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Commuting Toeplitz operators on the bidisk ☆

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Abstract

A necessary and sufficient condition is obtained for two Toeplitz operators to be commuting on the Hardy space of the bidisk. The main tool is the Berezin transform and the harmonic extension.

Keywords: Toeplitz operators; Hardy space; Bidisk; Berezin transform

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Its boundary is the unit circle $\mathbb{T}$. The bidisk $\mathbb{D}^2$ and the torus $\mathbb{T}^2$ are the subsets of $\mathbb{C}^2$ which are Cartesian products of two copies $\mathbb{D}$ and $\mathbb{T}$, respectively. Let $d\sigma(z)$ be the normalized Haar measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{D}^2)$ is the closure of the analytic polynomials in $L^2(\mathbb{T}^2, d\sigma)$ (or $L^2(\mathbb{T}^2)$). Let P be the orthogonal

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projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$. The Toeplitz operator with symbol f in $L^\infty(\mathbb{T}^2)$ is defined by

$$T_f h = P(fh),$$

for all $h \in H^2(\mathbb{D}^2)$.

On the Hardy space of the unit disk, Brown and Halmos [2] first showed that two Toeplitz operators are commuting if and only if either both symbols of these operators are analytic, or both symbols of these operators are co-analytic, or a nontrivial linear combination of the symbols of these operators is constant. Axler and Cuckvoic obtained the analogous result for Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk [1]. A natural question is to characterize commuting Toeplitz operators on the Hardy space of bidisk.

Question. For which functions f and g,

$$T_f T_g = T_g T_f?$$

The above question is equivalent to the question that the commutator $T_f T_g - T_g T_f$ equals zero. On the other hand, the commutator $T_f T_g - T_g T_f$ equals the difference of two semi-commutators $T_f T_g - T_{fg}$ and $T_g T_f - T_{gf}$. In [7], Gu and the third author showed that the semi-commutator $T_f T_g - T_{fg}$ equals zero if and only if for each $i = 1, 2$, either $\bar{f}(z_1, z_2)$ or $g(z_1, z_2)$ is analytic in z_i. Recently, in [9], Lee had made the progress on the above question for the special case when one symbol of two Toeplitz operators T_f and T_g is in the form

$$h(z_1) z_2^\alpha + k(z_1) \bar{z}_2^\beta.$$

Later, in [10], Lee obtained some results to address the above question on the Hardy space of the polydisk. The first author worked on the question in [4,3,5]. In this paper, we answer the above question by completely characterizing commuting Toeplitz operators on the Hardy space of the bidisk. The main idea is to use the Berezin transform and the harmonic extension. Even on the unit disk, the idea is new.

To state our results and to reformulate the Brown–Halmos theorem [2] in a different form, we need some notation. Let $K_{z_1}(w_1)$ denote the reproducing kernel

$$\frac{1}{1 - \bar{z}_1 w_1}$$

of Hardy space $H^2(\mathbb{D})$ at the point $z_1 \in \mathbb{D}$ and $k_{z_1}(w_1)$ the normalized reproducing kernel $\frac{(1-|z_1|)^{\frac{1}{2}}}{1-\bar{z}_1 w_1}$ of $H^2(\mathbb{D})$ at the point $z_1 \in \mathbb{D}$. Clearly, the reproducing kernel of $H^2(\mathbb{D}^2)$ at the point z with coordinates (z_1, z_2) in $\mathbb{D}^2$ is given by

$$K_z(w) = \prod_{i=1}^{2} K_{z_i}(w_i).$$

Thus the normalized reproducing kernel $k_z(w)$ of $H^2(\mathbb{D}^2)$ is in the form

$$k_z(w) = \prod_{i=1}^{2} k_{z_i}(w_i).$$

Given $f \in L^1(\mathbb{T}^2)$, the harmonic extension of f is given by

$$\begin{aligned}\hat{f}(z) &= \int_{\mathbb{T}^2} f(\zeta) \prod_{j=1}^{2} \frac{1-|z_j|^2}{|1-z_j\overline{\zeta_j}|^2}\, d\sigma(\zeta)\\ &= \int_{\mathbb{T}^2} f(\zeta)\big|k_z(\zeta)\big|^2\, d\sigma(\zeta)\\ &= \langle f k_z, k_z\rangle.\end{aligned}$$

Let ∂_i denote $\frac{\partial}{\partial z_i}$ and $\bar{\partial}_i$ denote $\frac{\partial}{\partial_i \bar{z}_i}$. The operator $\triangle_j$ is defined by

$$\triangle_j = \partial_j \bar{\partial}_j$$

for $j = 1, 2$. Clearly, $\hat{f}(z)$ is 2-harmonic function on $\mathbb{D}^2$. That is

$$\triangle_j \hat{f}(z) = 0$$

for $j = 1, 2$.

For a bounded operator S on $H^2(\mathbb{D}^2)$, the Berezin transform of S is the function $\tilde{S}$ on $\mathbb{D}^2$ defined by

$$\begin{aligned}\tilde{S}(z) &= \langle S k_z, k_z\rangle\\ &= \int_{\mathbb{T}^2} S k_z(\xi)\overline{k_z(\xi)}\, d\sigma(\xi).\end{aligned}$$

Thus the harmonic extension $\hat{f}(z)$ of f is the Berezin transform $\widetilde{T_f}$ of the Toeplitz operator T_f with symbol f. First we state the Brown and Halmos theorem in [2] as follows.

Theorem 1.1. *(See Brown and Halmos [2].) Let $f, g \in L^\infty(\mathbb{T})$. Then $T_f T_g = T_g T_f$ if and only if*

(a) *both f and g are analytic*; *or*
(b) *both f and g are co-analytic*; *or*
(c) *there are constants a and b with $|a| + |b| \neq 0$ such that $af + bg$ is constant.*

On the unit disk, for f in $L^1(\mathbb{T})$, we still use $\hat{f}(z)$ to denote the harmonic extension of f at a point z in the unit disk $\mathbb{D}$. The result of Brown and Halmos is reformulated in the following form.

Theorem 1.2. *Let $f, g \in L^\infty(\mathbb{T})$. Then $T_f T_g = T_g T_f$ if and only if*

$$\partial \hat{f}(z)\bar{\partial}\hat{g}(z) = \partial \hat{g}(z)\bar{\partial}\hat{f}(z)$$

on $\mathbb{D}$.

The form in the above theorem not only combines three conditions in the original form of the Brown and Halmos theorem [2] into one condition, but also makes sense on the bidisk. A proof of Theorem 1.2 will be given in Section 2.

To motivate readers to the first version of our main result, we reformulate the result in [7] in the similar way as the above theorem. To simplify notation we use $f(z_1, z_2)$ to denote the harmonic extension $\hat{f}(z_1, z_2)$.

Theorem 1.3. *(See [7].) Let f and g be two functions in $L^\infty(\mathbb{T}^2)$. The semi-commutator $T_f T_g - T_{fg}$ equals zero on the Hardy space $H^2(\mathbb{D}^2)$ of the bidisk $\mathbb{D}^2$ if and only if*

(a) *for almost all ς_2 in $\mathbb{T}$,*

$$\partial_1 f(z_1, \varsigma_2)\bar{\partial}_1 g(z_1, \varsigma_2) = 0$$

for z_1 in $\mathbb{D}$, and

(b) *for almost all ς_1 in $\mathbb{T}$,*

$$\partial_2 f(\varsigma_1, z_2)\bar{\partial}_2 g(\varsigma_1, z_2) = 0$$

for z_2 in $\mathbb{D}$,

(c) *for all $z_1, z_2 \in \mathbb{D}$,*

$$\partial_1\partial_2 f(z_1, z_2)\bar{\partial}_1\bar{\partial}_2 g(z_1, z_2) = 0.$$

Inspired by the above results, we obtain the following version of our main result, whose proof will be given in Section 4.

Theorem 1.4 *(First version). Let $f, g \in L^\infty(\mathbb{T}^2)$. The Toeplitz operator T_f commutes with the Toeplitz operator T_g on the Hardy space of the bidisk if and only if the following conditions hold.*

(a) *For almost all ς_2 in $\mathbb{T}$,*

$$\partial_1 f(z_1, \varsigma_2)\bar{\partial}_1 g(z_1, \varsigma_2) = \partial_1 g(z_1, \varsigma_2)\bar{\partial}_1 f(z_1, \varsigma_2)$$

for z_1 in $\mathbb{D}$, and

(b) *for almost all ς_1 in $\mathbb{T}$,*

$$\partial_2 f(\varsigma_1, z_2)\bar{\partial}_2 g(\varsigma_1, z_2) = \partial_2 g(\varsigma_1, z_2)\bar{\partial}_2 f(\varsigma_1, z_2)$$

for z_2 in $\mathbb{D}$, and

(c) *for all* $z_1, z_2 \in \mathbb{D}$,

$$\partial_1\partial_2 f(z_1, z_2)\bar{\partial}_1\bar{\partial}_2 g(z_1, z_2) = \partial_1\partial_2 g(z_1, z_2)\bar{\partial}_1\bar{\partial}_2 f(z_1, z_2).$$

In order to state the second version of our main result, which is analogous to the Brown and Halmos theorem [2], we introduce some decompositions of functions in $L^2(\mathbb{T}^2)$. As in [7], for each f in $\bigcap_{1<q<\infty} L^q(\mathbb{T}^2)$, we write the power series expansion of the harmonic extension $\hat{f}(z)$ of f as follows:

$$f = \sum_{m\in Z^2} f_m z^m = f_{++} + f_{+-} + f_{-+} + f_{--},$$

where

$$f_{++}(z) := \sum_{m\in Z_+\times Z_+} f_m z^m,$$

$$f_{+-}(z) := \sum_{m\in Z_+\times Z_-} f_m z^m,$$

$$f_{-+}(z) := \sum_{m\in Z_-\times Z_+} f_m z^m,$$

$$f_{--}(z) := \sum_{m\in Z_-\times Z_-} f_m z^m$$

and $z^{(m_1,m_2)} = z_1^{m_1} z_2^{m_2}$. Moreover $z_1^{m_1}$ is the m_1th power of z_1 if m_1 is nonnegative and $z_1^{m_1}$ is the $|m_1|$th power of $\bar{z}_1$ if m_1 is negative.

The following is the second version of our main result. It is analogous to the Brown and Halmos theorem [2]. Its proof will be given in Section 5.

Theorem 1.5 *(Second version). Let f and g be in $L^\infty(\mathbb{T}^2)$. The Toeplitz operator T_f commutes with the Toeplitz operator T_g on the Hardy space of the bidisk if and only if the following three conditions hold*:

(a) *For almost all $\varsigma_2 \in \mathbb{T}$,*
   (a1) *$f(z_1, \varsigma_2)$ and $g(z_1, \varsigma_2)$ are both analytic in variable z_1 on $\mathbb{D}$, or*
   (a2) *$f(z_1, \varsigma_2)$ and $g(z_1, \varsigma_2)$ are both co-analytic in variable z_1 on $\mathbb{D}$, or*
   (a3) *there are $a_1(\varsigma_2)$ and $b_1(\varsigma_2)$, not both zero, such that*

$$a_1(\varsigma_2)f(z_1, \varsigma_2) + b_1(\varsigma_2)g(z_1, \varsigma_2)$$

   is a constant in variable z_1 on $\mathbb{D}$.

(b) *For almost all $\varsigma_1 \in \mathbb{T}$,*
   (b1) *$f(\varsigma_1, z_2)$ and $g(\varsigma_1, z_2)$ are both analytic in variable z_2 on $\mathbb{D}$, or*
   (b2) *$f(\varsigma_1, z_2)$ and $g(\varsigma_1, z_2)$ are both co-analytic in variable z_2 on $\mathbb{D}$, or*

(b3) *there are* $a_2(\varsigma_1)$ *and* $b_2(\varsigma_1)$*, not both zero, such that*

$$a_2(\varsigma_1)f(\varsigma_1, z_2) + b_2(\varsigma_1)g(\varsigma_1, z_2)$$

is a constant in variable z_2 *on* $\mathbb{D}$.

(c) *One of the following conditions holds:*

(c1)

$$f_{++}(z_1, z_2) = f_1(z_1) + f_2(z_2),$$
$$g_{++}(z_1, z_2) = g_1(z_1) + g_2(z_2),$$

where f_1, f_2, g_1 *and* g_2 *are in* $H^q(\mathbb{D})$ *for every* $q > 1$.

(c2)

$$f_{--}(z_1, z_2) = \overline{f_1(z_1)} + \overline{f_2(z_2)},$$
$$g_{--}(z_1, z_2) = \overline{g_1(z_1)} + \overline{g_2(z_2)},$$

where f_1, f_2, g_1 *and* g_2 *are in* $H^q(\mathbb{D})$ *for every* $q > 1$.

(c3) *There exist constants* a, b*, not both zero, such that*

$$af_{++}(z_1, z_2) + bg_{++}(z_1, z_2) = h_1(z_1) + h_2(z_2),$$
$$af_{--}(z_1, z_2) + bg_{--}(z_1, z_2) = \overline{r_1(z_1)} + \overline{r_2(z_2)},$$

where h_1, h_2, r_1 *and* r_2 *are in* $H^q(\mathbb{D})$ *for every* $q > 1$.

2. The Brown–Halmos theorem via the Berezin transform

The harmonic extension will play an important role in this section. For $f \in L^1(\mathbb{T})$, $\hat{f}(z)$ is harmonic on $\mathbb{D}$ and

$$\lim_{r\to 1} \hat{f}(r\varsigma) = f(\varsigma)$$

for almost everywhere $\varsigma \in \mathbb{T}$. Conversely, if $f(z)$ is a harmonic function in the unit disk $\mathbb{D}$, one asks when f has boundary values, and how f is determined by its boundary values. The following theorem [8, p. 38, Corollary 2] gives a nice answer.

Theorem 2.1. *(See [8].) Let* f *be a complex-valued harmonic function in the unit disk and suppose that the integrals*

$$\int_{\mathbb{T}} \left|f(r\varsigma)\right|^p \, d\sigma(\varsigma)$$

are bounded as $r \to 1$ *for some* p*,* $1 \leqslant p < \infty$*. Then for almost every* ς *the radial limit*

$$f^*(\varsigma) = \lim_{r\to 1} f(r\varsigma)$$

exists and defines a function f^* in L^p of the circle $\mathbb{T}$. If $p > 1$, then $f(z)$ is the harmonic extension of f^*.

For $f \in L^1(\mathbb{T}^2)$ and fixed $z_i \in \mathbb{D}$, we defined

$$P_i f|_{\xi_i = z_i} = \int_{\mathbb{T}} f(\xi) k_{z_i}(\xi_i)\, d\sigma(\xi_i), \quad 1 \leqslant i \leqslant 2,$$

$$L^q_{ai}(\mathbb{T}^2) = \{ f \in L^q(\mathbb{T}^2)\colon\ f \text{ is analytic in variable } z_i \}.$$

Using the boundedness of P_i on L^q for $q > 1$, one can easily verify the following facts:

- P_1 commutes with P_2, and $P = P_1 P_2$ is a bounded linear operator from $L^q(\mathbb{T}^2)$ to $L^q_a(\mathbb{T}^2)$ for every $q > 1$.
- $\bigcap_{1<q<\infty} L^q$ is an algebra, i.e., both fg and $f + g$ are in $\bigcap_{1<q<\infty} L^q$ if f and g are in $\bigcap_{1<q<\infty} L^q$. In addition, f_+ and f_- are in $\bigcap_{1<q<\infty} L^q$ if $f \in \bigcap_{1<q<\infty} L^q$.
- If f and g belong to $\bigcap_{1<q<\infty} L^q(\mathbb{T})$, then $T_f T_g$ is an operator densely defined on $H^2(\mathbb{D})$.

Although our main concern is with bounded Toeplitz operators, we will need to make use of densely defined unbounded Toeplitz operators. Given two operators S_1 and S_2 densely defined on $H^2(\mathbb{D})$, we say that $S_1 = S_2$ if

$$S_1 p = S_2 p$$

for each p in the set $\mathcal{P}$ of analytic polynomials.

In 1998, Stroethoff obtained a characterization of $f, g, u, v \in L^\infty(\mathbb{T})$ for which $T_f T_g + T_u T_v$ is a Toeplitz operator in [11]. Although the proof of the following theorem may be known, we will include a proof for completeness since ideas in the proof will play an important role on the bidisk later. In fact, the proof gives another proof of the Brown–Halmos theorem via the Berezin transform. Let P be the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. For each function f in $L^2(\mathbb{T})$, let

$$f_+ = P(f),$$
$$f_- = (1 - P)(f).$$

Then we write

$$f = f_+ + f_-.$$

Theorem 2.2. *Let functions f, g, u, v be in $\bigcap_{1<q<\infty} L^q(\mathbb{T})$. Then*

$$T_f T_g = T_u T_v$$

holds on $H^2(\mathbb{D})$ if and only if

$$f_+(z)g_-(z) - u_+(z)v_-(z)$$

is harmonic and

$$f(\varsigma)g(\varsigma) = u(\varsigma)v(\varsigma)$$

for almost all ς *on the unit circle* $\mathbb{T}$.

Proof. In the case that the four functions f, g, u, v are in $L^\infty(\mathbb{T})$, if

$$T_f T_g = T_u T_v,$$

using the symbol mapping [6] from the C^*-algebra generated by bounded Toeplitz operators to $L^\infty(\mathbb{T})$ we have that operators in both sides of the above equality have the same symbol immediately to get

$$f(\varsigma)g(\varsigma) = u(\varsigma)v(\varsigma)$$

for almost all ς on the unit circle $\mathbb{T}$.

We will use the Berezin transform to settle the general case. Noting that for analytic function h in $H^2(\mathbb{D})$,

$$T_h k_z = h k_z$$

and

$$T_h^* k_z = \bar{h}(z) k_z,$$

taking the Berezin transform of the operator $T_f T_g - T_u T_v$ we obtain that for every $z \in \mathbb{D}$,

$$\begin{aligned}
\langle [T_f T_g - T_u T_v]k_z, k_z\rangle &= \langle [T_{(f_+ + f_-)} T_{(g_+ + g_-)} - T_{(u_+ + u_-)} T_{(v_+ + v_-)}]k_z, k_z\rangle \\
&= \langle \big[(T_{f_+} T_{g_+} + T_{f_-} T_{g_-}) + (T_{f_-} T_{g_+} + T_{f_+} T_{g_-})\big]k_z, k_z\rangle \\
&\quad - \langle \big[(T_{u_+} T_{v_+} + T_{u_-} T_{v_-}) + (T_{u_-} T_{v_+} + T_{u_+} T_{v_-})\big]k_z, k_z\rangle \\
&= \langle \big[(T_{f_+ g_+} + T_{f_- g_-}) + T_{f_- g_+}\big]k_z, k_z\rangle + \langle T_{g_-} k_z, T_{f_+}^* k_z\rangle \\
&\quad - \big[\langle \big[(T_{u_+ v_+} + T_{u_- v_-}) + T_{u_- v_+}\big]k_z, k_z\rangle + \langle T_{v_-} k_z, T_{u_+}^* k_z\rangle\big] \\
&= \big[f_+(z)g_+(z) + f_-(z)g_-(z)\big] + \widehat{f_- g_+}(z) + f_+(z)g_-(z) \\
&\quad - \big[u_+(z)v_+(z) + u_-(z)v_-(z)\big] - \widehat{u_- v_+}(z) - u_+(z)v_-(z).
\end{aligned}$$

If $T_f T_g = T_u T_v$, the above equalities give

$$\begin{aligned}
&u_+(z)v_-(z) - f_+(z)g_-(z) \\
&\quad = \big[f_+(z)g_+(z) + f_-(z)g_-(z)\big] + \widehat{f_- g_+}(z) - \big[u_+(z)v_+(z) + u_-(z)v_-(z)\big] - \widehat{u_- v_+}(z).
\end{aligned}$$

Each term in the right hand side of the above equation is harmonic. Thus $f_+(z)g_-(z) - u_+(z)v_-(z)$ is harmonic as desired. Writing z in the polar coordinates $r\varsigma$ and taking limit as r goes to 1^-, since each terms are the products of harmonic extensions of some functions in $\bigcap_{q>1} L^q(\mathbb{T})$, we obtain that for almost all ς on $\mathbb{T}$,

$$\begin{aligned} u_+(\varsigma)v_-(\varsigma) - f_+(\varsigma)g_-(\varsigma) = {}& \big[f_+(\varsigma)g_+(\varsigma) + f_-(\varsigma)g_-(\varsigma)\big] + f_-(\varsigma)g_+(\varsigma) \\ & - \big[u_+(\varsigma)v_+(\varsigma) + u_-(\varsigma)v_-(\varsigma)\big] - u_-(\varsigma)v_+(\varsigma). \end{aligned}$$

This gives

$$f(\varsigma)g(\varsigma) = u(\varsigma)v(\varsigma)$$

for almost all ς on $\mathbb{T}$.

Conversely, suppose

$$f(\varsigma)g(\varsigma) = u(\varsigma)v(\varsigma)$$

for almost all ς on $\mathbb{T}$ and

$$f_+(z)g_-(z) - u_+(z)v_-(z)$$

is harmonic. Let

$$V(z) = f_+(z)g_-(z) - u_+(z)v_-(z).$$

First we need to verify the conditions in Theorem 2.1 for the function V. Since f_+, g_-, u_+, v_- are in $L^q(\mathbb{T})$ for every $1 < q < \infty$, $f_+g_- - u_+v_-$ is in $\bigcap_{1<q<\infty} L^q(\mathbb{T})$ because $\bigcap_{1<q<\infty} L^q(\mathbb{T})$ is an algebra. On the other hand, the Cauchy–Schwarz inequality gives

$$\begin{aligned} \int_{\mathbb{T}} \big|V(r\varsigma)\big|^2 &\leqslant \int_{\mathbb{T}} \big|f_+(r\varsigma)g_-(r\varsigma)\big|^2 + \int_{\mathbb{T}} \big|u_+(r\varsigma)v_-(r\varsigma)\big|^2 \\ &\leqslant \Big\{\int_{\mathbb{T}} \big|f_+(\varsigma)\big|^4 \, d\sigma(\varsigma)\Big\}^{\frac{1}{2}} \Big\{\int_{\mathbb{T}} \big|g_-(\varsigma)\big|^4 \, d\sigma(\varsigma)\Big\}^{\frac{1}{2}} \\ &\quad + \Big\{\int_{\mathbb{T}} \big|u_+(\varsigma)\big|^4 \, d\sigma(\varsigma)\Big\}^{\frac{1}{2}} \Big\{\int_{\mathbb{T}} \big|v_-(\varsigma)\big|^4 \, d\sigma(\varsigma)\Big\}^{\frac{1}{2}}. \end{aligned}$$

Thus Theorem 2.1 gives

$$\begin{aligned} \lim_{r\to 1} V(r\varsigma) &= \lim_{r\to 1} f_+(r\varsigma)g_-(r\varsigma) - u_+(r\varsigma)v_-(r\varsigma) \\ &= f_+(\varsigma)g_-(\varsigma) - u_+(\varsigma)v_-(\varsigma) = V(\varsigma). \end{aligned}$$

By Theorem 2.1 again, we have

$$\begin{aligned} f_+(z)g_-(z) - u_+(z)v_-(z) &= V(z) \\ &= \hat{V}(z) \\ &= [f_+\widehat{g_- - u}_+v_-](z). \end{aligned}$$

An easy calculation gives that the harmonic extension of $fg - uv$ equals

$$\begin{aligned} (\widehat{fg - uv})(z) &= \big\langle \big[(f_+g_+ + f_-g_-) - (u_+v_+ + u_-v_-) \\ &\quad + (f_+g_- + f_-g_+) - (u_+v_- + u_-v_+) \big] k_z, k_z \big\rangle \\ &= \big(f_+(z)g_+(z) + f_-(z)g_-(z) \big) - \big(u_+(z)v_+(z) + u_-(z)v_-(z) \big) \\ &\quad + (f_+\widehat{g_- - u}_+v_-)(z) + (f_-\widehat{g_+ - u}_-v+)(z) \\ &= \big(f_+(z)g_+(z) + f_-(z)g_-(z) \big) - \big(u_+(z)v_+(z) + u_-(z)v_-(z) \big) \\ &\quad + f_+(z)g_-(z) - u_+(z)v_-(z) + (f_-\widehat{g_+ - u}_-v+)(z). \end{aligned}$$

Thus we have

$$\big\langle [T_fT_g - T_uT_v]k_z, k_z \big\rangle = (\widehat{fg - uv})(z).$$

So the Berezin transform of the operator $T_fT_g - T_uT_v$ equals zero on the unit disk. We conclude

$$T_fT_g = T_uT_v$$

as Berezin transform is one-to-one. This completes the proof. □

Proof of Theorem 1.2. By Theorem 2.2, T_f commutes with T_g if and only if $f_+(z)g_-(z) - g_+(z)f_-(z)$ is harmonic on the unit disk. Applying the Laplace operator to the harmonic function implies that this is equivalent to

$$f'_+(z)g'_-(z) = g'_+(z)f'_-(z).$$

Thus this is also equivalent to

$$\partial\hat{f}(z)\bar{\partial}\hat{g}(z) = \partial\hat{g}(z)\bar{\partial}\hat{f}(z)$$

on $\mathbb{D}$ as

$$\begin{aligned} \partial\hat{f}(z) &= f'_+(z), \\ \bar{\partial}\hat{g}(z) &= g'_-(z), \\ \partial\hat{g}(z) &= g'_+(z), \quad \text{and} \\ \bar{\partial}\hat{f}(z) &= f'_-(z). \end{aligned}$$

This completes the proof of Theorem 1.2. □

3. Reduction to one variable case

In this section, freezing one variable we reduce two variables problem to one variable conditions.

If we directly follow the idea in the previous section, as in [7], for each f in $\bigcap_{1<q<\infty} L^q(\mathbb{T}^2)$, we write f as

$$f = f_{++} + f_{+-} + f_{-+} + f_{--},$$

where $f_{++} = Pf$, $f_{+-} = P_1(1 - P_2)f$, $f_{-+} = (1 - P_1)P_2 f$ and $f_{--} = (1 - P_1)(1 - P_2)f$. Thus we have

$$T_f T_g = [T_{f_{++}} + T_{f_{+-}} + T_{f_{-+}} + T_{f_{--}}][T_{g_{++}} + T_{g_{+-}} + T_{g_{-+}} + T_{g_{--}}].$$

So we face the difficulty that

$$T_f T_g = T_g T_f$$

is an equation containing 32 terms of products of two Toeplitz operators. Hence we need to introduce a simpler decomposition of symbols. To do so, for each function f in $\bigcap_{1<q<\infty} L^q(\mathbb{T}^2)$, we write f as

$$f = f_{1+}(z_1, z_2) + f_{1-}(z_1, z_2) = \sum_{i=0}^{+\infty} a_i(z_2) z_1^i + \sum_{i=1}^{+\infty} a_{-i}(z_2) \bar{z}_1^i,$$

where

$$f_{1+}(z_1, z_2) = P_1 f = \sum_{i=0}^{+\infty} a_i(z_2) z_1^i$$

is analytic in variable z_1 and

$$f_{1-}(z_1, z_2) = (I - P_1) f = \sum_{i=1}^{+\infty} a_{-i}(z_2) \bar{z}_1^i$$

is co-analytic in variable z_1. Similarly, we decompose $f(z_1, z_2)$ with respect to the second variable z_2 as follows

$$f = f_{2+}(z_1, z_2) + f_{2-}(z_1, z_2) = \sum_{i=0}^{+\infty} b_i(z_1) z_2^i + \sum_{i=1}^{+\infty} b_{-i}(z_1) \bar{z}_2^i.$$

Since the operators P_i are bounded on each $L^q(\mathbb{T}^2)$, we obtain that f_{i+} and f_{i-} belong to $\bigcap_{1<q<\infty} L^q(\mathbb{T}^2)$.

For each function f in $\bigcap_{1<q<\infty} L^q(\mathbb{T}^2)$ and for fixed $z_1 \in \mathbb{D}$, let $T_{f(z_1,\cdot)}$ denote the Toeplitz operator on $H^2(\mathbb{D})$ given by

$$T_{f(z_1,\cdot)}u = P\big[f(z_1,\cdot)u\big]$$

for $u \in H^2(\mathbb{D})$ and for fixed $z_2 \in \mathbb{D}$, let $T_{f(\cdot,z_2)}$ denote the Toeplitz operator on $H^2(\mathbb{D})$ given by

$$T_{f(\cdot,z_2)}u = P\big[f(\cdot,z_2)u\big]$$

for $u \in H^2(\mathbb{D})$.

The following is the reduction procedure.

Theorem 3.1. *Let $f, g \in L^\infty(\mathbb{T}^2)$. Then $T_f T_g = T_g T_f$ if and only if the following two conditions hold*:

(1) *For $z_1, z_2 \in \mathbb{D}$*

$$[T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} \widetilde{-\,} T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}](z_2)$$

is harmonic in variable z_1.

(2) *For $z_1, z_2 \in \mathbb{D}$*

$$[T_{f_{2+}(\cdot,z_2)}T_{g_{2-}(\cdot,z_2)} \widetilde{-\,} T_{g_{1+}(\cdot,z_2)}T_{f_{1-}(\cdot,z_2)}](z_1)$$

is harmonic in variable z_2.

Proof. Assuming that $T_f T_g = T_g T_f$, we need to show that Conditions (1) and (2) hold. Clearly, it is sufficient to prove that Condition (1) holds. To do so, write $f = f_{1+} + f_{1-}$ and $g = g_{1+} + g_{1-}$. Using the same idea to calculate the Berezin transform as one in the proof of Theorem 2.2, we calculate the Berezin transform of the operator $T_f T_g - T_g T_f$ to get

$$\begin{aligned}
&\big\langle (T_f T_g - T_g T_f)k_z, k_z\big\rangle \\
&\quad = \big\langle [T_{f_{1+}+f_{1-}}T_{g_{1+}+g_{1-}} - T_{g_{1+}+g_{1-}}T_{f_{1+}+f_{1-}}]k_z, k_z\big\rangle \\
&\quad = \Bigg\langle \Bigg[\int_{\mathbb{T}} \big(\big(f_{1-}(\varsigma_1,\varsigma_2)P_2 g_{1+}(\varsigma_1,\varsigma_2) \\
&\qquad - \big(g_{1-}(\varsigma_1,\varsigma_2)P_2 f_{1+}(\varsigma_1,\varsigma_2)\big)\big)k_{z_2}(\varsigma_2)\big)\overline{k_{z_2}(\varsigma_2)}\,d\sigma(\varsigma_2)\Bigg]k_{z_1}, k_{z_1}\Bigg\rangle \\
&\qquad + \big\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1+}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1+}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle \\
&\qquad + \big\langle [T_{f_{1-}(z_1,\varsigma_2)}T_{g_{1-}(z_1,\varsigma_2)} - T_{g_{1-}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle \\
&\qquad + \big\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle.
\end{aligned}$$

Since T_f commutes with T_g, the above equalities give

$$
\begin{aligned}
&-\big\langle [T_{f_{1+}(z_1,\varsigma_2)}T_{g_{1-}(z_1,\varsigma_2)} - T_{g_{1+}(z_1,\varsigma_2)}T_{f_{1-}(z_1,\varsigma_2)}]k_{z_2}, k_{z_2}\big\rangle \\
&\quad = \bigg\langle \bigg[\int_{\mathbb{T}} \Big(\big(f_{1-}(\varsigma_1,\varsigma_2)P_2 g_{1+}(\varsigma_1,\varsigma_2) \\
&\quad\quad - \big(g_{1-}(\varsigma_1,\varsigma_2)P_2 f_{1+}(\varsigma_1,\varsigma_2)\big)\big)k_{z_2}(\varsigma_2)\Big)\overline{k_{z_2}(\varsigma_2)}\,d\sigma(\varsigma_2)\bigg]k_{z_1}, k_{z_1}\bigg\rangle \\
&\quad\quad + \big\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1+}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1+}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle \\
&\quad\quad + \big\langle [T_{f_{1-}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} - T_{g_{1-}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle.
\end{aligned}
$$

Noting that each term in the right hand side of the above equation is harmonic with respect to z_1, we obtain the function

$$
\begin{aligned}
&\big\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle \\
&\quad = [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} \widetilde{-} T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}](z_2)
\end{aligned}
$$

is harmonic in variable z_1 as desired.

Conversely assuming that Conditions (1) and (2) hold, we will show that T_f commutes with T_g. To do so it is sufficient to show that the Berezin transform of the commutator $T_fT_g - T_gT_f$ vanishes on the bidisk since the Berezin transform is injective.

First we verify that conditions hold in Theorem 2.1. Since $\bigcap_{1<q<\infty} L^q(\mathbb{T})$ is an algebra and $f_{j+}, \overline{f_{j-}}$ and $g_{j+}, \overline{g_{j-}}$ belong to $\bigcap_{1<q<\infty} H^q(\mathbb{D})$, we have

$$
\begin{aligned}
&\int_{\mathbb{T}} \big|\langle T_{f_{1+}(r\varsigma_1,\varsigma_2)}T_{g_{1-}(r\varsigma_1,\varsigma_2)}k_{z_2}, k_{z_2}\rangle\big|^2\,d\sigma(\varsigma_1) \\
&\quad \leqslant \int_{\mathbb{T}} \big\| f_{1+}(r\varsigma_1,\cdot)P_2 g_{1-}(r\varsigma_1,\cdot)k_{z_2}\big\|^2\,d\sigma(\varsigma_1) \\
&\quad = \int_{\mathbb{T}}\int_{\mathbb{T}} \big|f_{1+}(r\varsigma_1,\varsigma_2)P_2 g_{1-}(r\varsigma_1,\varsigma_2)k_{z_2}(\varsigma_2)\big|^2\,d\sigma(\varsigma_2)d\sigma(\varsigma_1) \\
&\quad \leqslant C(z_2)\bigg\{\int_{\mathbb{T}^2} \big|f_{1+}(\varsigma)\big|^4 d\sigma(\varsigma)\bigg\}^{\frac{1}{2}}\bigg\{\int_{\mathbb{T}^2} \big|g_{1-}(\varsigma)\big|^4\,d\sigma(\varsigma)\bigg\}^{\frac{1}{2}},
\end{aligned}
$$

where $C(z_2)$ is a constant for fixed z_2. Thus

$$
\varlimsup_{r\to 1^-}\int_{\mathbb{T}} \big|\langle T_{f_{1+}(r\varsigma_1,\varsigma_2)}T_{g_{1-}(r\varsigma_1,\varsigma_2)}k_{z_2}, k_{z_2}\rangle\big|^2\,d\sigma(\varsigma_1) < \infty.
$$

If we write

$$
g_{1-}(z_1,\varsigma_2) = \sum_{n=1}^{+\infty} \hat{g}_{1-}(n,\varsigma_2)\bar{z}_1^n
$$

and

$$f_{1+}(z_1,\varsigma_2)=\sum_{m=0}^{+\infty}\hat{f}_{1+}(m,\varsigma_2)z_1^m,$$

we have

$$\langle f_{1+}(z_1,\varsigma_2)P_2 g_{1-}(z_1,\varsigma_2)k_{z_2},k_{z_2}\rangle=\sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty}\langle \hat{f}_{1+}(m,\varsigma_2)P_2\hat{g}_{1-}(n,\varsigma_2)k_{z_2},k_{z_2}\rangle z_1^m\bar{z}_1^n.$$

Using the polar coordinates for $z_1=r\varsigma_1$ and taking the limit as r tends to 1 give

$$\begin{aligned}
&\lim_{r\to 1}\langle f_{1+}(r\varsigma_1,\cdot)P_2 g_{1-}(r\varsigma_1,\cdot)k_{z_2},k_{z_2}\rangle\\
&\quad=\lim_{r\to 1}\sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty}\langle \hat{f}_{1+}(m,\cdot)P_2\hat{g}_{1-}(n,\cdot)k_{z_2},k_{z_2}\rangle(r\varsigma_1)^m(r\bar{\varsigma}_1)^n\\
&\quad=\sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty}\langle \hat{f}_{1+}(m,\cdot)P_2\hat{g}_{1-}(n,\cdot)k_{z_2},k_{z_2}\rangle(\varsigma_1)^m(\bar{\varsigma}_1)^n\\
&\quad=\langle f_{1+}(\varsigma_1,\cdot)P_2 g_{1-}(\varsigma_1,\cdot)k_{z_2},k_{z_2}\rangle.
\end{aligned}$$

Thus we obtain

$$\lim_{r\to 1}\langle T_{f_{1+}(r\varsigma_1,\cdot)}T_{g_{1-}(r\varsigma_1,\cdot)}k_{z_2},k_{z_2}\rangle=\langle T_{f_{1+}(\varsigma_1,\cdot)}T_{g_{1-}(\varsigma_1,\cdot)}k_{z_2},k_{z_2}\rangle.$$

Letting

$$F(\varsigma_1)=\langle [T_{f_{1+}(\varsigma_1,\cdot)}T_{g_{1-}(\varsigma_1,\cdot)}-T_{g_{1+}(\varsigma_1,\cdot)}T_{f_{1-}(\varsigma_1,\cdot)}]k_{z_2},k_{z_2}\rangle,$$

by Theorem 2.1, we have that the harmonic extension of $F(\varsigma_1)$ is given by

$$\hat{F}(z_1)=\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)}-T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2},k_{z_2}\rangle.$$

An easy calculation gives that the Berezin transform of the commutator $T_fT_g-T_gT_f$ is equal to

$$\begin{aligned}
&\langle [T_fT_g-T_gT_f]k_z,k_z\rangle\\
&\quad=\int_{\mathbb{T}}\langle [T_{f_{1-}(\varsigma)}T_{g_{1+}(\varsigma)}-T_{g_{1-}(\varsigma)}T_{f_{1+}(\varsigma)}]k_{z_2},k_{z_2}\rangle\left|k_{z_1}(\varsigma_1)\right|^2d\sigma(\varsigma_1)\\
&\qquad+\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1+}(z_1,\cdot)}-T_{g_{1+}(z_1,\cdot)}T_{f_{1+}(z_1,\cdot)}]k_{z_2},k_{z_2}\rangle\big|_{H^2(\mathbb{D})}\\
&\qquad+\langle [T_{f_{1-}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)}-T_{g_{1-}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2},k_{z_2}\rangle\big|_{H^2(\mathbb{D})}\\
&\qquad+\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)}-T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2},k_{z_2}\rangle\big|_{H^2(\mathbb{D})}
\end{aligned}$$

$$= \int_{\mathbb{T}} \big\langle [T_{f(\varsigma_1,\cdot)}T_{g(\varsigma_1,\cdot)} - T_{g(\varsigma_1,\cdot)}T_{f(\varsigma_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle |k_{z_1}|^2\, d\sigma(\varsigma_1) \tag{$*$}$$

$$= \int_{\mathbb{T}} \big[f_{2+}(\varsigma_1, z_2)g_{2-}(\varsigma_1, z_2) - g_{2+}(\varsigma_1, z_2)f_{2-}(\varsigma_1, z_2)\big]|k_{z_1}|^2\, d\sigma(\varsigma_1)$$

$$+ \int_{\mathbb{T}^2} \big[f_{2-}(\varsigma)g_{2+}(\varsigma) - g_{2-}(\varsigma)f_{2+}(\varsigma)\big]|k_z|^2\, d\sigma(\varsigma).$$

The last two equalities follow from calculating the action of the commutator of two Toeplitz operators on reproducing kernels on the Hardy space of the unit circle. By the same argument, we have

$$\big\langle [T_f T_g - T_g T_f]k_z, k_z\big\rangle = \int_{\mathbb{T}} \big\langle [T_{f(\cdot,\varsigma_2)}T_{g(\cdot,\varsigma_2)} - T_{g(\cdot,\varsigma_2)}T_{f(\cdot,\varsigma_2)}]k_{z_1}, k_{z_1}\big\rangle \big|k_{z_2}(\varsigma_2)\big|^2\, d\sigma(\varsigma_2)$$

is harmonic in variable z_2. Thus we have that

$$\int_{\mathbb{T}} \big[f_{2+}(\varsigma_1, z_2)g_{2-}(\varsigma_1, z_2) - g_{2+}(\varsigma_1, z_2)f_{2-}(\varsigma_1, z_2)\big]|k_{z_1}|^2\, d\sigma(\varsigma_1)$$

is also harmonic in variable z_2. Differentiating under the integral sign, we obtain

$$\int_{\mathbb{T}} \big\{\triangle_2\big[f_{2+}(\varsigma_1, z_2)g_{2-}(\varsigma_1, z_2) - g_{2+}(\varsigma_1, z_2)f_{2-}(\varsigma_1, z_2)\big]\big\}|k_{z_1}|^2\, d\sigma(\varsigma_1) = 0.$$

Since the Berezin transform is one-to-one, it follows

$$\triangle_2\big[f_{2+}(\varsigma_1, z_2)g_{2-}(\varsigma_1, z_2) - g_{2+}(\varsigma_1, z_2)f_{2-}(\varsigma_1, z_2)\big] = 0.$$

This implies that $f(\varsigma_1, z_2)_{2+}g(\varsigma_1, z_2)_{2-} - g(\varsigma_1, z_2)_{2+}f(\varsigma_1, z_2)_{2-}$ is harmonic in variable z_2. By Theorem 2.2,

$$T_{f(\varsigma_1,\cdot)}T_{g(\varsigma_1,\cdot)} = T_{g(\varsigma_1,\cdot)}T_{f(\varsigma_1,\cdot)}.$$

By Eq. $(*)$, we have

$$\big\langle [T_f T_g - T_g T_f]k_z, k_z\big\rangle = \int_{\mathbb{T}} \big\langle [T_{f(\varsigma_1,\cdot)}T_{g(\varsigma_1,\cdot)} - T_{g(\varsigma_1,\cdot)}T_{f(\varsigma_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle |k_{z_1}|^2\, d\sigma(\varsigma_1) = 0.$$

Since the Berezin transform is one-to-one, it follows that

$$T_f T_g = T_g T_f.$$

This completes the proof. □

4. Proof of the first version of the main result

Theorem 3.1 gives a necessary and sufficient condition for two Toeplitz operators to be commuting on the Hardy space of the bidisk in terms of the Berezin transform by reducing it to the disk. In this section, using Theorem 3.1 we will give the proof of Theorem 1.4.

Proof of Theorem 1.4. First we assume that the Toeplitz operator T_f commutes with the Toeplitz operator T_g on the Hardy space of the bidisk. By Theorem 3.1, we obtain

$$\big\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle$$

is harmonic in variable z_1. Taking differentiation under the integral gives

$$\begin{aligned} 0 &= \triangle_1\big\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle \\ &= \big\langle [T_{\partial_1 f_{1+}(z_1,\cdot)}T_{\bar\partial_1 g_{1-}(z_1,\cdot)} - T_{\partial_1 g_{1+}(z_1,\cdot)}T_{\bar\partial_1 f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\big\rangle. \end{aligned}$$

Since the Berezin transform is injective, this implies that for a fixed z_1 in the unit disk,

$$T_{\partial_1 f_{1+}(z_1,\cdot)}T_{\bar\partial_1 g_{1-}(z_1,\cdot)} - T_{\partial_1 g_{1+}(z_1,\cdot)}T_{\bar\partial_1 f_{1-}(z_1,\cdot)} = 0$$

on $H^2(\mathbb{D})$. Theorem 2.2 gives that for almost all $\varsigma_2 \in \mathbb{T}$,

$$\partial_1 f(z_1,\varsigma_2)\bar\partial_1 g(z_1,\varsigma_2) = \partial_1 g(z_1,\varsigma_2)\bar\partial_1 f(z_1,\varsigma_2)$$

holds for $z_1 \in \mathbb{D}$, and

$$\big(\partial_1 f_{1+}(z_1,z_2)\big)_{2+}\big(\bar\partial_1 g_{1-}(z_1,z_2)\big)_{2-} - \big(\partial_1 g_{1+}(z_1,z_2)\big)_{2+}\big(\bar\partial_1 f_{1-}(z_1,z_2)\big)_{2-}$$

is harmonic in variable z_2. Thus Condition (a) holds. Applying the Laplacian operator with respect to the variable z_2 to the above function gives

$$\partial_2\big(\partial_1 f_{1+}(z_1,z_2)\big)_{2+}\bar\partial_2\big(\bar\partial_1 g_{1-}(z_1,z_2)\big)_{2-} - \partial_2\big(\partial_1 g_{1+}(z_1,z_2)\big)_{2+}\bar\partial_2\big(\bar\partial_1 f_{1-}(z_1,z_2)\big)_{2-} = 0.$$

Since $f_{1+} = f_{++} + f_{+-}$, $f_{1-} = f_{--} + f_{-+}$, $g_{1+} = g_{++} + g_{+-}$ and $g_{1-} = g_{--} + g_{-+}$, we have

$$\begin{aligned} \partial_2\big(\partial_1 f_{1+}(z_1,z_2)\big)_{2+} &= \partial_1\partial_2 f_{++}(z_1,z_2), \\ \bar\partial_2\big(\bar\partial_1 g_{1-}(z_1,z_2)\big)_{2-} &= \bar\partial_1\bar\partial_2 g_{--}(z_1,z_2), \\ \partial_2\big(\partial_1 g_{1+}(z_1,z_2)\big)_{2+} &= \partial_1\partial_2 g_{++}(z_1,z_2), \\ \bar\partial_2\big(\bar\partial_1 f_{1-}(z_1,z_2)\big)_{2-} &= \bar\partial_1\bar\partial_2 f_{--}(z_1,z_2), \end{aligned}$$

so from the equation we get

$$\partial_1\partial_2 f_{++}(z_1,z_2)\bar\partial_1\bar\partial_2 g_{--}(z_1,z_2) = \partial_1\partial_2 g_{++}(z_1,z_2)\bar\partial_1\bar\partial_2 f_{--}(z_1,z_2).$$

This implies Condition (c):

$$\partial_1\partial_2 f(z_1,z_2)\bar\partial_1\bar\partial_2 g(z_1,z_2)=\partial_1\partial_2 g(z_1,z_2)\bar\partial_1\bar\partial_2 f(z_1,z_2)$$

since

$$\begin{aligned}\partial_1\partial_2 f_{++}(z_1,z_2)&=\partial_1\partial_2 f(z_1,z_2),\\ \bar\partial_1\bar\partial_2 f_{--}(z_1,z_2)&=\bar\partial_1\bar\partial_2 f(z_1,z_2),\\ \bar\partial_1\bar\partial_2 g_{--}(z_1,z_2)&=\bar\partial_1\bar\partial_2 g(z_1,z_2),\\ \partial_1\partial_2 g_{++}(z_1,z_2)&=\partial_1\partial_2 g(z_1,z_2).\end{aligned}$$

Using the second condition in Theorem 3.1 that

$$\langle[T_{f_{2+}(\cdot,z_2)}T_{g_{2-}(\cdot,z_2)}-T_{g_{1+}(\cdot,z_2)}T_{f_{1-}(\cdot,z_2)}]k_{z_1},k_{z_1}\rangle$$

is harmonic in variable z_2, as we did as above, we derive that for almost all $\varsigma_1\in T$,

$$\partial_2 f(\varsigma_1,z_2)\bar\partial_2 g(\varsigma_1,z_2)=\partial_2 g(\varsigma_1,z_2)\bar\partial_2 f(\varsigma_2,z_2)$$

holds for $z_2\in\mathbb{D}$. Thus Condition (b) holds.

Conversely, suppose Conditions (a), (b) and (c) hold. For z_1 in $\mathbb{D}$, and ς_2 in $\mathbb{T}$,

$$\partial_1 f(z_1,\varsigma_2)\bar\partial_1 g(z_1,\varsigma_2)=\partial_1 g(z_1,\varsigma_2)\bar\partial_1 f(z_1,\varsigma_2)$$

and for z_2 in $\mathbb{D}$, ς_1 in $\mathbb{T}$,

$$\partial_2 f(\varsigma_1,z_2)\bar\partial_2 g(\varsigma_1,z_2)=\partial_2 g(\varsigma_1,z_2)\bar\partial_2 f(\varsigma_1,z_2).$$

For z_1,z_2 in $\mathbb{D}$,

$$\partial_1\partial_2 f(z_1,z_2)\bar\partial_1\bar\partial_2 g(z_1,z_2)=\partial_1\partial_2 g(z_1,z_2)\bar\partial_1\bar\partial_2 f(z_1,z_2).$$

This gives

$$\big(\partial_1 f_{1+}(z_1,z_2)\big)_{2+}\big(\bar\partial_1 g_{1-}(z_1,z_2)\big)_{2-}-\big(\partial_1 g_{1+}(z_1,z_2)\big)_{2+}\big(\bar\partial_1 f_{1-}(z_1,z_2)\big)_{2-}$$

is harmonic in variable z_2 and

$$\partial_1 f_{1+}(z_1,\varsigma_2)\bar\partial_1 g_{1-}(z_1,\varsigma_2)=\partial_1 g_{1+}(z_1,\varsigma_2)\bar\partial_1 f_{1-}(z_1,\varsigma_2).$$

By Theorem 2.2, we obtain

$$T_{\partial_1 f_{1+}(z_1,\cdot)}T_{\bar\partial_1 g_{1-}(z_1,\cdot)}-T_{\partial_1 g_{1+}(z_1,\cdot)}T_{\bar\partial_1 f_{1-}(z_1,\cdot)}=0.$$

This implies that

$$\langle [T_{f_{1+}(z_1,\cdot)}T_{g_{1-}(z_1,\cdot)} - T_{g_{1+}(z_1,\cdot)}T_{f_{1-}(z_1,\cdot)}]k_{z_2}, k_{z_2}\rangle$$

is harmonic in variable z_1.

On the other hand,

$$\partial_1\partial_2 f_{++}(z_1,z_2)\bar{\partial}_1\bar{\partial}_2 g_{--}(z_1,z_2) = \partial_1\partial_2 g_{++}(z_1,z_2)\bar{\partial}_1\bar{\partial}_2 f_{--}(z_1,z_2)$$

implies that

$$\big(\partial_2 f_{2+}(z_1,z_2)\big)_{1+}\big(\bar{\partial}_2 g_{2-}(z_1,z_2)\big)_{1-} - \big(\partial_2 g_{2+}(z_1,z_2)\big)_{1+}\big(\bar{\partial}_2 f_{2-}(z_1,z_2)\big)_{1-}$$

is harmonic in variable z_1 and

$$\partial_2 f_{2+}(\varsigma_1,z_2)\bar{\partial}_2 g_{2-}(\varsigma_1,z_2) = \partial_2 g_{2+}(\varsigma_1,z_2)\bar{\partial}_2 f_{2-}(\varsigma_1,z_2).$$

By Theorem 2.2 again, we have

$$T_{\partial_2 f_{2+}(\cdot,z_2)}T_{\bar{\partial}_2 g_{2-}(\cdot,z_2)} - T_{\partial_2 g_{2+}(\cdot,z_2)}T_{\bar{\partial}_2 f_{2-}(\cdot,z_2)} = 0$$

to get that

$$\langle [T_{f_{2+}(\cdot,z_2)}T_{g_{2-}(\cdot,z_2)} - T_{g_{2+}(\cdot,z_2)}T_{f_{2-}(\cdot,z_2)}]k_{z_1}, k_{z_1}\rangle$$

is harmonic in variable z_2. Theorem 3.1 gives that T_f commutes with T_g. This completes the proof. □

5. Proof of the second version of the main result

In this section we will prove the second version of the main result by using Theorem 1.4. At the end of the section we will obtain a characterization of normal Toeplitz operators.

Proof of Theorem 1.5. First we observe that if Condition (a) holds, then for z_1 in $\mathbb{D}$ and almost all ς_2 in $\mathbb{T}$,

$$\partial_1 f(z_1,\varsigma_2)\bar{\partial}_1 g(z_1,\varsigma_2) = \partial_1 g(z_1,\varsigma_2)\bar{\partial}_1 f(z_1,\varsigma_2);$$

if Condition (b) holds, then for z_2 in $\mathbb{D}$ and almost all ς_1 in $\mathbb{T}$,

$$\partial_2 f(\varsigma_1,z_2)\bar{\partial}_2 g(\varsigma_1,z_2) = \partial_2 g(\varsigma_1,z_2)\bar{\partial}_2 f(\varsigma_1,z_2);$$

and if Condition (c) holds, then for all $z_1, z_2 \in \mathbb{D}$,

$$\partial_1\partial_2 f(z_1,z_2)\bar{\partial}_1\bar{\partial}_2 g(z_1,z_2) = \partial_1\partial_2 g(z_1,z_2)\bar{\partial}_1\bar{\partial}_2 f(z_1,z_2).$$

By Theorem 1.4, we have that T_f commutes with T_g.

Conversely, assuming that T_f commutes with T_g, we will derive three conditions.

We will show that Condition (c) holds. By Theorem 1.4, we have that for (z_1, z_2) in $\mathbb{D}^2$,

$$\partial_1\partial_2 f_{++}(z_1, z_2)\bar{\partial}_1\bar{\partial}_2 g_{--}(z_1, z_2) = \partial_1\partial_2 g_{++}(z_1, z_2)\bar{\partial}_1\bar{\partial}_2 f_{--}(z_1, z_2). \tag{5.1}$$

Now we consider two cases.

First if one of $\partial_1\partial_2 f_{++}(z_1, z_2)$, $\bar{\partial}_1\bar{\partial}_2 g_{--}(z_1, z_2)$, $\partial_1\partial_2 g_{++}(z_1, z_2)$ and $\bar{\partial}_1\bar{\partial}_2 f_{--}(z_1, z_2)$ identically equals zero on the bidisk $\mathbb{D}^2$, for simplicity, we may assume

$$\partial_1\partial_2 f_{++}(z_1, z_2) \equiv 0$$

on $\mathbb{D}^2$, then

$$f_{++}(z_1, z_2) = f_1(z_1) + f_2(z_2)$$

on $\mathbb{D}^2$ for two functions f_1 and f_2 in H^q for every $q > 1$. Moreover Eq. (5.1) gives that either

$$\partial_1\partial_2 g_{++}(z_1, z_2) = 0$$

or

$$\bar{\partial}_1\bar{\partial}_2 f_{--}(z_1, z_2) = 0$$

on $\mathbb{D}^2$. Thus this implies that

$$g_{++}(z_1, z_2) = g_1(z_1) + g_2(z_2)$$

or

$$f_{--}(z_1, z_2) = \overline{h_1(z_1)} + \overline{h_2(z_2)}$$

where g_1, g_2, h_1 and h_2 are in H^q for every $q > 1$. So Condition (c1) holds or Condition (c3) holds.

Next if none of $\partial_1\partial_2 f_{++}(z_1, z_2)$, $\bar{\partial}_1\bar{\partial}_2 g_{--}(z_1, z_2)$, $\partial_1\partial_2 g_{++}(z_1, z_2)$ and $\bar{\partial}_1\bar{\partial}_2 f_{--}(z_1, z_2)$ identically equals zero on the bidisk $\mathbb{D}^2$, then

$$\frac{\partial_1\partial_2 f_{++}(z_1, z_2)}{\partial_1\partial_2 g_{++}(z_1, z_2)} = \frac{\bar{\partial}_1\bar{\partial}_2 f_{--}(z_1, z_2)}{\bar{\partial}_1\bar{\partial}_2 g_{--}(z_1, z_2)}.$$

The right hand side of the above equation is analytic in both z_1 and z_2 in the bidisk and the left hand side of the equation is co-analytic in both z_1 and z_2 in the bidisk except for a zero set of an analytic function of z_1 and z_2. Thus it must be a constant C. So this implies that

$$f_{++} - Cg_{++} = h_1(z_1) + h_2(z_2)$$

and

$$f_{--} - Cg_{--} = \overline{r_1(z_1)} + \overline{r_2(z_2)}$$

where $h_i, r_i \in H^q(\mathbb{D})$ for every $q > 1$. Thus Condition (c3) holds. So we have proved that Condition (c) holds.

Now we turn to the proof of that Conditions (a) and (b) hold if T_f commutes with T_g and Condition (c) holds. We need only prove that Condition (a) holds. The same argument will derive Condition (b). We consider three cases.

First assume Condition (c1) holds:

$$f_{++}(z_1, z_2) = f_1(z_1) + f_2(z_2)$$

and

$$g_{++}(z_1, z_2) = g_1(z_1) + g_2(z_2),$$

where $f_1, f_2, g_1, g_2 \in H^q(\mathbb{D})$ for every $q > 1$. Thus we have

$$\begin{aligned}
\partial_1 f(z_1, \varsigma_2) &= \partial_1\big[f_{++}(z_1, \varsigma_2) + f_{+-}(z_1, \varsigma_2) + f_{-+}(z_1, \varsigma_2) + f_{--}(z_1, \varsigma_2)\big] \\
&= \partial_1\big[f_{++}(z_1, \varsigma_2) + f_{+-}(z_1, \varsigma_2)\big] \\
&= \partial_1\big[f_1(z_1) + f_2(\varsigma_2) + f_{+-}(z_1, \varsigma_2)\big] \\
&= \partial_1\big[f_1(z_1) + f_{+-}(z_1, \varsigma_2)\big].
\end{aligned}$$

So $\partial_1 f(z_1, \varsigma_2)$ is in $\overline{H^2(\mathbb{D})}$ for the second variable ς_2. Similarly, we also have that

$$\partial_1 g(z_1, \varsigma_2) = \partial_1\big[g_1(z_1) + g_{+-}(z_1, \varsigma_2)\big]$$

is in $\overline{H^2(\mathbb{D})}$ for the second variable ς_2.

If there is a positive measure set $E \subset \mathbb{T}$ such that for $\varsigma_2 \in E$,

$$\partial_1 f(z_1, \varsigma_2) = 0$$

for $z_1 \in \mathbb{D}$, we have

$$\partial_1 f(z_1, \varsigma_2) = 0$$

for z_1 in $\mathbb{D}$ and for almost all $\varsigma_2 \in \mathbb{T}$ since $\partial_1 f(z_1, \varsigma_2)$ is in $\overline{H^2(\mathbb{D})}$ for the second variable ς_2. By Theorem 1.4, we have

$$\partial_1 f(z_1, \varsigma_2)\bar{\partial}_1 g(z_1, \varsigma_2) = \partial_1 g(z_1, \varsigma_2)\bar{\partial}_1 f(z_1, \varsigma_2). \tag{5.2}$$

This implies

$$\partial_1 g(z_1, \varsigma_2) \equiv 0$$

for z_1 in $\mathbb{D}$ and for almost all ς_2 in $\mathbb{T}$ or

$$\bar{\partial}_1 f(z_1, \varsigma_2) \equiv 0$$

for z_1 in $\mathbb{D}$ and for almost all ς_2 in $\mathbb{T}$ since $\partial_1 g(z_1, \varsigma_2)$ is in $\overline{H^2(\mathbb{D})}$ for the second variable ς_2. Thus $f(z_1, \varsigma_2)$ and $g(z_1, \varsigma_2)$ are both co-analytic in variable z_1, which gives Condition (a2), or $f(z_1, \varsigma_2)$ is a constant in variable z_1. The later case implies that $1 \cdot f(z_1, \varsigma_2) + 0 \cdot g(z_1, \varsigma_2)$ is constant in variable z_1, which gives Condition (a3).

By the same argument, if $\partial_1 g(z_1, \varsigma_2)$ identically equals zero for z_1 in $\mathbb{D}$ and for ς_2 on a positive measure subset of $\mathbb{T}$, then Condition (a) holds.

Next assume that none of $\partial_1 f(z_1, \varsigma_2)$ and $\partial_1 g(z_1, \varsigma_2)$ identically equals zero for z_1 in $\mathbb{D}$ and for ς_2 on a positive measure subset of $\mathbb{T}$. Let

$$E_g = \{\varsigma_2 \in \mathbb{T}\colon\ \bar{\partial}_1 g(z_1, \varsigma_2) \equiv 0 \text{ for } z_1 \text{ in } \mathbb{D}\}.$$

If $\sigma(E_g) = \sigma(\mathbb{T})$, then for almost all ς_2 in $\mathbb{T}$,

$$\bar{\partial}_1 g(z_1, \varsigma_2) \equiv 0$$

for z_1 in $\mathbb{D}$. Thus (5.2) gives that for almost ς_2 in $\mathbb{T}$,

$$\bar{\partial}_1 f(z_1, \varsigma_2) \equiv 0$$

for z_1 in $\mathbb{D}$. This implies that for almost all ς_2 in $\mathbb{T}$, both $g(z_1, \varsigma_2)$ and $f(z_1, \varsigma_2)$ are analytic in z_1. Thus we obtain Condition (a1).

If $\sigma(E_g) < \sigma(\mathbb{T})$, (5.2) gives that for almost all ς_2 in the complement E_g^c of E_g,

$$\frac{\partial_1 f(z_1, \varsigma_2)}{\partial_1 g(z_1, \varsigma_2)} = \frac{\bar{\partial}_1 f(z_1, \varsigma_2)}{\bar{\partial}_1 g(z_1, \varsigma_2)}$$

on $\mathbb{D}$. The left hand side of the above equation is analytic in variable z_1 and the right hand side of the equation is co-analytic in variable z_1. Thus it must be constant with respect to the variable z_1. So for almost all $\varsigma_2 \in E_g^c$, there is a function $a(\varsigma_2)$ of ς_2 such that

$$\frac{\partial_1 f(z_1, \varsigma_2)}{\partial_1 g(z_1, \varsigma_2)} = \frac{\bar{\partial}_1 f(z_1, \varsigma_2)}{\bar{\partial}_1 g(z_1, \varsigma_2)} = a(\varsigma_2).$$

This gives that for almost all ς in $\mathbb{T}$,

$$\chi_{E_g^c}(\varsigma_2) f(z_1, \varsigma_2) - a(\varsigma_2) \chi_{E_g^c}(\varsigma_2) g(z_1, \varsigma_2)$$

is a constant in variable z_1 and the function $a(\varsigma_2)$ is defined to be equal to zero on E_g. Hence Condition (a3) holds.

Next assume Condition (c2) holds. By the same argument as in the previous case, we obtain that Condition (a) holds.

Finally assume Condition (c3) holds. In this case, there exist constants a, b, not both zero, such that

$$af_{++}(z_1, z_2) + bg_{++}(z_1, z_2) = h_1(z_1) + h_2(z_2)$$

and

$$af_{--}(z_1, z_2) + bg_{--}(z_1, z_2) = \overline{r_1(z_1)} + \overline{r_2(z_2)},$$

where $h_1, h_2, r_1, r_2 \in H^q(\mathbb{D})$ for every $q > 1$.

If one of a and b equals zero, then we can use the same method as in the first case to obtain that Condition (a) holds.

Now we assume that $a \neq 0$ and $b \neq 0$. We have

$$f_{++}(z_1, z_2) = -\frac{b}{a} g_{++}(z_1, z_2) + H_1(z_1) + H_2(z_2)$$

and

$$f_{--}(z_1, z_2) = -\frac{b}{a} g_{--}(z_1, z_2) + \bar{R}_1(z_1) + \bar{R}_2(z_2),$$

where $H_i = -\frac{b}{a} h_i$ and $R_i = -\frac{b}{a} r_i$. Let

$$F(z_1, z_2) = f(z_1, z_2) + \frac{b}{a} g(z_1, z_2).$$

Since T_f commutes with T_g, we obtain

$$T_F T_g = T_g T_F.$$

Also we have

$$F_{++}(z_1, z_2) = f_{++}(z_1, z_2) + \frac{b}{a} g_{++}(z_1, z_2) = H_1(z_1) + H_2(z_2)$$

and

$$F_{--}(z_1, z_2) = f_{--}(z_1, z_2) + \frac{b}{a} g_{--}(z_1, z_2) = \bar{R}_1(z_1) + \bar{R}_2(z_2)$$

to get

$$\partial_1 F(z_1, \varsigma_2) = \partial_1 \big[H_1(z_1) + F_{+-}(z_1, \varsigma_2) \big]$$

is analytic in z_1 and co-analytic in ς_2, and

$$\bar{\partial}_1 F(z_1, \varsigma_2) = \bar{\partial}_1 \big[\bar{R}_1(z_1) + F_{-+}(z_1, \varsigma_2) \big]$$

is co-analytic in z_1 and analytic in ς_2. We observe that for almost each ς_2 in $\mathbb{T}$, the zero set of $\partial_1 F(z_1, \varsigma_2)$ or $\bar{\partial}_1 F(z_1, \varsigma_2)$ is discrete or equal to the unit disk. Let

$$E_1 = \big\{ \varsigma_2 \in \mathbb{T}\colon\ \partial_1 F(z_1, \varsigma_2) = 0 \text{ for countable many } z_1 \in \mathbb{D} \big\}$$

and

$$E_2 = \left\{\varsigma_2 \in \mathbb{T}\colon\ \bar{\partial}_1 F(z_1, \varsigma_2) = 0 \text{ for countable many } z_1 \in \mathbb{D}\right\}.$$

Theorem 1.4 gives that for almost all ς_2 in $\mathbb{T}$,

$$\partial_1 g(z_1, \varsigma_2)\bar{\partial}_1 F(z_1, \varsigma_2) = \bar{\partial}_1 g(z_1, \varsigma_2)\partial_1 F(z_1, \varsigma_2) \tag{5.3}$$

holds on $\mathbb{D}$. Thus for ς_2 in $E_1 \cap E_2$, we have that

$$\frac{\partial_1 g(z_1, \varsigma_2)}{\partial_1 F(z_1, \varsigma_2)} = \frac{\bar{\partial}_1 g(z_1, \varsigma_2)}{\bar{\partial}_1 F(z_1, \varsigma_2)}$$

holds for all except for countable many z_1 in $\mathbb{D}$. The left hand side of the above equality is analytic in z_1 and the right hand side of the above equality is co-analytic in z_1. Thus it must be a constant function of z_1 and so it depends only on ς_2 and is denoted by $A(\varsigma_2)$. The above equation gives that on $E_1 \cap E_2$

$$g(z_1, \varsigma_2) - A(\varsigma_2)F(z_1, \varsigma_2)$$

is a function of ς_2. If the Lebesgue measure of $E_1 \cap E_2$ equals the measure of $\mathbb{T}$, this implies Condition (a3).

If the Lebesgue measure of $E_1 \cap E_2$ is less than the measure of $\mathbb{T}$, then one of the measures of E_1 and E_2 is less than the measure of $\mathbb{T}$. We may assume that the measure of E_1 is less than the measure of $\mathbb{T}$. For ς_2 in the complement E_1^c of E_1, we have

$$\partial_1 F(z_1, \varsigma_2) \equiv 0$$

for z_1 in the unit disk $\mathbb{D}$. Since $\partial_1 F(z_1, \varsigma_2)$ is analytic in z_1, for a fixed z_1, $\partial_1 F(z_1, \varsigma_2)$ is $\overline{H^2(\mathbb{D})}$ in variable ς_2 and E_1^c has positive measure, we have that for almost all ς_2 in $\mathbb{T}$, we have

$$\partial_1 F(z_1, \varsigma_2) \equiv 0$$

for z_1 in the unit disk $\mathbb{D}$. Thus by (5.3), we have that for almost all ς_2 in $\mathbb{T}$,

$$\partial_1 g(z_1, \varsigma_2)\bar{\partial}_1 F(z_1, \varsigma_2) \equiv 0$$

for z_1 in the unit disk. This implies that for almost all ς_2 in $\mathbb{T}$,

$$\partial_1 g(z_1, \varsigma_2) \equiv 0$$

for z_1 in $\mathbb{D}$ or for almost all ς_2 in $\mathbb{T}$,

$$\bar{\partial}_1 F(z_1, \varsigma_2) \equiv 0$$

for z_1 in the unit disk $\mathbb{D}$ since $\bar{\partial}_1 F(z_1, \varsigma_2)$ is co-analytic in z_1 and for a fixed z_1, $\bar{\partial}_1 F(z_1, \varsigma_2)$ is $H^2(\mathbb{D})$ in variable ς_2. That means that for almost all ς_2 in $\mathbb{T}$, either both $F(z_1, \varsigma_2)$ and $g(z_1, \varsigma_2)$ are co-analytic in z_1 or $F(z_1, \varsigma_2)$ is constant in z_1. The first case gives Condition (a2) since

$$F(z_1, \varsigma_2) = f(z_1, \varsigma_2) + \frac{b}{a} g(z_1, \varsigma_2).$$

The later case gives that for almost all ς_2 in $\mathbb{T}$,

$$F(z_1, \varsigma_2) = f(z_1, \varsigma_2) + \frac{b}{a} g(z_1, \varsigma_2)$$

is a function ς_2. This is Condition (a3). This completes the proof. □

We immediately get the following corollary to characterize normal Toeplitz operators T_f. That is,

$$T_f T_f^* = T_f^* T_f.$$

Corollary 5.1. *Let $f \in L^\infty(\mathbb{T}^2)$. Then T_f is normal operator if and only if the following hold*:

(a) *For almost all $\varsigma_2 \in \mathbb{T}$, there are $a_1(\varsigma_2)$ and $b_1(\varsigma_2)$, not both zero, such that*

$$a_1(\varsigma_2) f(z_1, \varsigma_2) + b_1(\varsigma_2) \overline{f(z_1, \varsigma_2)}$$

is a constant in variable z_1.

(b) *For almost all $\varsigma_2 \in \mathbb{T}$, there are $a_2(\varsigma_1)$ and $b_2(\varsigma_1)$, not both zero, such that*

$$a_2(\varsigma_1) f(\varsigma_1, z_2) + b_2(\varsigma_1) \overline{f(\varsigma_1, z_2)}$$

is a constant in variable z_2.

(c) *There are constants a and b, not both zero, such that*

$$a f_{++}(z_1, z_2) + b \overline{f_{--}(z_1, z_2)} = h_1(z_1) + h_2(z_2)$$

and

$$a f_{--}(z_1, z_2) + b \overline{f_{++}(z_1, z_2)} = \bar{r}_1(z_1) + \bar{r}_2(z_2),$$

where $h_i, r_i \in H^2(\mathbb{D})$.

The following example gives that even if T_f commutes with T_g, but for any constants a, b, not both zero, $af(z_1, z_2) + bg(z_1, z_2)$ is not a constant in variable z_1.

Example. For $h(\varsigma_1) \in L^\infty(\mathbb{T})$ and $a(\varsigma_2) \in L^\infty(\mathbb{T})$, let

$$f(\varsigma_1, \varsigma_2) = h(\varsigma_1),$$
$$g(\varsigma_1, \varsigma_2) = h(\varsigma_1) a(\varsigma_2).$$

Then

(a) for $z_1 \in \mathbb{D}$ and $\varsigma_2 \in \mathbb{T}$,

$$a(\varsigma_2) f(z_1, \varsigma_2) - g(z_1, \varsigma_2) = 0,$$

(b) for $z_1 \in \mathbb{D}$ and $\varsigma_2 \in \mathbb{T}$,

$$1 \cdot f(\varsigma_1, z_2) + 0 \cdot g(\varsigma_1, z_2)$$

is a constant in variable z_2,

(c3)

$$1 \cdot f_{++}(z_1, z_2) + 0 \cdot g_{++}(z_1, z_2) = h_+(z_1)$$

and

$$1 \cdot f_{--}(z_1, z_2) + 0 \cdot g_{--}(z_1, z_2) = h_-(z_1).$$

Thus Theorem 1.5 gives that

$$T_f T_g = T_g T_f.$$

But there are no constants a, b, not both zero, such that $af(z_1, z_2) + bg(z_1, z_2)$ is a constant in variable z_1.

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