The inverse of band preserving and disjointness preserving operators

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Communicated by Prof. A.C. Zaanen at the meeting of February 24, 1992

Recently it was shown by C.B. Huijsmans and B. de Pagter [6] (see also the book by P. Meyer-Nieberg [10, Theorem 3.1.10]) that if $T$ is an orthomorphism on an Archimedean vector lattice $E$ (denoted by $T \in \text{Orth}(E)$ and meaning that $T$ is order bounded and band preserving) and $T$ is bijective (i.e., $T$ is invertible in the algebra $L(E)$ of all linear operators on $E$), then $T^{-1} \in \text{Orth}(E)$ as well. In other words, Orth($E$) is a full subalgebra of $L(E)$. Since a result due to Y.A. Abramovich, A.I. Veksler and A.V. Koldunov [2] (see also [8] or [12]) states that band preserving mappings from a Banach lattice $E$ into itself are automatically order (and norm) bounded (actually, in this case Orth($E$) = $Z(E)$, the center of $E$), we get as a corollary that the inverse of a bijective band preserving operator on a Banach lattice is band preserving as well.

However, operators between Banach lattices which are merely disjointness preserving need not be order or norm bounded (see [1] for a counterexample). Note in this connection that Y.A. Abramovich showed in [1] (see also [12]) that any norm bounded disjointness preserving operator between two Banach lattices is order bounded. Furthermore, it is of independent interest to observe that a disjointness preserving operator $T$ between two Archimedean vector lattices $E$ and $F$ which is, in addition, $\sigma$-order continuous is automatically order bounded. Indeed, it was shown by P.T.N. McPolin and A.W. Wickstead in [9, Theorem 2.1] that if $T : E \to F$ is disjointness preserving and satisfies (*):

(*) $\bigwedge_{n=1}^{\infty} |Tx_n| = 0$ for all sequences $\{x_n\}_{n=1}^{\infty}$ in $E'$ converging relatively uniformly to 0,
then $T$ is order bounded. Obviously, (*) is satisfied if $T$ is $\sigma$-order continuous. Consequently, a band preserving operator on an Archimedean vector lattice is order bounded if and only if it is ($\sigma$-) order continuous. Notice also that any norm bounded operator between two Banach lattices satisfies (*), so a norm bounded disjointness preserving operator between two Banach lattices is order bounded (see above).

It was shown by W. Arendt in [3, Proposition 2.7] that a bijective order bounded disjointness preserving operator between two Banach lattices has an order bounded disjointness preserving inverse. This result can easily be extended to Archimedean vector lattices.

**Theorem 1.** Let $E, F$ be Archimedean vector lattices and $T : E \to F$ be a bijective order bounded disjointness preserving operator. Then $T^{-1} : F \to E$ is also order bounded and disjointness preserving. Moreover, $|T|$ and $|T^{-1}|$ exist and satisfy $|T^{-1}| = |T|^{-1}$.

**Proof.** It is a well-known result due to M. Meyer [10] (see also S.J. Bernau [5] or B. de Pagter [12]) that under the present hypotheses $T^+, T^-$ and $|T|$ exist, are lattice homomorphisms and satisfy

\begin{align*}
(1) \quad & (Tx)^+ = T^+x, \quad (Tx)^- = T^-x \quad (x \in E^+) \\
(2) \quad & |Tx| = |T||x| = |T||x| \quad (x \in E).
\end{align*}

We show first that $|T|$ is injective. Indeed, if $|T|x = 0$, then by (2), $Tx = 0$, so $x = 0$. Next we claim that $|T|$ is surjective. To this end, take $y \in F$. There exist $x_1, x_2 \in E$ such that $Tx_1 = y^+, Tx_2 = y^-$ and thus

\[
|T|(|x_1| - |x_2|) = |Tx_1| - |Tx_2| = y^+ - y^- = y,
\]

which proves the claim. We have therefore that $|T|$ is a lattice isomorphism, so $|T|^{-1}$ is a lattice isomorphism as well and consequently $|T|^{-1} \geq 0$. In order to verify that $T^{-1}$ is disjointness preserving it has to be shown that $|Tx_1| \land |Tx_2| = 0$ in $F$ implies $|x_1| \land |x_2| = 0$ in $E$. This follows immediately from

\[
|x_1| \land |x_2| = |T|^{-1}|T||x_1| \land |T|^{-1}|T||x_2| \\
= |T|^{-1}|Tx_1| \land |T|^{-1}|Tx_2| \\
= |T|^{-1}(|Tx_1| \land |Tx_2|) \\
= |T|^{-1}(0) = 0.
\]

Next, if $|x| \leq z (z \in E^+)$, then $|T|^{-1}|x| \leq |T|^{-1}z$, as $|T|^{-1} \geq 0$. Put $y = T^{-1}x (x = Ty)$. Since $|x| = |T||y|$, we have $|T^{-1}x| = |y| = |T|^{-1}|x| \leq |T|^{-1}z$ for all $x \in E$ satisfying $|x| \leq z$, showing that $T^{-1}$ is order bounded. Consequently, $|T^{-1}|$ exists by Meyer's theorem. Hence, it follows from

\[
|x| = |TT^{-1}x| = |T||T^{-1}||x|
\]
that \( |T||T^{-1}| = I_I \). Similarly, \( |T^{-1}| = I_{E} \) and therefore \( |T^{-1}| = |T|^{-1} \). The proof is complete. \( \square \)

On account of the above observations the following two questions seem natural to ask:

(a) if \( E \) is an Archimedean vector lattice and \( T : E \rightarrow E \) is a bijective band preserving operator in \( E \), is \( T^{-1} \) band preserving as well?

(\( \beta \)) if \( E \) and \( F \) are Archimedean vector lattices (or even Banach lattices) and \( T : E \rightarrow F \) is bijective and disjointness preserving, is \( T^{-1} : F \rightarrow E \) also disjointness preserving? Problem \( (\beta) \) was proposed by Y.A. Abramovich in the problem section of [7].

The purpose of this note is to show that (a) has an affirmative answer in the following two cases:

(i) \( E \) has the principal projection property

(ii) \( E \) is relatively uniformly complete.

Of course, (ii) generalizes the fore-mentioned result that the inverse of an invertible band preserving operator on a Banach lattice is also band preserving.

**Theorem 2.** Let \( E \) be a vector lattice with the principal projection property and \( T : E \rightarrow E \) be an invertible band preserving operator on \( E \). Then \( T^{-1} \) is also band preserving.

**Proof.** Observe that \( T^{-1} \) is band preserving if and only if \( T^{-1}x \in \{x\}^{dd} \) for all \( x \in E \). Decompose

\[
T^{-1}x = x_1 + x_2(x_1 \in \{x\}^{dd}, x_2 \in \{x\}^{d}).
\]

By hypothesis, \( T(\{x\}^{dd} \subset \{x\}^{dd} \) and \( T(\{x\}^{d}) \subset \{x\}^{d} \). Hence, \( Tx_1 \in \{x\}^{dd} \) and \( Tx_2 \in \{x\}^{d} \). It follows from \( Tx_2 = x - Tx_1 \) that \( Tx_2 \in \{x\}^{dd} \) as well. Hence, \( Tx_2 = 0 \), so injectivity of \( T \) yields \( x_2 = 0 \). Therefore \( T^{-1}x = x_1 \in \{x\}^{dd} \) and we are done. Observe that we did not use the surjectivity of \( T \) in the proof. \( \square \)

Before stating and proving the next theorem we first remind the reader of the following result due to P.T.N. McPolin and A.W. Wickstead [9, Theorem 2.2]: If \( E \) is an Archimedean relatively uniformly complete vector lattice and \( T \) is a non-order bounded band preserving operator on \( E \), then \( E \) contains a universally \( \sigma \)-complete principal projection band \( B \) such that the restriction \( T/B \) is not order bounded. Recall that a vector lattice is termed universally \( \sigma \)-complete whenever it is Dedekind \( \sigma \)-complete and laterally \( \sigma \)-complete (the latter meaning that every disjoint sequence of positive elements has a supremum).

**Theorem 3.** Let \( E \) be an Archimedean relatively uniformly complete vector lattice and \( T : E \rightarrow E \) bijective and band preserving. Then \( T^{-1} \) is also band preserving.
PROOF. It suffices to show that $Tx \perp y$ implies $x \perp y$. Since $Tx^+ \perp Tx^-$ we have
\[ |Tx| = |Tx^+ - Tx^-| = |Tx^+ + Tx^-| = |T|x||. \]
Replacing therefore, if necessary, $x$ by $|x|$ and $y$ by $|y|$ we may assume without loss of generality that $x, y \in E^+$. Write $z = x \wedge y$. Since $T$ is band preserving, $T$ leaves $\{z\}_{dd}$ invariant. Therefore, the restriction mapping $S = T/\{z\}_{dd}$ maps $\{z\}_{dd}$ into itself. We distinguish two cases:

(I) $S$ is order bounded (so $S \in \text{Orth} (\{z\}_{dd})$). Fix $n \in \mathbb{N}$. We assert that $\{nz \wedge x\}_{dd} = \{z\}_{dd}$. Evidently, $0 \leq nz \wedge x \leq nz$ implies $\{nz \wedge x\}_{dd} \subseteq \{z\}_{dd}$. Conversely, if $u \vee nz \wedge x = 0$, then $u \wedge nz \wedge x = u \cap nz = 0$, so $u \in \{z\}_{dd}$, showing that $\{nz \wedge x\}_{dd} \subseteq \{z\}_{dd}$. This proves the assertion. Since $x - (nz \wedge x) = (nz - x)^- \perp (nz - x)^+$, the band preserving property of $T$ yields $T(x - (nz \wedge x)) \perp (nz - x)^+$. But $Tx \perp y$ and $0 \leq z \leq y$ implies $Tx \perp z$. Consequently, $Tx \perp (nz - x)^+$. Combining these two results we get
\[ T(nz \wedge x) = S(nz \wedge x) \perp (nz - x)^+. \]
Since $(nz - x)^+ = nz - nz \wedge x$ and $\{nz \wedge x\}_{dd} = \{z\}_{dd}$ we have that $(nz - x)^+ \in \{nz \wedge x\}_{dd}$. Order continuity of $S$ gives $Sp \perp (nz - x)^+$ for all $p \in \{nz \wedge x\}_{dd}$. In particular, $S(nz - x)^+ \perp (nz - x)^+$. Another use of the band preserving property of $S$ yields $S(nz - x)^+ \perp S(nz - x)^+$, i.e., $S(nz - x)^+ = 0$. This shows $(nz - x)^+ = 0$, as $S$ is injective, i.e., $nz \leq x$. This holds for all $n \in \mathbb{N}$, so by the Archimedean property $z = x \wedge y = 0$ which was to be proved.

(II) $S$ is not order bounded (so clearly $z > 0$). We will show that this case cannot occur. By the previously cited McPolin-Wickstead result [9, Theorem 2.2], $\{z\}_{dd}$ contains a universally $\sigma$-complete band $B \neq \{0\}$. In [4, Theorem 1] S.J. Bernau presents an elementary proof of a result due to A.I. Veksler and V.A. Geiler [13], viz. that an Archimedean laterally $\sigma$-complete vector lattice has the principal projection property. Actually, a close inspection of Bernau’s proof shows that any laterally $\sigma$-complete principal band in an Archimedean vector lattice is a projection band. Particularly, the above band $B$ is a projection band in $E$. Denote the band projection of $E$ onto $B$ by $P$.

Since $\{z\}_{dd} \subseteq B^d$ and $Tx \perp z$ we have $PTx = 0$. Also, $Px \neq 0$, because $B \subseteq \{z\}_{dd} \subseteq \{x\}_{dd}$ implies $x \in B^d$. For any $y \in E$ we have $TPy \in B$ and $T(I - P)y \in B^d$, so
\[ Ty = TPy + T(I - P)y \]
is the decomposition of $Ty$ into an element of $B$ and an element of $B^d$. It follows immediately that $TPy = PTy$ for all $y \in E$. Specifically, $TPx = PTx = 0$. Injectivity of $T$ yields $Px = 0$. This is the desired contradiction so that the second case cannot happen and we are through. 

REFERENCES