triangles, but it is not obvious if Σ admits a proper non-obtuse triangulation. Saraf proved in [15] the following theorem.

Theorem 2 (*Saraf 2009*)**.** *Every polyhedral surface Σ has a proper non-obtuse triangulation.*

Let us roughly outline Saraf's proof. For a point P of a polyhedral surface Σ, a *disk of radius ρ centered at P* means a set of points on Σ within geodesic distance ρ from P.

Suppose that each face of Σ is a triangle, and the length of minimum edge is 10. Let θ be the minimum of interior angles of the triangles, and let $t = \sin\frac{\theta}{2}$. At each vertex of Σ, put a disk of radius 1, and cover the remaining part of edges by disks of radii approximately t as in Fig. 1 in such a way that the distances of the centers of the mutually overlapping disks of radii $\approx t$ lie between t and $\sqrt{2}t$. For each face σ (triangle) of Σ, let Q_σ denote the uncovered part of σ. Then, it is possible to take a set V of points on the boundary of Q_σ with including black dots $\bullet$ shown in Fig. 1 so that (1) the polygon P_V obtained by connecting the vertices in V consecutively along the boundary of Q_σ has a non-obtuse triangulation with no other vertices on the boundary of P_V, and (2) $\sigma \setminus int(P_V)$ has an acute triangulation with using only the centers $\circ$ and V as the vertex set. (To prove (1), Saraf used a sufficiently fine rectangular grid imposed on Q_σ, see [15] for details.) Thus, all faces of Σ have non-obtuse triangulations, which fit with each other on original edges of Σ, and constitute a non-obtuse triangulation of Σ. □

Saraf's proof shows further that Σ has a proper non-obtuse triangulation satisfying the following condition:

$*$ Every triangle that has at least one vertex on the boundary of a face of Σ is an acute triangle.

Let us call a proper non-obtuse triangulation of Σ that satisfies this condition a *Saraf-type triangulation* of Σ.

Proposition 1. *If Σ has a Saraf-type triangulation with ν non-obtuse triangles, then Σ has a proper acute triangulation with at most 12ν triangles.*

Corollary 1. *Every polyhedral surface admits a proper acute triangulation.* □

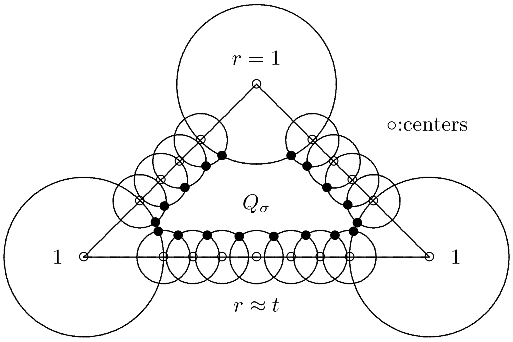


Fig. 1. Cover edges by circles of radii 1 and t.

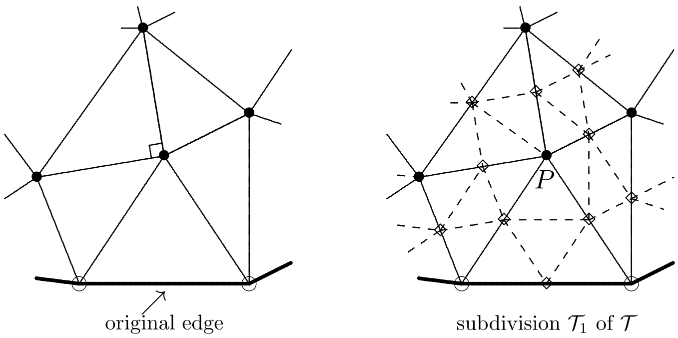


Fig. 2. A part of $\mathcal{T}$ (and $\mathcal{T}_1$) on a face of Σ.

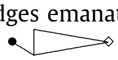
Proof of Proposition 1. Let $\mathcal{T}$ be a Saraf-type triangulation of Σ with ν triangles. Every vertex of $\mathcal{T}$ that lies inside a face of Σ is called a *black vertex* and denoted by $\bullet$, while every vertex lying on the boundary of a face is denoted by $\circ$, see Figure 2-left, which shows a part of $\mathcal{T}$ on a face of Σ. Then, by the above condition $*$, *every right triangle in $\mathcal{T}$ has three black vertices*.

For each edge of $\mathcal{T}$, take its midpoint (denoted by $\diamond$), and then divide each triangle of $\mathcal{T}$ in the following way:

For a non-obtuse triangle ABC of $\mathcal{T}$, let L, M, N be the midpoints of the edges opposite to A, B, C, respectively. If ABC is an acute triangle, then divide ABC by adding edges LM, MN, NL. If ABC is a right triangle with $\angle A = \pi/2$, then divide ABC by adding edges AL, LM, LN.

Then we get a subdivision $\mathcal{T}_1$ of $\mathcal{T}$, see Fig. 2-right. Note that concerning $\mathcal{T}_1$, the following holds:

(1) $\mathcal{T}_1$ is a Saraf-type triangulation of Σ with 4ν triangles.
(2) Triangles in $\mathcal{T}_1$ that have no black vertex are acute triangles.
(3) Each triangle in $\mathcal{T}_1$ has at most one black vertex.
(4) If a triangle in $\mathcal{T}_1$ has a black vertex, then its interior angle at the black vertex is less than $\pi/2$.

Now, take a black vertex P in $\mathcal{T}_1$ and let k be its degree, PM_i $(i = 1, 2, \ldots, k)$ be the edges emanating from P. Take a small circle centered at P, and circumscribe a k-gon $A_1A_2 \ldots A_k$ to this circle in such a way that k edges of the k-gon perpendicularly cut the k edges emanating from P, see Fig. 3. And then, replace each edge emanating from P in $\mathcal{T}_1$ (the dashed lines in Fig. 2) with  as shown in Fig. 3. Then, since PA_1 bisects $\angle M_1PM_2$ and PA_2 bisects $\angle M_2PM_3$, it follows that $\angle A_1PA_2 = (\angle M_1PM_2 + \angle M_2PM_3)/2 < \pi/2$ by (4). Hence the triangle A_1PA_2 is an acute triangle. Similarly k triangles around P are all acute triangles. It will be obvious from (1) (3) (4) that if the circle centered at P is very small (and hence the circumscribed k-gon is very small), then the triangles that surround the k-gon $A_1A_2 \ldots A_k$ become acute triangles by such replacement. Since no two black vertices are adjacent, we can apply such replacement operation independently to every black vertex, and then get a new triangulation $\mathcal{T}_2$ of Σ. It follows now from (2) that all triangles in $\mathcal{T}_2$ are acute triangles. Note that since no black vertex lies on the boundary of a face of Σ, all original edges of Σ still remain in $\mathcal{T}_2$ in subdivided forms. Thus $\mathcal{T}_2$ is a proper acute triangulation of Σ. Clearly the number of triangles in $\mathcal{T}_2$ is at most $3(4\nu) = 12\nu$. □

Unfortunately, Saraf's non-obtuse triangulation in [15] is not suitable to estimate the number of triangles in the triangulation. So, we modify the disk-packing method in [1] to get an appropriate Saraf-type triangulation in Section 4.

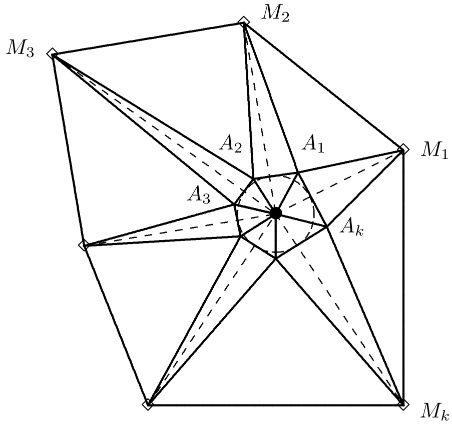


Fig. 3. Replacement around a black vertex P.

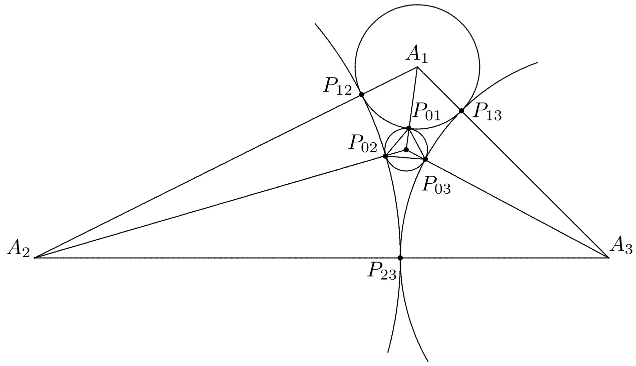


Fig. 4. Four disks.

3. A cycle of disks

A cyclic sequence of k disks $D_0, D_1, \ldots, D_{k-1}$ ($k \geq 3$) in a plane is called a *cycle* of k disks if the k disks are mutually non-overlapping, and each D_i is tangent to only D_{i-1} and D_{i+1} (where the subscripts are taken modulo k). For a cycle of k disks, a k-gon is obtained by connecting the centers of mutually tangent disks, which is called the *k-gon determined by the cycle of k disks*. Each edge of such k-gon contains the *contact point* of two disks with centers at the endpoints of the edge. All edges of such k-gon are covered by the k disks, and the uncovered part inside the k-gon (the remainder in the k-gon) is called a *k-side arc polygon*.

Let us bound the number of triangles in a non-obtuse triangulation of a k-gon determined by a cycle of k disks.

Lemma 1. *Every triangle determined by a cycle of three disks has an acute triangulation with at most* 10 *triangles, in which newly introduced vertices on the boundary of the triangle are only the three contact points on the edges and all other newly introduced vertices lie inside the triangle.*

Proof. Let D_1, D_2, D_3 be a cycle of three disks with centers A_1, A_2, A_3. Inscribe a disk D_0 in the 3-side arc polygon determined by D_1, D_2, D_3 as shown in Fig. 4, and let A_0 be the center of D_0. Let P_{ij} ($0 \leq i < j \leq 3$) be the contact point of D_i and D_j. We show that the following 10 triangles are all acute triangles:

$$\triangle P_{01}P_{02}P_{03},\ \triangle P_{12}P_{01}P_{02},\ \triangle P_{23}P_{02}P_{03},\ \triangle P_{13}P_{01}P_{03},$$
$$\triangle A_1P_{01}P_{12},\ \triangle A_1P_{01}P_{13},\ \triangle A_2P_{02}P_{12},\ \triangle A_2P_{02}P_{23},\ \triangle A_3P_{03}P_{23},\ \triangle A_3P_{03}P_{13}.$$

(1) Since P_{0i} lies on A_0A_i, $i = 1, 2, 3$, the line segment $P_{0i}P_{0j}$ lies in $\triangle A_iA_0A_j$, and hence A_0 lies inside $\triangle P_{01}P_{02}P_{03}$. Thus, the circumcenter of $\triangle P_{01}P_{02}P_{03}$ lies inside $\triangle P_{01}P_{02}P_{03}$, and hence it is an acute triangle.
(2) Let $x = \angle A_0P_{02}P_{03},\ y = \angle A_2P_{23}P_{02},\ z = \angle A_3P_{03}P_{23}$. Then, in $\triangle A_0A_2A_3$,

$$\pi = \angle A_0 + \angle A_2 + \angle A_3 = \pi - 2x + \pi - 2y + \pi - 2z = 3\pi - 2(x + y + z).$$

Therefore $x + y + z = \pi$, and in $\triangle P_{02}P_{23}P_{03}$,

$$\angle P_{02} = \pi - (x + y) = z, \qquad \angle P_{23} = \pi - (y + z) = x, \qquad \angle P_{03} = \pi - (x + z) = y.$$

Since $x, y, z < \pi/2$, the triangle $\triangle P_{02}P_{23}P_{03}$ is an acute triangle. Similarly, $\triangle P_{01}P_{12}P_{02}$, $\triangle P_{03}P_{13}P_{01}$ are acute triangles.
(3) Let r_i be the radius of D_i ($i = 0, 1, 2, 3$). Then $r_0 < \min\{r_1, r_2, r_3\}$. Since $|A_iA_j| = r_i + r_j$, the edge A_iA_j is the longest edge in $\triangle A_0A_iA_j$, which implies that $\angle A_iA_0A_j$ is the largest angle in $\triangle A_0A_iA_j$. This implies that $\angle A_0A_iA_j < \pi/2$ and $\angle A_0A_jA_i < \pi/2$. Hence

$$\triangle A_1P_{01}P_{12},\ \triangle A_1P_{01}P_{13},\ \triangle A_2P_{02}P_{12},\ \triangle A_2P_{02}P_{23},\ \triangle A_3P_{03}P_{23},\ \triangle A_3P_{03}P_{13}$$

are acute triangles. □

The next lemma is a consequence of Lemmas 4, 5, and 7 in [1].

Lemma 2. *Every quadrilateral determined by a cycle of four disks has a non-obtuse triangulation with at most $56 (= 28 + 28)$ triangles, in which newly added vertices on the boundary of the quadrilateral are only the four contact points on the edges, and all other new vertices lie inside the quadrilateral.* □

The next lemma is also proved in [1, Lemma 1].

Lemma 3. *In every k-side arc polygon, $k > 4$, it is possible to pack at most $k - 4$ disks so that the remaining part in the k-side arc polygon splits into at most $2k - 7$ arc polygons each having three or four sides.* □

From Lemmas 1–3, we have the following.

Corollary 2. *A k-gon ($k \geq 4$) determined by a cycle of k disks has a non-obtuse triangulation with at most $56 \times (2k - 7)$ triangles, in which newly introduced vertices on the edges of the k-gon are only the k contact points.* □

4. Saraf-type triangulation induced by disk packing

Let Σ be a polyhedral surface, and V, E, F denote the set of vertices, the set of edges, and the set of faces of Σ, respectively. Then, $|V| - |E| + |F| = \chi$, the Euler characteristic of Σ. For a vertex $v \in V$, let $\Theta(v)$ denote the sum of the face angles at v. For example, if Σ is the surface of a cube, then $\Theta(v) = 3\pi/2$ for every $v \in V$. Recall that a disk on Σ centered at a vertex v, with radius ρ means a set of points on Σ within geodesic distance ρ from v. Thus, the perimeter of a disk of radius $\rho < \delta$ centered at a vertex $v \in V$ is equal to $\rho \cdot \Theta(v)$.
Define $K(v)$ in the following way:

$$K(v) = \begin{cases} \pi - \Theta(v) & \text{if } v \text{ lies on the boundary } \partial\Sigma \text{ of } \Sigma \\ 2\pi - \Theta(v) & \text{otherwise.} \end{cases}$$

Then the following *Polyhedral Gauss–Bonnet Theorem* holds. For a proof, see e.g. [14].

Lemma 4. $\sum_{v \in V} K(v) = 2\pi\chi$. □

Corollary 3. $\sum_{v \in V} \Theta(v) \leq 2\pi(|E| - |F|) < 2\pi|E|$. □

Lemma 5. *Let Γ and γ be externally tangent circles; Γ has center O and radius R; γ has center P radius r. Let OA, OB be two tangent lines of γ, tangent to γ at A, B, respectively, and let $\varphi = \angle AOB$. Then $\frac{2r}{R+r} < \varphi < \frac{2r}{R}$.*

Proof. Since $\sin(\varphi/2) < \varphi/2 < \tan(\varphi/2)$, we have $\frac{r}{R+r} < \varphi/2 < \frac{r}{|OA|} < \frac{r}{R}$. Multiplying by 2, we have the lemma. □

Now, Theorem 1 follows from Proposition 1 and the following.

Proposition 2. *Every polyhedral surface Σ has a Saraf-type triangulation with at most $CLn/(\delta\theta)$ triangles, where C is a constant.*

Proof. We may consider the case $\theta \leq 1$. Let $R = \delta/3$ and $r = R\theta/5$. Denote the length of an edge e of Σ by $l(e)$. First, we cover all edges of Σ by mutually non-overlapping disks of radii R and approximately r in the following way.

(1) For each vertex of Σ, place a disk of radius R with center at the vertex.
(2) For every edge e, cover its uncovered part by placing $\lfloor (l(e)/2 - R)/r \rfloor$ disks of radius $\frac{l(e)/2-R}{\lfloor (l(e)/2-R)/r \rfloor}$ $(\approx r)$ with centers on the edge e.

These disks are called *edge-cover disks*. Denote the set of edge-cover disks by $\mathcal{E}$, see Fig. 5. The sizes R and r are chosen so that the edge-cover disks never overlap with each other.
Then the uncovered part of each face of Σ becomes an *arc polygon*. Next, in each arc polygon, pack disks (*disks of types a, b, c*, denoted by $\mathcal{D}_{abc}$) in the following way:

(a) First, for every pair of adjacent arcs of the same radius $\approx r$, place a disk of the same radius tangent to both arcs. This type of disks are called *type a*, see Fig. 6.

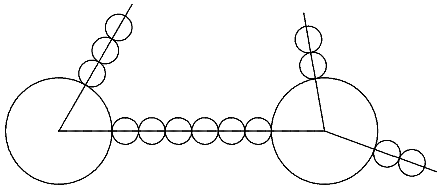


Fig. 5. $\mathcal{E} = \{\text{edge-cover disks}\}$.

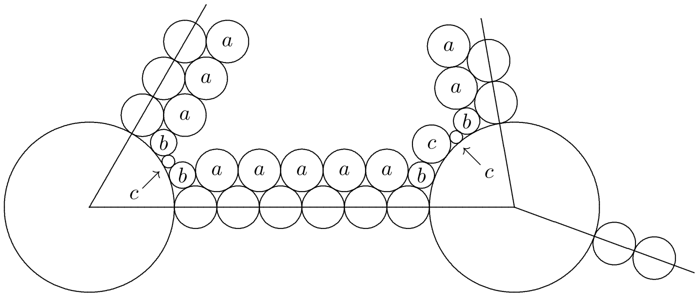


Fig. 6. $\mathcal{D}_{abc} = \{\text{disks of types } a,\ b,\ c\}$.

(b) Next, for every pair of adjacent arcs of radius R and radius $\approx r$, place a disk that is tangent to both arcs, and also tangent to a disk of type a. This kind of disks are called *type b*, see Fig. 6. The diameter of a disk of type b is obviously greater than r, and it is less than $1.221\,r$ (if $r = R/5$, then the diameter is approximately $1.2207\,r$). Since $r = R\theta/5$, any two disks of type b are disjoint by Lemma 5.

(c) Finally, pack disks (disks of type c) of radii at most r tangent to the arcs of radius R so that all disks of types $a,\ b,\ c$ make a cycle of disks. (Make the number of disks of type c as fewer as possible. Then the radii of the disks of type c tangent to an arc of R are all r except the last one whose radius is adjusted to be inscribed in the remaining small gap.)

It is clearly possible to pack disks in each arc polygon in the above way. Let us estimate here $|\mathcal{D}_{abc}|$, the number of disks of types $a,\ b,\ c$. The number of disks of type a is at most $2 \times \frac{L-2R-2r}{2r} \times n$. Tangent to a disk of radius R centered at $v \in V$, there are at most $\Theta(v) \times \frac{R+r}{2r}$ disks of radius $\approx r$ by Lemma 5 (among them, $\deg(v)$ disks are edge-cover disks), at most $2\deg(v)$ disks of type b, and at most $\deg(v)$ disks of type c packed in small gaps. Thus, around a disk of radius R centered at v, there are at most $2\deg(v) + \Theta(v) \times \frac{R+r}{2r}$ disks of types b and c. Therefore, $|\mathcal{D}_{abc}|$ is at most

$$\frac{(L-2R-2r)n}{r} + 2\sum_{v \in V} \deg(v) + \sum_{v \in V} \Theta(v) \times \frac{R+r}{2r}.$$

Since $\sum \Theta(v) < 2\pi n$ by Corollary 3, and $\sum \deg(v) = 2n$, we have

$$|\mathcal{D}_{abc}| < \frac{Ln}{r} + \frac{(\pi-2)Rn}{r} + (\pi+2)n < 26\left(\frac{L}{\delta\theta}\right)n.$$

Now, each cycle of disks of types $a,\ b,\ c$ determines a polygon. Let $m = |F|$, the number of faces of Σ, and let $k_1, \ldots, k_m$ be the numbers of sides of m polygons determined by m cycles of disks of types a, b, c. Then, the line segments connecting the centers of mutually tangent disks in $\mathcal{E} \cup \mathcal{D}_{abc}$ divide Σ into planar triangles and k_i-gon, $i = 1, 2, \ldots, m$. By Lemma 1, each triangle has an acute triangulation with at most 10 acute triangles, in which newly added vertices on the boundary of the triangle are only contact points. And by Corollary 2, each k_i-gon ($k \geq 4$) has a non-obtuse triangulation with at most $56(2k_i - 7)$ triangles, in which newly added vertices on the edges of the k_i-gon are only contact points. Thus, noting that $\sum(2k_i - 7) < 2|\mathcal{D}_{abc}|$, there is a proper Saraf-type triangulation of Σ with at most

$$10 \times 4|\mathcal{D}_{abc}| + 56 \times 2|\mathcal{D}_{abc}| < 152 \times 26\left(\frac{L}{\delta\theta}\right)n$$

triangles. □

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