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Cyclizable matrix pairs over $\mathbb{C}[x]$ and a conjecture on Töplitz pencils☆

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Abstract

The 22-year-old conjecture is addressed which claims that any reachable matrix pair (A, B) from $\mathbb{C}[x]^{n\times(n+m)}$ is cyclizable, i.e. allows for F, u over $\mathbb{C}[x]$ s.t. $(A + BF)^k Bu$, $k = 0, \ldots, n-1$, is a basis for $\mathbb{C}[y]^n$. It is shown that for a whole class of pairs the correctness of the conjecture is a consequence of the correctness of a conjecture on certain Töplitz pencils. An algebraic computable test for the validity of the latter is given. Based on these results the validity of the conjecture on cyclizability can be extended up to dimension 5.

Keywords: Systems over rings; Töplitz pencils; Cyclizability; BSSV-Conjecture

1. Introduction

One of the basic results in early control theory for linear multivariable state space systems is Heymann's Lemma [5,6]. It states that the following *cyclization property* is true, when R is a field.

Cyclization Property 1.1. *Let R be a commutative ring. For all $n, m \in \mathbb{N}_+$ and all $A \in R^{n\times n}$, $B \in R^{n\times m}$ s.t.*

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$$\mathscr{R}(A, B) := [B, AB, \ldots, A^{n-1}B] \text{ is right invertible}, \tag{1}$$

there exists $u \in R^m$, $F \in R^{m\times n}$ *s.t.*

$$[Bu, (A + BF)Bu, \ldots, (A + BF)^{n-1}Bu] \text{ is invertible}. \tag{2}$$

In control theory the matrix $\mathscr{R}(A, B)$ is called reachability matrix (for the control process with parameters A, B) and in case (1) is valid, the pair (A, B) is called *reachable*.

In case (2) is valid for a specific pair (A, B), this pair is called *cyclizable*. The importance of the Cyclization Property in control theory lies in the fact, that it gives direct access to the so-called pole-shifting theorem. For further historic, control theoretic and algebraic background we refer to [13].

Rings with cyclization property for fixed n are called FC_n-rings (Feedback Cyclization) and rings which are FC_n for all $n \geqslant 1$ are called FC-rings according to [3, p. 115]. The search for FC_n-rings was to a great extent stimulated by [3,14] and the monograph [1]. Many rings are known to have the FC-property. Among the interesting ones are fields, finite rings, certain power series rings or more generally speaking rings with finitely many maximal ideals (see e.g. [1]), also 1-stable rings including standard rings of analytic functions (see [9] for references). Further examples can be found in [2, Theorem 5]. Already in [3, p. 124] it is demonstrated that the ring $\mathbb{Z}$ of integers and the ring $\mathbb{R}[y]$ of real polynomials are not FC_2-rings even and conjectured that nevertheless $\mathbb{C}[y]$ *is an FC-ring* (BSSV-conjecture).

The BSSV conjecture has been confirmed for $n = 2$ in [3], $n = 3$ in [8] and $n = 4$ in [9]. In [10,12] the conjecture could be confirmed for arbitrary n but with the exception of two specific families of "exceptional" pairs. One can deduce from the results that $\mathbb{C}[y]$ is "at least" generically an FC_n-ring for $n \geqslant 1$.

In this article we will show in Section 3 for one of the two exceptional families that the cyclization property holds if and only if the following conjecture for a certain class of Töplitz pencils is true.

Töplitz-Pencil-conjecture 1.2 (TP-conjecture). *For any* $n \geqslant 4$ *and any nonzero complex numbers* $c_1, \ldots, c_{n-1}$ *and for an indeterminate* x *over* $\mathbb{C}$, *if*

$$M_{n,0} = \begin{bmatrix} c_2 & c_1 & x & 0 & \cdots & 0 \\ & & & & & \vdots \\ \vdots & & & & & 0 \\ & & & \ddots & & x \\ c_{n-2} & c_{n-3} & & & & c_1 \\ c_{n-1} & c_{n-2} & & \cdots & & c_2 \end{bmatrix} \tag{3}$$

does not have full rank, then the first two columns are linearly dependent (*the converse being trivial*).

By this result it is no longer necessary to perform tedious feedback constructions in order to confirm the cyclization property. It can shift the effort to a problem in a well-established classical mathematical surrounding [11].

We will give the easy proof for $n = 5$ and develop an algebraic test which allows to confirm the conjecture for arbitrary n. With this test the TP-conjecture has been confirmed up to $n = 7$ so far. See end of Section 3 for more details.

In Section 4 the cyclization property is proved for all remaining exceptional pairs of dimension 5. Together with the TP-conjecture for $n = 5$ this establishes the BSSV-conjecture for $n = 5$.

2. Mathematical background

The arguments to follow in Sections 3 and 4 will be founded on the following result which can be extracted from [10, pp. 552–555].

Theorem 2.1. *Let*

$$C(f) = \begin{bmatrix} 0 & & \cdots & & 0 \\ 1 & & & & 0 \\ \vdots & \ddots & & & 0 \\ 0 & \cdots & 1 & & 0 \\ 0 & \cdots & 0 & f & 0 \end{bmatrix} \in \mathbb{C}[y]^{n\times n},$$

$$D(h_1, \ldots, h_{n-2}, g) = \begin{bmatrix} 1 & 0 \\ 0 & h_1 \\ \vdots & \\ 0 & h_{n-2} \\ 0 & g \end{bmatrix} \in \mathbb{C}[y]^{n\times 2}. \tag{4}$$

$\mathbb{C}[y]$ is an FC-ring if and only if for all $n \geqslant 2$ and all nonconstant and coprime $a, b \in \mathbb{C}[y]$ and all $c_1, \ldots, c_{n-2} \in \mathbb{C}[y]$ one can find $v_1, \ldots, v_{n-2}, u \in \mathbb{C}[y]$ and $\mathbb{Q} \in \mathbb{C}[y]^{2\times 2}$ with $\det Q = 1$ *s.t.*

$$\det \mathscr{R}(C(a), D(c_1, \ldots, c_{n-2}, b))\,\mathrm{diag}(Q, \ldots, Q)^{\mathrm{t}} \times \mathscr{R}(C(1), D(v_1, \ldots, v_{n-2}, u)) = 1. \tag{5}$$

We call this a "symmetric approach to FC". Recall the definition of $\mathscr{R}$ from (1). Despite the transparent structure of Eq. (5) the number of terms still "explodes" with growing n.

Note that for $n = 1$ the cyclization property is trivially true, for $n = 2$ it has been proved in [3] and for $n = 3$ in [8]. From a control point of view it makes sense to try cyclization assuming

$$v_1 = \cdots = v_{n-3} = 0. \tag{6}$$

In this case the equation in (5) has been solved in [9] for $n = 4$. These equations—still assuming (6)—can be expanded for $n \geqslant 5$ as follows (see also [10, p. 555])

$$a\, q_{11}^n + d_u\, u + d_v\, v + d_{vv}\, v^2 = 1\,, \tag{7}$$

where

$$\left.\begin{array}{l} d_u = (-1)^{n-2} \det[B, ABq^{(1)}, \ldots, A^{n-2}Bq^{(1)}], \\ d_v = 2(-1)^{n-3} \det[B, ABq^{(1)}, \ldots, A^{n-3}Bq^{(1)}, A^{n-1}Bq^{(1)}], \\ d_{vv} = -\det[B, AB, A^2Bq^{(1)}, \ldots, A^{n-3}Bq^{(1)}], \end{array}\right\} \tag{8}$$

and where $A = C(a)$, $B = D(c_1, c_2, \ldots, c_{n-2}, b)$, $Q = [q^{(1)}, q^{(2)}] = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$.

The following results have been obtained so far:

Theorem 2.2 [10]. *If* $\boldsymbol{d_{vv} \neq 0}$ (*as a polynomial in* q_{11}, q_{21}) *and if*

$$\textit{a has a simple prime divisor not dividing } c_1, \tag{9}$$

solutions to (5) *or* (7) *can be constructed for arbitrary* n.

Theorem 2.3 [12]. *If for* $1 \leqslant j \leqslant n-3$ *one has*

$$b^j c_{n-2-j} = a^j c_{n-2}^{j+1}, \tag{10}$$

then (5) (*or equivalently* (7)) *is solvable*.

It is easily checked that under condition (10) one has $d_{vv} = 0$ as a polynomial in q_{11}, q_{21}, the case excluded in Theorem 2.2.

The converse, that $d_{vv} = 0$ implies relations (10), is true for $n = 4$ and will be proved later for $n = 5$. For $n = 6$ the converse may be wrong if some of the c_i are zero. See [12, p. 3] for an example. Fortunately in proving the FC_n-property there is no loss in assuming

$$c_1 \neq 0, \ldots, c_{n-2} \neq 0, a \neq 0, b \neq 0 \tag{11}$$

(apply e.g. Proposition 3.8 in [12]).

Thus the question arises whether (11) and $d_{vv} = 0$ imply (10). This will be investigated in the following section.

Conditions (9) and (10) describe the two remaining (and intersecting) families of pairs, for which the cyclization property has not yet been proved and which have been mentioned in the introduction.

3. Consequences of $d_{vv} = 0$ and the TP-conjecture

Throughout we assume $c_1, \ldots, c_{n-2}, a, b, q_{11}, q_{21}$ to be nonzero polynomials in $\mathbb{C}[y]$. From (8), putting $c_{n-1} := \frac{b}{a}$, $x = \frac{q_{11}}{q_{21}}$ one derives over $\mathbb{C}(y)$:

Observation 3.1. *$d_{vv} = 0$ as polynomial in q_{11}, q_{21} over $\mathbb{C}[y]$ if and only if* $\det M_n = 0$ *as a polynomial in x over $\mathbb{C}(y)$, where*

$$M_n = \begin{bmatrix} c_2 & c_1 & x & & 0 \\ & & & \ddots & \\ \vdots & & \ddots & & x \\ & & & & c_1 \\ c_{n-1} & c_{n-2} & & & c_2 \end{bmatrix} \tag{12}$$

is a Töplitz pencil. Note that here the c_k are polynomials from $\mathbb{C}[y]$.

We are now ready to establish a link between the TP-conjecture and the case where $d_{vv} = 0$.

Proposition 3.2. *Let $n \geqslant 4$.*

(a) *The following statements are equivalent:*
  - (i) *for all nonzero $c_1, \ldots, c_{n-1} \in \mathbb{C}[y]$ (or $\mathbb{C}(y)$) and if* $\det M_n = 0$, *then the first two columns of M_n are dependent over $\mathbb{C}[y]$ (or $\mathbb{C}(y)$).*
  - (ii) *The TP-conjecture is true for n.*

(b) *If the TP-conjecture is correct for n, then for any system of dimension n of type* (4), (6), (11) *and with $d_{vv} = 0$ Eq.* (5) (*or equivalently* (7)) *is solvable.*

Proof. (a) The implication (i) $\Rightarrow$ (ii) is obvious. For the converse let us assume (ii) and $\det M_n = 0$. Let P be the finite set of potential poles and zeros of the c_i, then for any $\xi \in \mathbb{C} \backslash P$ inserted for y one obtains $\det(M_n(\xi)) = 0$. Then (ii) gives the dependence of the first two columns of $M_n(\xi)$ over $\mathbb{C}$. Since the c_i are determined by finitely many points, dependence must be valid over $\mathbb{C}[y]$ (or $\mathbb{C}(y)$).

(b) Let the TP-conjecture be correct for n and let us consider a pair of type (4), (6) and (11) such that $d_{vv} = 0$. Then by Observation 3.1 we must have $\det M_n = 0$. But then by (a) the first two columns of M_n are dependent. Since (11) is assumed, this is equivalent to

$$\frac{c_1}{c_2} = \frac{c_2}{c_3} = \cdots = \frac{c_{n-2}}{c_{n-1}} \tag{13}$$

and the latter will be shown to be equivalent to

$$\frac{c_{n-3}}{c_{n-2}} = \frac{c_{n-2}}{c_{n-1}}, \frac{c_{n-4}}{c_{n-2}} = \left(\frac{c_{n-2}}{c_{n-1}}\right)^2, \ldots, \frac{c_1}{c_{n-2}} = \left(\frac{c_{n-2}}{c_{n-1}}\right)^{n-3}. \tag{14}$$

Note that (14) is nothing else than (10), when replacing c_{n-1} by $\frac{b}{a}$.

Thus by Theorem 2.3 one obtains cyclizability and it remains to verify the equivalence of (13) and (14). Given (13) one has for $2 \leqslant k \leqslant n-3$,

$$\frac{c_{n-2-k}}{c_{n-2}} = \frac{c_{n-3}}{c_{n-2}} \frac{c_{n-4}}{c_{n-3}} \cdots \frac{c_{n-2-k}}{c_{n-2-(k-1)}} = \left(\frac{c_{n-2}}{c_{n-1}}\right)^k \tag{15}$$

and on the other hand, assuming (14) and dividing (15) for k by (15) for $k-1$, we are lead to (13). □

As an example we give the easy proof of the TP-conjecture for $n = 5$, a result applied in the following section.

Proof of the TP-conjecture for *n* = 5. Let $M_{5,0}$ be as in (3), then

$$\det M_{5,0} = \left(c_3{}^2 - c_2c_4\right)x + \left(c_2{}^3 - 2c_1c_2c_3 + c_4c_1{}^2\right).$$

If $\det M_{5,0} = 0$ in $\mathbb{C}[x]$, then $c_4 = \frac{c_3{}^2}{c_2}$. Inserted into the constant term this gives $\frac{1}{c_2}(c_2{}^2 - c_1c_3)^2 = 0$. Thus two of the 2×2-minors of the first columns of $M_{5,0}$ vanish, which means, that these two columns are linearly dependent. □

We also have at hand a (considerably more involved) proof for $n = 6$. No proof is known so far for $n = 7$.

Nevertheless it was possible to confirm the truth of the TP-conjecture for $n = 7$ by the following computable algebraic test, see below for details.

To derive the test, we replace c_i by z_i in the matrix $M_{n,0}$ for $1 \leqslant i \leqslant n-1$ and consider the new matrix $M_n(z)$ as a matrix over the polynomial ring $\mathbb{C}[z_1, \dots, z_{n-1}, x]$ with independent variables. Apparently $\det M_n(z)$ has degree $n-4$ as a polynomial in x. Assume therefore

$$\det M_n(z) = f_{n-4}x^{n-4} + \cdots + f_1x + f_0.$$

Let

$$r_{ij} = \det \begin{bmatrix} z_{i+1} & z_i \\ z_{j+1} & z_j \end{bmatrix} \quad \text{for } 1 \leqslant i < j \leqslant n-2$$

be the 2×2-minors of the first two columns of $M_n(z)$. Let furthermore $S = \mathbb{C}[z_0, \dots, z_{n-1}]$ be a polynomial ring with additional variable z_0 and put

$$I = \left\langle f_0, \dots, f_{n-4}, 1 - z_0 \cdots z_{n-1} \right\rangle_S,$$
$$J = \left\langle \{r_{ij} : 1 \leqslant i < j \leqslant n-2\} \right\rangle_S,$$

and let $\mathscr{V}(I)$, $\mathscr{V}(J)$ be the corresponding varieties in $\mathbb{C}^n$.

Now the TP-conjecture is equivalent to the conjecture

$$\mathscr{V}(I) \subseteq \mathscr{V}(J). \tag{16}$$

The converse inclusion is trivial.

Note that the equation $1 - z_0 \cdots z_{n-1}$ is included because of the requirement in 1.1 that all $c_i \neq 0$ and also that linear dependence of the first two columns of $M_{n,0}$ is equivalent to the vanishing of the determinants

$$\begin{bmatrix} c_{i+1} & c_i \\ c_{j+1} & c_j \end{bmatrix} \quad \text{for } 1 \leqslant i < j \leqslant n-2.$$

Note also that because of Hilbert's Nullstellensatz (16) is equivalent to saying $J \subseteq \sqrt{I}$.

Test for the TP-conjecture 3.3. *Let z be an additional variable. Compute bases for the test-ideals $\mathscr{T}_k = \langle I, 1 - zr_{1k}\rangle_{S[z]}$ for $2 \leqslant k \leqslant n-2$. If all $n-3$ bases contain a constant, then* (16) *and thus also the TP-conjecture are true.*

Proof. If $\mathscr{T}_k$ has a constant in its basis, then $\mathscr{V}(\mathscr{T}_k) = \emptyset$ in $\mathbb{C}^{n+1}$. This means, that no solution $\left(\frac{1}{c_1 \cdots c_{n-1}}, c_1, \ldots, c_{n-1}\right)$ from $\mathscr{V}(I)$ can be extended to one in $\mathscr{V}(\mathscr{T}_k)$. If there exist a solution in $\mathscr{V}(I)$ s.t. $r_{1k}(c_1, c_2, c_k, c_{k+1}) \neq 0$, then

$$\left(\frac{1}{c_1 \cdots c_{n-1}}, c_1, \ldots, c_{n-1}, \frac{1}{r_{1k}(c_1, c_2, c_k, d_{k-1})}\right) \in \mathscr{V}(\mathscr{T}_k),$$

a contradiction. This argument applied for $2 \leqslant k \leqslant n-2$ proves $\mathscr{V}(I) \subseteq \mathscr{V}(r_{12}, \ldots, r_{1n-2})$. Since $J = \langle r_{12}, \ldots, r_{1n-2}\rangle_S$ (see e.g.: [15, Example 1.4]) this proves (17). □

Based on the preceding algebraic test the TP-conjecture could be verified up to $n = 7$ with Maple 9 on a PC under Windows-XP. Since for $n = 7$ computing time with appropriately weighted monomial orders for the test-ideals the computing times became long (>5 minutes), the results have been confirmed with the help of Singular 2-3-0 [7] on the same PC-installation (<2 seconds). For $n = 8$ only the first test-ideal could be checked successfully, whereas for the second one with Singular 2-3-0 after hours there was still no result. The limit $n = 7$ could not be broken even on a Pentium PC, 1 GB RAM, 2.4 GigaHertz under Linux.

4. $\mathbb{C}[y]$ is an FC_5-ring

Since the TP-conjecture can be proved for $n = 5$ the case $d_{vv} = 0$ is completely covered. When $d_{vv} \neq 0$, then the condition (9) is required in [10] to guarantee that d_u, d_v, d_{vv} do not have certain common transcendental zeros to be defined below. We will explain this now and show how to avoid such zeros. Once this is achieved (7) can constructively be solved exactly as in [10, p. 557 ctd.].

For $n=5$ the polynomials are as follows:

$$\left.\begin{aligned} d_u &= {q_{11}}^3 b + \left(-2ac_1c_3 - a{c_2}^2\right){q_{11}}^2 q_{21} + 3q_{11}{q_{21}}^2 a{c_1}^2 c_2 - {q_{21}}^3 a{c_1}^4, \\ d_v &= 2aq_{11}\left(c_3{q_{11}}^2 - 2c_2q_{21}c_1q_{11} + {c_1}^3{q_{21}}^2\right), \\ d_{vv} &= \left({c_3}^2 a - c_2 b\right)q_{11} + \left({c_1}^2 b - 2c_1c_2ac_3 + a{c_2}^3\right)q_{21}. \end{aligned}\right\} \tag{17}$$

The construction of a solution $q_{11}, q_{21}, u, v \in \mathbb{C}[y]$ for (7) proceeds as follows. Let g be the normed gcd of all coefficients of d_u, d_v, d_{vv} considered as polynomials in q_{11}, q_{22} and put

$$g\overline{d}_u = d_u, \quad g\overline{d}_v = d_v, \quad g\overline{d}_{vv} = d_{vv}. \tag{18}$$

Since b is always a coefficient of d_u and since a, b are coprime, automatically $\gcd(a, g) = 1$ and therefore (see [8]) one can find $q_{11}, \lambda \in \mathbb{C}[y]$ s.t.

$$a{q_{11}}^5 + g\lambda = 1. \tag{19}$$

Moreover, q_{11} can be chosen to be coprime to any prescribed polynomial and if $\gcd(q_{11}, a) = 1$ then d_u is not the zero polynomial independently of q_{21}. By these facts it is sufficient to solve the equation

$$\overline{d}_u u + \overline{d}_v v + \overline{d}_{vv} v^2 = \lambda$$

locally at zeros of $\overline{d}_u$ if possible and later lift the results. This is only feasible if there are no common zeros of $\overline{d}_u, \overline{d}_v, \overline{d}_{vv}$ which might not be a zero of λ at the same time.

We now choose $q_{21} := t \in \mathbb{C}$ to be a constant polynomial and t to be transcendental over the subfield Z of $\mathbb{C}$ which is generated by the coefficients of $a, b, c_1, c_2, c_3, q_{11}$. We also choose q_{11} to be coprime to the coefficients of $\overline{d}_u, \overline{d}_v, \overline{d}_{vv}$ as polynomials in q_{11}, q_{21} and to a. Automatically q_{11}, q_{21} are coprime in $\mathbb{C}[y]$. Suppose y_0 is a common zero of $\overline{d}_u, \overline{d}_v, \overline{d}_{vv}$. There are two cases:

Case 1. y_0 is algebraic over Z. Then $q_{11}(y_0) \neq 0$, otherwise by $d_u(y_0) = 0$ one would have $a(y_0)c_1(y_0)^4 = 0$. But a, c_1, q_{11} are coprime. Since t is transcendental over Z and $q_{11}(y_0) \neq 0$, all coefficients of $\overline{d}_u, \overline{d}_v, \overline{d}_{vv}$ as polynomials in q_{11}, q_{21} must vanish, which is impossible by the choice of g.

Case 2. y_0 is transcendental over Z. Note that y_0 is always algebraic over $Z(t)$. It is not obvious whether such zeros are possible at all, but we will show now how to avoid this case by manipulating the matrix pair.

Lemma 4.1. *Given $A = C(a)$, $B = D(c_1, c_2, c_3, b)$ from* (4), *then there are unimodular matrices $P \in \mathbb{C}[y]^{5\times5}$, $Q \in \mathbb{C}[y]^{2\times2}$ and a matrix $F \in \mathbb{C}[y]^{2\times5}$ such that*

$$P(A + BF)P^{-1} = \widetilde{A} \quad \textit{and} \quad PBQ = \widetilde{B}, \tag{20}$$

where $\widetilde{A} = C(\widetilde{a})$ *and* $\widetilde{B} = D(\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3, \widetilde{b})$ *with*

$$\begin{aligned} &\widetilde{c}_1 = {c_3}^3\varphi^2 + 2c_2c_3\varphi + c_1, \quad \widetilde{c}_2 = \varphi {c_3}^2 + c_2, \\ &\widetilde{c}_3 = c_3, \quad \widetilde{b} = b, \quad \widetilde{a} = a + b\varphi \end{aligned} \tag{21}$$

and where $\varphi \in \mathbb{C}[y]$ *can be chosen arbitrarily.*

Proof. The following matrices do the job and $\varphi \in \mathbb{C}[y]$ can be chosen freely:

$$P = \begin{bmatrix} 1 & \varphi c_3 & \varphi^2 {c_3}^2 + \varphi c_2 & \varphi c_3(\varphi^2 {c_3}^2 + \varphi c_2) + \varphi^2 c_2 c_3 + \varphi c_1 & 0 \\ 0 & 1 & \varphi c_3 & \varphi^2 {c_3}^2 + \varphi c_2 & 0 \\ 0 & 0 & 1 & \varphi c_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} -\varphi c_3 & -\varphi^2 {c_3}^2 - \varphi c_2 & -\varphi^3 {c_3}^3 - 2\varphi^2 c_2 c_3 - \varphi c_1 & F_{14} & 0 \\ 0 & 0 & 0 & \varphi & 0 \end{bmatrix},$$

where

$$F_{14} = -\varphi^4 {c_3}^4 - 3\varphi^3 c_2 {c_3}^2 + (-2c_3c_1 - {c_2}^2)\varphi^2,$$

$$Q = \begin{bmatrix} 1 & -{c_3}^4\varphi^3 - 3{c_3}^2 c_2 \varphi^2 - (2c_3c_1 + {c_2}^2)\varphi \\ 0 & 1 \end{bmatrix}. \qquad \square$$

Transformations of the type (20) for general matrix pairs (A, B) of fixed dimension form the so-called feeback group [3, p. 116]. In our situation we want the transformations to maintain the special structure of (A, B). Such transformations have been used systematically for the first time in [4] and later in [10]. In [12] explicit formulae are derived for the entries in Lemma 4.1 and arbitrary n.

It is a well-known fact that a pair (A, B) is cyclizable if and only if any pair obtained from (A, B) by operations of the feedback group is cyclizable. Therefore later on we examine $\overline{d}_u, \overline{d}_v, \overline{d}_{vv}$ for the new polynomials $\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3, \widetilde{a}, \widetilde{b}$ and choose φ appropriately.

Consider now the expression

$$\begin{aligned} d_{\text{new}} &= 2q_{11}d_u + c_1q_{21}d_v \\ &= 2{q_{11}}^4 b - 2{q_{11}}^3 q_{21} a c_1 c_3 - 2{q_{11}}^3 q_{21} a {c_2}^2 + 2{q_{11}}^2 {q_{21}}^2 a {c_1}^2 c_2 \end{aligned}$$

which results from (17) and is of degree 2 in $q_{21} = t$. If y_0 is a common zero of $\overline{d}_u, \overline{d}_v, \overline{d}_{vv}$, then also $d_{\text{new}}(y_0) = 0$. If y_0 is transcendental over Z, then this quadratic equation in $q_{21} = t$ is nontrivial, since $q_{11}, a, b, c_1, c_2 \neq 0$. Its discriminant is

$$\left({c_3}^2{c_1}^2 + 2c_3c_1{c_2}^2 + {c_2}^4\right)a^2 - 4c_2{c_1}^2 ab \tag{22}$$

and t is algebraic over $Z(y_0)$. Inserting the new polynomials from Lemma 4.1 into (22) gives a new discriminant whose leading φ-term is

$$4bc_3{}^6\left(ac_3{}^2 - bc_2\right)\varphi^5 \tag{23}$$

if the coefficient of φ^5 is nonzero. Assume $ac_3{}^2 - c_2b = 0$, then this equality is valid in $\mathbb{C}[x]$, since y_0 is transcendental over Z and c_1, c_2, c_3, a, b are all nonzero. But then also the coefficients of q_{21} in d_{vv} must be the zero polynomial (see (17)), which means $d_{vv} = 0$, contradicting our assumption.

This shows that by suitable choice of the degree of φ one can make the degree of the discriminant uneven. We now choose such a φ with coefficients from Z. Then t cannot be from $Z(y_0)$ contradicting the linear equation $d_{vv}(y_0) = 0$.

Altogether we have shown that we can choose the pair (A, B) via the feedback group s.t. Case 2 also cannot occur.

Now (7) can be solved along the lines in [10] and the pair (A, B) becomes cyclizable also when $d_{vv} \neq 0$. As a consequence $\mathbb{C}[y]$ is an FC_5-ring.

References

[1] J. Brewer, J. Bunce, F. van Vleck, Linear Systems Over Commutative Rings, Dekker, 1986.

[2] J. Brewer, D. Katz, W. Ullery, Pole assignability in polynomial rings, power series rings, and Prüfer-domains, J. Algebra 106 (1987) 256–286.

[3] R. Bumby, E.D. Sontag, H.J. Sussmann, W. Vasconcelos, Remarks on the pole-shifting problem over commutative rings, J. Pure Appl. Algebra 20 (1981) 113–127.

[4] J. Dübelde, Normformenproblem und Koeffizientenzuweisung bei Systemen über Ringen, Diplomarbeit, University of Oldenburg, 1994.

[5] M.L.J. Hautus, A simple proof of Heymann's Lemma, IEEE Trans. Autom. Control 22 (1977) 885–886.

[6] M. Heymann, Comments on pole assignment in multi-input controllable linear systems, IEEE Trans. Autom. Control AC-13 (1968) 748–749.

[7] H.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 2.0.3, A computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern, 2002. Available from <http://www.singular.uni-kl.de>.

[8] W. Schmale, Three-dimensional feedback cyclization over $\mathbb{C}[y]$, Syst. Control Lett. 12 (1989) 327–330.

[9] W. Schmale, A symmetric approach to cyclization for systems over $\mathbb{C}[y]$, Linear Algebra Appl. 263 (1997) 221–232.

[10] W. Schmale, Matrix cyclization over complex polynomials, Linear Algebra Appl. 275/276 (1998) 551–562.

[11] W. Schmale, P. Sharma, Singularity of a Töplitz pencil, Image, Bull. Int. Linear Algebra Soc. 30 (2003) 36.

[12] K. Rajeshwari, K. Santaram, P. Sharma, A note on feedback cyclicity over $\mathbb{C}[y]$, Linear Algebra Appl. 320 (2000) 1–13.

[13] E.D. Sontag, Mathematical Control Theory, Deterministic Finite Dimensional Systems, Springer, 1990.

[14] E.D. Sontag, Linear systems over commutative rings: a survey, Ricerce di Automatica 7 (1976) 1–34.

[15] B. Sturmfels, Gröbner Bases and Convex Polytopes, AMS University Lecture Series, vol. 8, 1996.