An Af*-Af geometry of order q is a residually connected rank three geometry where planes are dual affine planes and stars of points are affine planes of order q. We prove that such a geometry is necessarily obtained from the Klein quadric $Q^*_q$ of $PG(5, q)$ deleting the points of a hyperplane and considering as points the elements of one of the two systems of maximal subspaces of $Q = Q^*_q$, as lines the points of $Q$, and as planes the elements of the other system.

The deleted hyperplane is tangent to $Q$ if and only if the Af*-Af geometry obtained satisfies property (PL₁) (i.e. there is a unique plane on every point-line antiflag).

When (PL₁) is satisfied, some generalizations are obtained for L*-L (resp. N*-L) geometries (i.e. residually connected rank three geometries where planes are dual linear spaces (resp. dual nets) and stars of points are linear spaces). In particular, this yields a characterization of $H^* q$ and $T^* q(K)$, where $K = AG(2, q)$, in the context of rank 3 partial geometries. Furthermore, it leads to some classification results for other rank $n > 3$ diagrams related to what we call the rank $n H^* q$ ($n \geq 3$, see Examples 5.1 and 5.4 for the definition).

1. Introduction

We consider special rank 3 partial geometries using the terminology of Hughes [14, 15]: the $pG.L$ geometries of order $(r, s, t)$ and parameter $x$ (for previous investigations of special rank 3 partial geometries, see for instance [16, 17]). These are residually connected Buekenhout geometries belonging to the following diagram:

$$
(pG.L) \quad \circ \quad \circ \quad \circ \quad L \quad (1 \leq r, s, t < \infty),
$$

points lines planes
where the stroke

\[ \begin{array}{ccc}
  \circ & r & s \\
  \circ & \circ & s
\end{array} \]

denotes the class of partial geometries of order \((r, s)\) and parameter \(\alpha\) (\(pG_\alpha (r, s)\) for short) while

\[ \begin{array}{ccc}
  \circ & L & s \\
  \circ & \circ & t
\end{array} \]

is the class of 2-(\(st+s+1, s+1, 1\)) designs, namely linear spaces of order \((s, t)\), or, equivalently, partial geometries of order \((s, t)\), with \(\alpha = s + 1\).

We recall that a partial geometry of order \((r, s)\) and parameter \(\alpha\) (briefly \(pG_\alpha (r, s)\)) is an incidence structure with \(r + 1\) points on a line, \(s + 1\) lines on a point, two points on at most one line, such that for every antiflag \((P, l)\), the number of points in the line \(l\) collinear with the point \(P\) is a constant \(\alpha \neq 0\).

It is easily seen that the following property always holds in \(\Gamma\), because of the very features of the diagram \(pG.L\), which \(\Gamma\) belongs to:

(\(LL\)) any two distinct points are incident with at most one common line.

As \((LL)\) holds, phrases as ‘the point \(P\) belongs to the line \(l\)’, ‘the line \(l\) passes through \(P\)’, etc. may be freely used. However, we will also use similar conventions for point–plane and line–plane incidences.

In particular, we shall be interested in the following cases:

(i) \(\alpha = r = q, s = q - 1\), for which we use the diagram

\[ \begin{array}{ccc}
  \circ & \circ & q \\
  \circ & q - 1 & q
\end{array} \]

where

\[ \begin{array}{ccc}
  \circ & \circ \\
  \circ & q - 1 & q
\end{array} \]

denotes the class of affine planes, as usual, while the symbol

\[ \begin{array}{ccc}
  \circ & \circ \\
  \circ & q & q - 1
\end{array} \]

denotes the class of dual affine planes of order \(q\).

(ii) \(\alpha = r = s + 1 = q\), in which we use the diagram

\[ \begin{array}{ccc}
  \circ & \circ & L \\
  \circ & q - 1 & t
\end{array} \]

(iii) \(\alpha = s + 1\), in which we use the diagram

\[ \begin{array}{ccc}
  \circ & \circ & L \\
  \circ & q - 1 & t
\end{array} \]
where

\[
\begin{array}{c}
\circ \quad \circ \\
L^* \\
r \quad \alpha - 1
\end{array}
\]

denotes the class of dual linear spaces of order \((r, \alpha - 1)\), or equivalently partial geometries of order \((r, s)\) with \(\alpha = s + 1\).

(iv) \(\alpha = r\), for which we use the diagram

\[
\begin{array}{c}
\circ \quad \circ \\
N^* \\
r \quad s
\end{array}
\]

L

\[
\begin{array}{c}
\circ \quad \circ \\
\alpha \quad s \quad t
\end{array}
\]

where

\[
\begin{array}{c}
\circ \quad \circ \\
N^* \\
r \quad s
\end{array}
\]

is the class of dual nets. Clearly,

\[
\begin{array}{c}
\circ \quad \circ \\
N \\
r \quad s
\end{array}
\]

denotes the class of nets of order \(s + 1\) and degree \(\alpha\) (deficiency \(s + 1 - \alpha\)).

If \(\Gamma\) is a \(pG.L\) geometry of order \((r, s, t)\) and parameter \(\alpha\) we denote by \(S(\Gamma)\) the point–line system of \(\Gamma\), namely the geometry of points and lines of \(\Gamma\) with the incidence inherited from \(\Gamma\). Clearly, (LL) amounts to saying that \(S(\Gamma)\) is a partial plane (as in [1, 2]). \(S(\Gamma)\) has \(r + 1\) points on each line and \(s(t + 1) + 1\) lines on each point, i.e. it has order \((r, s(t + 1))\). It is immediately seen that \(S(\Gamma)\) is a \(pG_s(r, s(t + 1))\) if and only if the following property holds in \(\Gamma\):

(PL, ) If \((P, l)\) is a nonincident point–line pair, then there is exactly one plane \(\pi = \langle P, l \rangle\) incident with both \(P\) and \(l\).

**Lemma 1.1.** Let \(\Gamma\) be a \(pG.L\) geometry. Property (PL, ) implies that the following property holds in \(\Gamma\):

(LH) If a line \(l\) has 2 points \(P\) and \(Q\) in common with a plane \(\pi\), then \(l\) is incident with \(\pi\).

**Proof.** Let \(R\) be a point of \(\pi\) collinear with both \(P\) and \(Q\) (\(R\) exists since the residue \(\Gamma^R\) of \(\pi\) is a \(PG_s, \alpha \geq 1\)). If \(R\) is on \(l\) then the line \(PR\) of \(\pi\) is necessarily \(l\), by (LL); hence, \(l\) is incident with \(\pi\). If \(R\) is not on \(l\), and \(l'\) is the line \(QR\), then in the residue \(\Gamma^Q\) of \(Q\) there is a unique plane \(\pi'\) on \(l\) and \(l'\). Clearly, \(\pi'\) and \(\pi\) both contain the point \(P\) of \(l\) and the line \(l'\). This implies \(\pi = \pi'\), since the plane on \(P\) and \(l'\) is unique by (PL, ).

Other equivalent formulations for the property of Lemma 1.1, denoted as (LH) after [22, Ch. 6], are in [9]. Here we just note that (LH) is a weak intersection property and that in the whole of this paper we do not assume the intersection property of [2]. Even (LH) is not assumed in the first part of the paper.

In the rest of the paper we shall need the following properties and definitions.
**Lemma 1.2.** Let $\Gamma$ be a $pG.L$ of order $(r, s, t)$ and parameter $\alpha$.

(i) Every plane $\pi$ of $\Gamma$ contains at most $m = 1 + rs/\alpha$ pairwise noncollinear points. A set of pairwise noncollinear points of $\pi$ has maximum size $m$ if and only if it is an ovoid of $\pi$ (i.e. it meets every line of $\pi$ in one point).

(ii) If $\Gamma$ satisfies $(PL_1)$, then on two noncollinear points $P$ and $Q$ there are exactly

$$\frac{s(t+1)+1}{s+1} = \frac{st}{s+1} + 1$$

planes. In particular $s+1$ divides $t$; hence, $s+1 \leq t$.

(iii) If $\Gamma$ satisfies $(PL_1)$, then two planes $\pi \neq \pi'$ with $|\pi \cap \pi'| \geq 2$ meet either in a line or in a set of pairwise noncollinear points.

**Proof.** (i) Every line of $\pi$ meets a given set $K$ of pairwise noncollinear points either in 0 or in 1 point (i.e. with the terminology of [8] or [7]: $K$ is a $(0,1)$-set in the $pG_s(r, s)$ defined by $\pi$).

Let $l$ be a line of $\pi$ on a point $P$ of $K$. On each of the $r$ points of $l$ different from $P$ there are $s$ lines different from $l$, each line with at most 1 point of $K$. Since every point of $K$ is counted at most $s$ times in the product $rs$ we have $|K| \leq 1 + rs/\alpha$, where equality holds if and only if there are no lines external to $K$ (i.e. $K$ is an ovoid of $\pi$).

(ii) We count pairs $(l, \pi)$ with $l$ a line on $Q$ and $\pi$ the plane on $l$ and $P$, which is unique, by $(PL_1)$.

(iii) Let $P$ and $Q$ be two distinct points in $\pi \cap \pi'$. If $P \sim Q$ (i.e. $P$ is collinear with $Q$), the line $l = PQ$ is in both $\pi$ and $\pi'$, by (LH) (see Lemma 1.1); hence, $l \subseteq \pi \cap \pi'$. Conversely, a point $R \in \pi \cap \pi'$ is necessarily on $l$, because otherwise there is a unique plane on $R$ and $l$ (by $(PL_1)$) so that $\pi = \pi'$, which is impossible. Therefore, $l = \pi \cap \pi'$. This completes the proof. 

**Definition 1.3.** A $pG.L$ geometry $\Gamma$ (of order $(r, s, t)$ and parameter $\alpha$) with $(PL_1)$ is said to be regular if it satisfies the following property:

(R) Two planes of $\Gamma$ not meeting in a line meet either in 0, 1 or in $m = 1 + rs/\alpha$ (pairwise noncollinear) points (i.e. in an ovoid).

Property (R), which plays an important role in this paper, generalizes to $pG.L$ geometries the notion of 'regularity' introduced in [26] for partial geometries satisfying the axiom of Pasch. We recall that the Pasch axiom can be stated as follows:

(P) If two lines $l_1 \neq l_2$ meet in a point $P$ and $m_1, m_2$ are two lines not on $P$, each meeting both $l_1$ and $l_2$, then $m_1$ and $m_2$ are concurrent.

Condition (R) is a generalization, since to obtain the regularity of [26] we also need the so-called weak linearity condition (see also [9]):

(WLC) If $P, Q, R$ are pairwise distinct points and $\pi_1, \pi_2, \pi_3$ are pairwise distinct planes such that all of $P, Q, R$ are incident with both $\pi_1$ and $\pi_2$ and all of $\pi_1, \pi_2, \pi_3$ are incident with both $P, Q$, then $R$ is incident with $\pi_3$.

In other words, if $\pi_3$ contains two distinct points of $\pi_1 \cap \pi_2$ and $\pi_1 \neq \pi_2$, then $\pi_1 \cap \pi_2 = \pi_3$. This justifies the following definition.
Definition 1.4. A $pG.L$ geometry $\Gamma$ (of order $(r,s,t)$ and parameter $\alpha$) with $(PL_1)$ is said to be the Thas–de Clerck-regular (briefly $TD$-regular) if it satisfies (R) and (WLC).

In Section 2, we classify $Af^*Af$ geometries. Namely, we show (see Theorem 2.5) that such a geometry, say $\Gamma$, is necessarily obtained from the Klein quadric $Q(q)$ of $PG(5,q)$ deleting the points of a hyperplane and considering as points the elements of one of the two systems of maximal subspaces of $\mathcal{P} = Q_q^5$, as lines the points of $\mathcal{P}$, and as planes the elements of the other system of maximal subspaces. Property $(PL_1)$ holds in $\Gamma$ if and only if the deleted hyperplane is tangent to $\mathcal{P}$.

Since $\mathcal{P}$ is the Grassmannian of the lines in $PG(3,q)$, the unique $Af^*Af$ geometry with $(PL_1)$ is also the geometry obtained from $PG(3,q)$ deleting a line $l$, all points and planes on $l$, and all lines meeting $l$. Then the point–line system is the well-known dual net $H_3^q$. Since $\mathcal{P}$ is the Grassmannian of the lines in $PG(3,q)$, the unique $Af^*Af$ geometry with $(PL_1)$ is also the geometry obtained from $PG(3,q)$ deleting a line $l$, all points and planes on $l$, and all lines meeting $l$. Then the point–line system is the well-known dual net $H_3^q$ satisfying (P) denoted by $H_3^q$.

It is natural to generalize and to consider the $Af^*L$ geometry $\mathcal{X}_q^\alpha$, satisfying $(PL_1)$, defined on the points and the lines of $PG(n,q)$ external to a fixed $PG(n-2,q) = W$ by the planes of $PG(n,q)$ meeting $W$ in a point. Now the point–line system $S(\Gamma)$ is the dual net $H_3^q$ satisfying (P), characterized by Thas and de Clerck [26].

In Section 3, we prove that every $L^*L$ geometry with $(PL_1)$ (thus, in particular, every $Af^*L$ geometry) satisfies (P). This, together with results in [26, Section 2], gives that partial geometries of order $(s,t)$ and parameter $\alpha$ satisfying (P) are exactly the point–line systems of $L^*L$ geometries of order $(s,\alpha-1,u)$ with $(PL_1)$, where $u = t/(\alpha-1) - 1$. Thas and de Clerck proved [26] (Section 4) that these always have subgeometries with $u = s$.

In Section 4, we find nonexistence conditions for $L^*L$ geometries of order $(s,\alpha-1,s)$ with $(PL_1)$. Incidentally, this proves that condition

$$2s > \left(\frac{s}{\alpha}\right)^4 - \left(\frac{s}{\alpha}\right)^3 + \left(\frac{s}{\alpha}\right)^2 + \left(\frac{s}{\alpha}\right) - 2$$

of the main characterization theorem of [26] is unessential in all cases, except $s = \alpha + \alpha^2$. Therefore, only the regularity condition of [26] (which is equivalent to our TD-regularity) must be assumed for a $pG_4(s,t)$ ($\alpha \neq 1$, $t + 1$, $s + 1$, $s \neq \alpha + \alpha^3$) satisfying (P) to be isomorphic to $H_3^q$.

In Section 5, we apply the results in Sections 3 and 4 to classify regular $L^*L$ geometries of order $(s,\alpha-1,u)$ with $(PL_1)$ and (WLC) (see Theorem 5.2). As an immediate consequence, we obtain that an $Af^*L$ geometry with $(PL_1)$ is always isomorphic to the above-defined rank 3 geometry $\mathcal{X}_q^\alpha$ (for some $n \geq 3$), and it is the $(0,1,2)$-truncation of a rank $n$ geometry $Af^*Af.A_{n-2}$

\[ \begin{array}{cccccccc}
Af^* & & & & & & & \\
\text{q} & \text{q}^{-1} & \text{q} & \text{q} & \text{q} & \text{q} & \text{q} & \text{q} \\
\end{array} \]

with $(PL_1)$ say $\Gamma$ (i.e. it is the geometry whose objects are the objects of $\Gamma$ with types of $\{0,1,2\}$).
We also classify in Section 5 $Af^*,Af,A_{n-2}$ geometries with $(PL_1)$; they are all isomorphic to the rank $n$ geometry denoted again by $\mathcal{H}^n_q$, by abuse of notation. This is obtained from the rank $3 \mathcal{H}^q_3$ defining as $j$-spaces, $j \geq 3$, the $j$-dimensional subspaces of $PG(n, q)$ meeting the fixed $PG(n-2, q)$ in a subspace of dimension $j - 2$.

Finally, in Section 6, we consider regular $N^*,L$ geometries with $(PL_1)$. This leads to a characterization of the well-known partial geometry $T_{2}(K)$ with $K = AG(2, q)$ as point–line system of the unique $N^*$. $Af$ geometry of order $(q-1, q-1, q)$ (see Example 6.3 and Theorem 6.4).

2. $Af^*,Af$ geometries

**Example 2.1.** Let $l$ be a line in $P = PG(3, q)$. Take as points and lines the points and lines of $P$ that do not meet $l$. Choose as planes the planes of $P$ meeting $l$ in a single point. The incidence is the natural one. The geometry $G_1$ obtained in this way belongs to the diagram

\[
\begin{array}{c}
Af^* \\
q
\end{array}
\hspace{1cm}
\begin{array}{c}
Af \\
q-1
\end{array}
\hspace{1cm}
\begin{array}{c}
\circ \\
1
\end{array}
\]

and is self-dual. Clearly, it satisfies $(PL_1)$ (hence $(LH)$) and the dual property of $(PL_1)$, namely:

$(PL_1)^*$ If $(\pi, l)$ is a nonincident plane–line pair, then there is exactly one point $P = \pi \cap l$ incident with both $P$ and $l$.

Furthermore, $G_1$ satisfies (R) and (WLC). The point–line system $S(G_1)$ of $G_1$ is a partial geometry $pG_q(q, q^2 - 1)$ satisfying (P). This well-known dual net is denoted by $H^3_q$.

Using the grassmannian of the lines in $P$ (i.e. the Klein quadric $\mathcal{Q} = Q^+(q)$), we can obtain Example 2.1 in the following way.

**Example 2.2.** Let $\mathcal{Q}$ be the hyperbolic quadric in $PG(5, q)$ and let $\mathcal{R}$ be the affine polar space obtained from $\mathcal{Q}$ excluding the tangent hyperplane $\tau = l^*$ at a point $l$ of $\mathcal{Q}$ (representative of a line $l$ of $PG(3, q)$). $\mathcal{R} = \mathcal{Q} - \tau$ belongs to the diagram

\[
\begin{array}{c}
Af \\
q-1
\end{array}
\hspace{1cm}
\begin{array}{c}
\circ \\
q
\end{array}
\hspace{1cm}
\begin{array}{c}
\circ \\
1
\end{array}
\]

In $\mathcal{R}$ we have two systems of planes, inherited from $\mathcal{Q}$, say $\mathcal{M}^+$ and $\mathcal{M}^-$ (representing stars of lines and ruled planes of $PG(3, q)$, respectively). The geometry $G_1$ is represented in $\mathcal{R}$ by the geometry $G_1'$ defined below.

Points, lines, and planes of $G_1'$ are the elements of $\mathcal{M}^+$, the points of $\mathcal{Q}$, and the elements of $\mathcal{M}^-$, respectively. A point (plane of $\mathcal{M}^+$) and a plane (plane of $\mathcal{M}^-$) of $G_1'$ are incident if and only if, as planes of $\mathcal{Q}$, they meet in $\mathcal{Q}$.
Clearly, \( \mathcal{G}_1 \cong \mathcal{G}_1 \); hence, \( \mathcal{G}_1 \) belongs to the diagram

\[
\begin{array}{c}
\mathbb{A}^* \quad \mathbb{A} \\
\circ \quad \circ \\
q \quad q-1 \quad q
\end{array}
\]

and property (PL_1) holds. In \( \mathcal{F} \), property (PL_1) has the following form:

(PL_1) If \((P, \pi)\) is a nonincident point–plane pair of \( \mathcal{F} \), then there is exactly one plane \( \pi' \) containing \( P \) and meeting \( \pi \) in a line.

We note that (PL_1) includes both properties (PL_1) and (PL_1)* of \( \mathcal{G}_1 \cong \mathcal{G}_1 \).

It is not difficult to see that there are \( q^4 \) lines in \( \mathcal{G}_1 \cong \mathcal{G}_1 \).

**Example 2.3.** Let \( \mathcal{F} = Q^+(q) - \sigma \) be the affine polar space obtained from the Klein quadric \( \mathcal{F} = Q^+_5(q) \) excluding a secant hyperplane \( \sigma \) of \( \mathbb{P}G(5, q) \).

We note that since \( \sigma \cap \mathcal{F} \) is now nonsingular \( Q_4(q) \), we are excluding from the \((q^2 + q + 1)(q^2 + 1)\) points of \( \mathcal{F} \) exactly \((q + 1)(q^2 + 1)\) points. Therefore, \( \mathcal{F} \) now has \( q^2(q^2 + 1) \) points.

As before, we have in \( \mathcal{F} \) two systems of planes, inherited from \( \mathcal{F} \), say \( M^+ \) and \( M^- \).

Define a geometry \( \mathcal{G}_2 \) as follows.

Points, lines, and planes of \( \mathcal{G}_2 \) are the elements of \( M^+ \), the points of \( \mathcal{F} \), and the elements of \( M^- \), respectively. A point (plane of \( M^+ \)) and a plane (plane of \( M^- \)) of \( \mathcal{G}_2 \) are incident if and only if as planes of \( \mathcal{F} \) they meet in a line of \( \mathcal{F} \).

Again, we have an \( \mathbb{A}^* \cdot \mathbb{A} \) geometry of order \((q, q-1, q)\). However, \( \mathcal{G}_2 \) has \( q^2(q^2 + 1) \) lines, while \( \mathcal{G}_1 \cong \mathcal{G}_1 \) has \( q^4 \) lines. Furthermore, (PL_1) and (PL_1)* do not hold in \( \mathcal{G}_2 \), or, equivalently, (PL_1) does not hold in \( \mathcal{F} \), as it is not difficult to verify directly (or see Example 2.4 and Corollary 2.6).

**Example 2.4.** Let \( W = W(q) \) be the symplectic variety in \( P = PG(3, q) \). Define a geometry \( \mathcal{G}_2 \) as follows. The points of \( \mathcal{G}_2 \) are the points of \( P \), the lines of \( \mathcal{G}_2 \) are the lines of \( P \) that are not on \( W \). The planes of \( \mathcal{G}_2 \) are the planes of \( P \). The incidence relation is the natural one, except that an incident point–plane pair \((P, \pi)\) of \( P \) is considered to be incident in \( \mathcal{G}_2 \) only if \( P \neq \pi \). The geometry \( \mathcal{G}_2 \) has diagram and parameters as follows:

\[
\begin{array}{c}
\mathbb{A}^* \quad \mathbb{A} \\
\circ \quad \circ \\
q \quad q-1 \quad q
\end{array}
\]

The point–line system of \( \mathcal{G}_2 \) is a well-known semi-partial geometry ([4, 6]). As for \( \mathcal{G}_2 \), we now have \( q^2(q^2 + 1) \) lines and (PL_1) does not hold (since if \((P, l)\) is an antiflag with \( l \subset P \), there is no plane incident with both \( P \) and \( l \)).

**Theorem 2.5.** If \( \Gamma \) is a geometry belonging to the diagram

\[
\begin{array}{c}
\mathbb{A}^* \quad \mathbb{A} \\
\circ \quad \circ \\
q \quad q-1 \quad q
\end{array}
\]
then one of the following holds:

(i) $\Gamma$ is isomorphic to the geometry $\mathcal{G}_1$ with $q^k$ lines of Example 2.2 ($\cong \mathcal{G}_1$, Example 2.1) and thus satisfies (PL$_1$),

(ii) $\Gamma$ is isomorphic to the geometry $\mathcal{G}_2$ with $q^2(q^2 + 1)$ lines of Example 2.3. Hence, $\Gamma$ does not satisfy (PL$_1$).

Proof. We consider the grassmannian with respect to the central node of the diagram of $\Gamma$, that is the geometry $Gr(\Gamma)$ defined as follows: points of the grassmannian are the lines of $\Gamma$, planes are planes of $\Gamma$ and stars of points of $\Gamma$, lines are the incident point–plane pairs $(P, \pi)$ of $\Gamma$. The point (line of $\Gamma$) $l$ of $Gr(\Gamma)$ is said to be incident to the line $(P, \pi)$ if and only if $P l \pi l \pi$ (where $l$ is the incidence in $\Gamma$).

Clearly, the diagram of $\Gamma$ implies that the grassmannian $Gr(\Gamma)$ belongs to the diagram $Af.C_2$ of order $(q - 1, q, 1)$, namely,

$\begin{array}{cccc}
& & Af & \\
q - 1 & q & 1 \\
& & Af & \\
\end{array}$

and satisfies (LL), as the following argument shows. If $(P, \pi)$ and $(R, \rho)$ are two lines incident with the two points $l$ and $l'$ in the grassmannian, then we have in the dual affine plane $\pi$ of $\Gamma$ two lines $l$ and $l'$ meeting in the points $P$ and $R$. Thus $P = R$. Now, since the residue $\Gamma_P$ of $P$ is an affine plane, there is a unique plane on $P$ containing the two lines $l$ and $l'$. This implies $\pi = \rho$; hence (LL) holds in the grassmannian.

We note that (LL) implies that (LH) (see Section 1) holds in $Gr(\Gamma)$, since the residue of a plane is an affine plane. Furthermore, for any $Af.C_2$ geometry, the following property is again a consequence of (LL), since two points of an affine plane are always collinear:

(HH) If two distinct planes meet in two distinct points, then these two points are collinear.

Clearly, (LL) and (HH) together for any $pG.L$ geometry yield that the intersection property (IP) of [2] (equivalently, Int of [27]) holds.

Hence we have in $Gr(\Gamma)$ the intersection property (IP), even when this does not hold in $\Gamma$. This allows us to apply the results in [21] (see also [3]) to conclude that $Gr(\Gamma)$ is either an affine polar space or a standard quotient of an affine polar space in the sense of [19, 20].

On the other side, in proper quotients of affine polar spaces obtained from $\mathcal{Z} = Q^+_z(q)$, the two systems of planes of $\mathcal{Z}$ are mixed together (see [20]), while this cannot happen in $Gr(\Gamma)$ (since it is obtained from $\Gamma$). Therefore, $Gr(\Gamma)$ is not a proper quotient of an affine polar space, so that it is affine polar space, obtained from $\mathcal{Z} = Q^+_z(q)$ excluding a hyperplane.

If the deleted hyperplane is tangent to $\mathcal{Z}$, then $\Gamma$ is isomorphic to the geometry $\mathcal{G}_1$ of Example 2.2.

We thank Frank de Clerck for many helpful comments and for pointing out that special cases of Theorem 2.5 were also considered in other situations and with different terminologies and methods in [10–12].
If the hyperplane is polar of a point not in $\mathcal{L}$, then $\Gamma$ is isomorphic to the geometry $\mathcal{G}_2'$ of Example 2.3. These are the only two possible examples of $\text{Af}^*$.Af geometries. □

Theorem 2.5 implies, in particular the following result, which is also an obvious consequence of the well-known representation of lines of $\text{PG}(3, q)$ in $\text{PG}(5, q)$ (see [13, part IV, Section 15.41] for a description).

**Corollary 2.6.** With the notation in Example 2.4 and Theorem 2.5, we have $\mathcal{G}_2 \cong \mathcal{G}_2'$.

**Remark 2.7.** If $q = 2$ the (IP) holds in $\mathcal{G}_2 \cong \mathcal{G}_2'$ (as two planes $\pi$ and $\rho$ which meet in an isotropic line $l$ still meet in a point $T$ of the geometry, since $l = \pi^+ \cup \rho^+ \cup T$). In fact, in this case $\mathcal{G}_2'$ is nothing but a truncation of the Coxeter complex of type $A_5$, by [22].

If $q > 2$ the above argument fails, since $|l| = q + 1 > 3$. Indeed, (IP) does not hold in any $\text{Af}^*$.Af geometry with $q > 2$; this also follows from [22]. The next corollary obviously follows from Examples 2.1, 2.2 and Theorem 2.5.

**Corollary 2.8.** An $\text{Af}^*$.Af geometry of order $(1, q - 1, q)$ satisfies $(\text{PL}_1)$ if and only if it is isomorphic to the geometry $\mathcal{G}_1 \cong \mathcal{G}_1'$ of Examples 2.1 and 2.2.

**Remark 2.9.** A direct proof of this corollary can also be given showing that an $\text{Af}^*$.Af geometry of order $(q, q - 1, q)$ with $(\text{PL}_1)$, say $\Gamma$, satisfies the assumptions in the main theorem of [26]; namely, the point–line system $S(\Gamma)$ is a partial geometry satisfying $(P)_1$, $(R)$, $(\text{WLC})$ and $(2)$ (see Remark 6.2).

Then $S(\Gamma)$ will be isomorphic to $H_3^q$, by the above-mentioned theorem of [26], and $\Gamma \cong \mathcal{G}_1 \cong \mathcal{G}_1'$, since $(\text{PL}_1)$ holds in $\Gamma$.

This direct proof is useful for suggesting generalizations which appear in the rest of the paper. The validity of the Pasch axiom for an $\text{Af}^*$.Af geometry with $(\text{PL}_1)$ will be proved more in general in Lemma 3.1 for any $L^*$.L geometry with $(\text{PL}_1)$. Condition $(R)$ will be proved in Remark 6.2 as a consequence of a more general result concerning $N^*$.Af geometries with $(\text{PL}_1)$ (see also [26, Section 3]). Condition $(\text{WLC})$ will then be a consequence of $(R)$ (since in every plane of an $N^*$.Af geometry the noncollinearity is an equivalence relation). Finally, condition 1.(2) in this case of a $\text{pG}_q(q, q^2 - 1)$ is trivial.

### 3. A few lemmas on $L^*$.L geometries

**Lemma 3.1.** Let $\Gamma$ be a rank 3 geometry with $(\text{PL}_1)$ for the diagram

\[
\begin{array}{c}
\circ & \circ & \circ \\
L & L & L \\
s & s + 1 & u \\
\end{array}
\]
where clearly \( \alpha \geq 2, u \geq 1. \) Then the point-line system \( S(\Gamma) \) is a \( pG_s(s, t) \) with
\[
t = (\alpha - 1)(u + 1),
\]
satisfying the axiom of Pasch, \((P)\). This implies \( u \geq s \) (i.e. \( t \geq (s + 1)(\alpha - 1) \)) and
\[
u = s \Leftrightarrow (PL_1)^* \text{ holds.}
\]

**Proof.** If two lines \( l_1 \neq l_2 \) meet in a point \( P \) and \( m_1, m_2 \) are two lines not on \( P \), each meeting both \( l_1 \) and \( l_2 \), in the residue \( \Gamma_r \) we have a unique plane \( \pi \) on \( l_1 \) and \( l_2 \). Now \((LH)\) (which holds by Lemma 1.1) implies that also \( m_1 \) and \( m_2 \) are in \( \pi \). In \( \pi \) the two lines \( m_1 \) and \( m_2 \) are concurrent, since the residue of \( \pi \) is a dual linear space.

Clearly, we have \( \alpha \neq 1, t + 1. \) Thus, given a plane \( \pi \), then a point \( P \notin \pi \) always exists. Each of the \( s(\alpha - 1) + \alpha \) lines of \( \pi \) contains \( \alpha \) points collinear with \( P \). Since each of these points is counted in the product \( \alpha[s(\alpha - 1) + \alpha] \) precisely \( \alpha \) times, we have exactly \( s(\alpha - 1) + \alpha \) lines on \( P \) meeting \( \pi \). Thus, \( s(\alpha - 1) + \alpha \leq t + 1 \) (the total number of lines on \( P \)). Therefore,
\[
t \geq (s + 1)(\alpha - 1),
\]
and equality holds if and only if \((PL_1)^* \) holds. \( \square \)

**Remark 3.2.** The converse of Lemma 3.1 is also true as it is implicitly proved in [26, Section 2], where for every pair of concurrent lines in a \( pG_s(s, t) \) satisfying \((P)\) is defined a structure which is the dual of a \( 2-(s(\alpha - 1) + \alpha, \alpha, 1) \), corresponding to our plane on the two lines. Also, [26] gives the inequality \( t \geq (s + 1)(\alpha - 1) \) for a \( pG_s(s, t) \) satisfying \((P)\).

Thus, from Lemma 3.1 and from the results in [26] it follows immediately that the point-line systems of rank 3 geometries \( L^*.L \) of order \( (s, \alpha - 1, [t/(\alpha - 1)] - 1) \), with \((PL_1)\), are precisely the \( pG_s(s, t) \) satisfying \((P)\).

We now want to compare our properties \((R)\) and \((WLC)\) with the regularity property considered in [26] (see also [28] for a study of ‘regular’ partial geometries in the spirit of combinatorial spaces of [23]).

**Lemma 3.3.** Every plane \( \pi \) of an \( L^*.L \) geometry of order \( (s, \alpha - 1, u) \) contains at most
\[
m = s + 1 - s/\alpha \text{ pairwise noncollinear points.}
\]

**Proof.** It is the particular case \((r, s, t) = (s, \alpha - 1, u)\) of Lemma 1.2(i). \( \square \)

From now on \( \Gamma \) will denote a geometry for the diagram

\[
\begin{array}{ccc}
\circ & \Leftrightarrow & \circ \\
L^* & L & L^* \\
\circ & \Leftrightarrow & \circ \\
\circ & \Leftrightarrow & \circ \\
L & L & L \\
\circ & \Leftrightarrow & \circ \\
\end{array}
\]
satisfying \((PL_1)\).
Lemma 3.4. On two noncollinear points $P$ and $Q$ of $\Gamma$ there are exactly
\[
\frac{t+1}{\alpha} = \frac{(s-1)(u+1)+1}{\alpha}
\]
planes.

Proof. See Lemma 1.2(ii), for $(r, s, t) = (s, \alpha - 1, u)$. \(\Box\)

Definition 3.5. Given two noncollinear points $P \neq Q$ of $\Gamma$, the intersection of all the $(t+1)/\alpha$ planes on $P$ and $Q$ is said to be the pseudo-line of $\Gamma$ joining $P$ and $Q$ and is denoted by $\langle P, Q \rangle$.

Clearly, the points of $\langle P, Q \rangle$ are pairwise noncollinear (see Lemma 1.2(iii)). Furthermore, $\langle P', Q' \rangle = \langle P, Q \rangle$ for every $P' \neq Q' \in \langle P, Q \rangle$ (see [26, Section 2], where our pseudo-lines are called lines of the second type).

It follows from Lemma 3.3 that $|\langle P, Q \rangle| \leq s + 1 - s/\alpha$.

Lemma 3.6. For an $L^*$. $L$ geometry of order $(s, \alpha - 1, u)$ with $(PL_1)$ the following ones are equivalent conditions:

(i) The intersection of all planes on any two noncollinear points consists of $s + 1 - s/\alpha$ points (pairwise noncollinear by $(PL_1)$).

(ii) There are $s + 1 - s/\alpha$ points on every pseudo-line of $\Gamma$.

(iii) Every pseudo-line $\langle P, Q \rangle$ is an ovoid in each plane containing $\langle P, Q \rangle$.

(iv) Properties $(R)$ and $(WLC)$ hold in $\Gamma$.

Proof. It is a trivial consequence of Lemmas 3.3, 3.4 and Definition 3.5 (see also the proof of Lemma 1.2(i)). \(\Box\)

Thas and de Clerck [26] call as regular a partial geometry satisfying Pasch and one of the equivalent conditions of Lemma 3.6.

The reason we chose to call as regular a $pG_{s}.L$ satisfying only $(R)$ (and not necessarily also $(WLC)$) will appear in Section 6. Consequently, according to Definition 1.4, we have, as a consequence the following corollary

Corollary 3.7. An $L^*$. $L$ geometry of order $(s, \alpha - 1, u)$ with $(PL_1)$ is TD-regular if and only if it satisfies one of the equivalent conditions of Lemma 3.6.

Corollary 3.8. A geometry $\Gamma$ for the diagram
\[
\circ - \circ \quad \circ
\]
\[
\alpha \quad \alpha - 1 \quad u
\]

satisfies $(R)$ and $(WLC)$, if $(PL_1)$ holds in it.
**Proof.** Lemma 3.1 implies that the point-line system $S(I)$ is a

$$pG_s(\alpha, (\alpha - 1)(u + 1)),$$

that is a dual net of order $\alpha + 1$ and deficiency $(\alpha - 1)(u + 1) - \alpha + 1 = u(\alpha - 1)$ satisfying

(P). It follows from the theorem in [26, Section 31] that then $I$ is TD-regular, hence satisfies (R) and (WLC).

4. The case $u = s$

In this section, $\Delta$ represents a geometry with (PL$_1$) and (WLC) for the diagram

$$\begin{array}{c}
O & L^* & O & L & O \\
s & \alpha - 1 & s
\end{array}$$

and, obviously, $\alpha \geq 2$.

By Lemma 3.1, the point-line system $S(\Delta)$ of $\Delta$ is a $pG_s(s, (\alpha - 1)(s + 1))$ satisfying (P). Furthermore, (PL$_1$)* holds in $\Delta$. Therefore, we have

$$v = (s + 1) \left[ \frac{s(\alpha - 1)(s + 1)}{\alpha} + 1 \right],$$

(6)

points and planes in $\Delta$.

**Lemma 4.1.** Two planes of $\Delta$ always meet in either a line or a pseudo-line, and $\Delta$ is TD-regular.

**Proof.** Let $\pi \neq \pi'$ be two planes of $\Delta$ not meeting in a line, and let $l'$ be a line in $\pi'$. By (PL$_1$)* there is a point $P = l' \cap \pi$; thus $|\pi \cap \pi'| = k \geq 1$. The $k\alpha$ lines of $\pi'$ on the $k$ points in $\pi \cap \pi'$ are all the lines of $\pi'$ meeting $\pi$.

On the other hand, every line of $\pi'$ must meet $\pi$, by (PL$_1$)*. Now $\pi'$ is a $pG_s(s, \alpha - 1)$; hence it has $s\alpha - s + \alpha$ lines. Thus, $k\alpha = s\alpha - s + \alpha$ or

$$k = s + 1 - \frac{s}{\alpha}.$$  

(7)

This implies $k \neq 1$, since otherwise $\alpha - 1 = 0$, while $\alpha \geq 2$. Clearly, these $k$ points are pairwise noncollinear (see Lemma 1.2(iii)). This proves that (R) holds in $I$. Since (WLC) is assumed, we have the statement.

We remark that, in particular, $\alpha$ divides $s$, thus

$$2 \leq \alpha \leq s.$$  

(8)
**Lemma 4.2.** If $\alpha < s$, then points, pseudo-lines, and planes of $\Delta$ are the elements of a rank 3 partial geometry $\Delta$ for the diagram

![Diagram](https://via.placeholder.com/150)

with

$$\beta = s + 1 - \alpha - \frac{s}{\alpha}. \quad (9)$$

If $\alpha = s$, then the pseudo-lines in a plane of $\Delta$ form a partition of that plane.

**Proof.** Every pseudo-line has $k = s - (s/\alpha) + 1$ points, since $\Delta$ is TD-regular by Lemma 4.1. Dually (or by Lemma 3.4) on every pseudo-line there are exactly

$$\frac{(\alpha-1)(s+1)+1}{\alpha} = \left(s - \frac{s}{\alpha} \right) + 1 = k \quad (10)$$

planes.

A line and a pseudo-line in a plane meet in exactly one point, by Lemma 3.3. This implies that the number of pseudo-lines of a plane $\pi$ through a point $P \in \pi$ is the same as the number of points noncollinear with $P$ on a line $l \not\parallel P$ of $\pi$, which is $s + 1 - \alpha$. If $(P, l)$ is a point-pseudo-line antiflag in a plane, then each of the $\alpha$ lines of $\pi$ through $P$ meets $l$ in only one point (since $l$ is an ovoid of $\pi$). Therefore, we have on $l$ exactly $\beta = s + 1 - (s/\alpha) \cdot \alpha$ points noncollinear with $P$ in $\Delta$. Each of these points gives a pseudo-line of $\pi$ through $P$, meeting $l$.

Dually, given a pseudo-line $l$ and a plane $\pi$, with $l \notin \pi$, both on the same point $P$, each of the $\alpha$ planes of $P$ in $\pi$ defines with $l$ a unique plane meeting $\pi$ in a line. The remaining $\beta = k - \alpha$ planes on $l$ meet $\pi$ in a pseudo-line on $P$. \(\square\)

**Theorem 4.3.** If a geometry $\Delta$ for the diagram

![Diagram](https://via.placeholder.com/150)

satisfies $(PL_1)$ and $(WLC)$, then either $s = \alpha$ and $\Delta$ is the geometry $\mathcal{D}_1 \simeq \mathcal{D}_4$ of Example 2.2 or $s = \alpha + \alpha^3$.

**Proof.** By Lemma 4.1, $\Delta$ is TD-regular. The number $N_P$ of pseudo-lines on a point $P$ is by duality, the same as the number of pseudo-lines on a plane. Since every plane of $\Delta$ is a $pG_\alpha(s, \alpha - 1)$ we have on every plane $\pi$

$$v_\pi = (s + 1)\left(\frac{s(\alpha - 1)}{\alpha} + 1\right) = (s + 1)\left(s - \frac{s}{\alpha} + 1\right) \quad (11)$$

points.
On every point of a plane \( \pi \) we have \( s+1-\alpha \) pseudo-lines of \( \pi \) (see Lemma 4.2). Counting point–pseudo-line flags \((P, l)\) of \( \pi \) gives

\[
N_p = \frac{v_u(s+1-\alpha)}{k} = (s+1)(s+1-\alpha),
\]

where \( k \) is given by (7).

Now, counting point–pseudo-line flags \((P, l)\) of \( A \) yields, with elementary computations (see (6), (7) and (12)), that the total number \( N \) of pseudo-lines of \( A \) is given by

\[
N = \frac{v_N p}{k} = (s+1)^3(s+1-\alpha) - \frac{s(s+1)^2(s+1-\alpha)}{(s+1-\alpha)}.
\]

Since \( N \) is an integer,

\[
s(s+1)^2(s+1-\alpha) \equiv 0 \left( \text{mod } s+1-\frac{s}{\alpha} \right),
\]

or

\[
\left( s - 1 \right) \left( \frac{s}{\alpha} - \alpha \right) = 0 \left( \text{mod } s+1-\frac{s}{\alpha} \right),
\]

because \( s \equiv (s/\alpha) - 1 \pmod{s+1-\alpha} \). Now \( s/\alpha \) divides \( s \) and \( s/\alpha \), thus \( s/\alpha \) is coprime with \( (s-s/\alpha) + 1 \). Hence, we deduce from the last congruence

\[
\left( s - 1 \right) \left( \frac{s}{\alpha} - \alpha \right) \equiv 0 \left( \text{mod } s+1-\frac{s}{\alpha} \right),
\]

If \( s = \alpha^2 \), we consider the point graph \( G(S(A)) \) of the \( pG_s(\alpha^2, (\alpha-1)(\alpha^2 + 1)) \) defined by \( S(A) \). Since the multiplicities of the eigenvalues of this graph are integers, we obtain

\[
(\alpha-1)(\alpha^2 + 1)^2 [(\alpha-1)(\alpha^2 + 1) + 1] \equiv 0 \pmod{\alpha^2 + (\alpha-1)(\alpha^2 + 1) + 1 - \alpha},
\]

or

\[
\alpha^3(\alpha-1)(\alpha^2 + 1)^2(\alpha^2 - \alpha + 1) \equiv 0 \pmod{\alpha^4},
\]

i.e.

\[
(\alpha-1)(\alpha^2 + 1)^2(\alpha^2 - \alpha + 1) \equiv 0 \pmod{\alpha}.
\]

But this is never satisfied for \( \alpha \neq 1 \).

If \( s = \alpha^2 \), it cannot be \( s < \alpha^2 \), otherwise \( (s/\alpha) < \alpha \) or \(-[(s/\alpha) - \alpha] > 0\), and \( s > \alpha \) (see (8)) or equivalently \( (s/\alpha) - 1 > 0 \), imply

\[
\left( \alpha - \frac{s}{\alpha} \right) \left( \frac{s}{\alpha} - 1 \right) > 0.
\]

This, by (15), gives

\[
s + 1 - \frac{s}{\alpha} \leq \left( \alpha - \frac{s}{\alpha} \right) \left( \frac{s}{\alpha} - 1 \right) = \frac{s^2}{\alpha^2} - \frac{\alpha + s}{\alpha},
\]
which yields
\[
\left( \frac{s}{x} \right)^2 - 2 \frac{s}{x} + x + 1 \leq 0,
\]
and this leads to a contradiction. Therefore, we have the divisibility condition (15) and
\[
s \neq x \implies s > x^2. \tag{17}
\]
Let \( d = (s + 1 - s/x, (s/x - 1)) \) be the greatest common divisor of \((s + 1 - s/x)\) and \((s/x - 1)\). Clearly, \(d\) divides \(s = (s + 1 - s/x) + (s/x - 1) = (s/x) \times \) and is coprime with \((s/x)\) (since it divides \((s/x - 1)\)). Hence, \(d\) divides \(x\).

If \((s + 1 - s/x) = df\), then \(f\) divides \((s/x) - x\), by (15), so that
\[
df = s + 1 - \frac{s}{x} \text{ divides } x \left( \frac{s}{x} - \frac{s}{x} \right) = s - x^2. \tag{18}
\]
Therefore, \(s - x^2 \equiv 0 \pmod{s + 1 - s/x}\), and since \(s \equiv s/x - 1 \pmod{s + 1 - s/x}\), we obtain
\[
\frac{s}{x} - 1 - x^2 \equiv 0 \pmod{s + 1 - \frac{s}{x}}. \tag{19}
\]
Clearly, (14) is satisfied for \(s = x + x^2\). When \(s = x\), the conclusion follows from Theorem 2.5. To finish the proof, it remains to show that it cannot be \(s \neq x, \alpha + x^3\).

If \(s > x + x^3\), then \(s/x - 1 - x^2 > 0\) and (19) implies \((s + 1 - s/x) \leq (s/x - 1 - x^2)\) or \(x^2 \leq (2s/x - s - 2)\), which yields \(4 \leq 2s/x - s - 2\). Hence,
\[
\alpha \leq \frac{2s}{s + 6} < 2,
\]
which is impossible, since \(\alpha \geq 2\).

If \(s < x + x^3, s \neq x\), then \(s/x - 1 - x^2 < 0\) and (19) gives
\[
s + 1 - \frac{s}{x} \leq -\frac{s}{x} + 1 + x^2.
\]
This implies \(s \leq x^2\), which is impossible, by (17). Therefore, we have the statement. \(\square\)

5. Regular \(L^*.L\) geometries, \(H^n_q\) and related topics

We start this section by pointing out the obvious generalizations of Examples 2.1 and 2.4 (equivalent to Examples 2.2 and 2.3, by Theorem 2.5) to \(Af^*.L\) geometries, or to rank \(n\) geometries.
**Example 5.1.** Let $S$ be a $PG(n-2, q)$ in $P = PG(n, q)$.

The points and the lines of $P$ which do not meet $S$, and the planes of $P$ meeting $S$ in a single point are the elements of a rank 3 geometry $\mathcal{H}_q^a$ belonging to the diagram $A\mathcal{I}^* L.$

$$
\begin{array}{cccc}
A\mathcal{I}^* & \rightarrow & L \\
q & \rightarrow & q-1 & \rightarrow & q^{a-2} + q^{a-3} + \cdots + q
\end{array}
$$

where the linear space residue of a point is a $\{0, 1\}$-truncation of $AG(n-1, q)$ (the projection from a point $P$ onto the $AG(n-1, q) = PG(n-1, q) - S$, $P \not\in PG(n-1, q) \supset S$), is an isomorphism between the residue $(\mathcal{H}_q^a)_p$ and the point–line system of $AG(n-1, q)$.

Clearly, $\mathcal{H}_q^a$ satisfies $\text{(PL}_1\text{)}$ (hence $\text{(LH)}$) (see Section 1), the regularity condition $(\text{R})$ and $\text{(WLC)}$ (see Definition 1.3, Corollaries 3.7 and 3.8).

We note that if two planes $\pi$ and $\rho$ of $\mathcal{H}_q^a$ meet in more than one point, then if the line $l = \pi \cap \rho$ of $P$ is not a line of $\mathcal{H}_q^a$, certainly $l$ meets $S = PG(n-2, q)$ in one point. Hence, in the projective completion of the dual affine planes $\pi$ and $\rho$ the $q$ pairwise noncollinear points of $\pi \cap \rho$ appear as a line on the 'point at infinity'.

The point–line system of $\mathcal{H}_q^a$ is a well-known partial geometry $pG_q(q, q^{-1} - 1)$, satisfying Pasch axiom $(P)$ (see Section 1 for the statement), denoted by $H_q^a$.

Clearly, the geometry $\mathcal{H}_1$ of Example 2.1 is the particular case $n = 3$.

**Theorem 5.2.** Let $\Gamma$ be a geometry with $(\text{PL}_1\text{)}$ and $(\text{WLC})$ for the diagram

$$
\begin{array}{cccc}
L^* & \rightarrow & L \\
s & \rightarrow & \alpha - 1 & \rightarrow & u
\end{array}
$$

($\alpha \geq 2, u \geq 1$). If $\Gamma$ is regular, then one of the following holds:

(i) $s = \alpha = q, u = q^{a-2} + q^{a-3} + \cdots + q$ for some integer $n \geq 3$, and $\Gamma$ is isomorphic to the rank 3 $\mathcal{H}_q^a$ of Example 5.1.

(ii) $s = u = \alpha + \alpha^3$.

**Proof.** Since $u \geq 1$, we may fix two lines $l$ and $l'$ with $l \cap l' = \emptyset$. Let $\mathcal{P}'$ be the set of all points in the $s + 1$ planes $l \cup X$ on $l$ and on a point $X \in l'$. The point–line system $S(\Gamma)$ satisfies $(P)$, by Lemma 3.1. Hence, the argument in [26, pp. 129–130] proves that any line of $\Gamma$ containing at least two points of $\mathcal{P}'$, is entirely contained in $\mathcal{P}'$. A sub-geometry $\mathcal{A}$ with $(\text{PL}_1\text{)}$ for

$$
\begin{array}{cccc}
L^* & \rightarrow & L \\
s & \rightarrow & \alpha - 1 & \rightarrow & s
\end{array}
$$

is defined as follows (see Lemma 3.1 and Remark 3.2 and [26, p. 130]): points of $\mathcal{A}$ are the points of $\mathcal{P}'$, lines of $\mathcal{A}$ are the lines of $\Gamma$ with two points in $\mathcal{P}'$, planes of $\mathcal{A}$ are the planes (uniquely defined, by $(\text{PL}_1\text{)})$ of the form $P \cup p$, where $P$ and $p$ are a point and a line of $\mathcal{A}$, respectively.
It follows then from Theorem 4.3 that \( s = \alpha + \alpha^3 \).

If \( s = \alpha \), Lemma 3.1 (see also Remark 3.2) and the argument of [26, pp.132 – 136] prove (i).

If \( s = \alpha + \alpha^3 \), then the Krein condition
\[
(s + 1 - 2\alpha)t \leq (s - 1)(s + 1 - \alpha^2)
\]
for the strongly regular graph of the \( pG(s,t) \) \((s = \alpha + \alpha^3, \, t = (\alpha - 1)(u + 1)) \) given by \( S(\Gamma) \) yields
\[
u + 1 \leq \frac{(\alpha^3 + \alpha - 1)(\alpha^3 + 1)^2}{(\alpha^2 - \alpha + 1)(\alpha - 1)}.
\]

Since \( S(\Delta) \) is a subgeometry \( pG(s',t') \) with \( s' = s, \, t' = (a - 1)(\alpha^3 + \alpha + 1) \), then either \( t = t' \) or \( t \geq s't' + \alpha - 1 \)
(see in [5, Theorem 6.2 and Remark 6.3]). In our case this yields
\[
\text{either } u + 1 = \alpha^3 + \alpha + 1 \text{ or } u + 1 \geq (\alpha^3 + \alpha)(\alpha^3 + \alpha + 1) + 1.
\]

If \( u + 1 \neq \alpha^3 + \alpha + 1 \), then (20) and (21) imply
\[
(\alpha^3 + \alpha)(\alpha^3 + \alpha + 1) + 1 \leq u + 1 \leq \frac{(\alpha^3 + \alpha - 1)(\alpha^3 + 1)^2}{(\alpha^2 - \alpha + 1)(\alpha - 1)},
\]
or equivalently
\[
\frac{(\alpha^3 + \alpha)(\alpha^3 + \alpha + 1) + 1}{(\alpha^3 - \alpha + 1)(\alpha - 1)} \leq u + 1 \leq \frac{(\alpha^3 + \alpha - 1)(\alpha^3 + 1)^2}{(\alpha^2 - \alpha + 1)(\alpha - 1)}.
\]

It is not difficult to see, by elementary computations, that inequality (22) is never satisfied for \( \alpha \geq 2 \). Therefore, (22) implies \( u = s = \alpha + \alpha^3 \). \( \square \)

**Corollary 5.3.** Every \( Af^*.\,L \) geometry with \((PL_1)\)

\[
\begin{array}{cccccc}
& & & & & \text{L} \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\alpha & \alpha - 1 & u \\
\end{array}
\]

has \( \alpha = q = p^h \) \((p \geq 2 \text{ prime and } h \text{ positive integer}), \, u = q^{n-2} + q^{n-3} + \cdots + q \) \( \text{for some } n \geq 3 \), and it is isomorphic to the rank 3 \( \mathcal{K}_q^n \) of Example 5.1.

**Proof.** It follows from Corollary 3.8 and Theorem 5.2. \( \square \)

Example 5.1 has an obvious rank \( n \) generalization that we shall denote again as \( \mathcal{K}_q^n \), by abuse of notation.

**Example 5.4.** Let \( S \) be a \( PG(n-2, q) \) in \( P = PG(n, q) \).

Define the rank \( n \) \( \mathcal{K}_q^n \) taking as points lines and planes, those of the rank 3 \( \mathcal{K}_q^3 \) defined in Example 5.1, and as \( i \)-spaces, \( i = 3, \ldots, n-1 \) the \( i \)-dimensional subspaces of \( P \) meeting the \((n-2)\)-dimensional subspace \( S \) in a subspace of dimension \( i - 2 \).
The rank $n$ geometry $\mathcal{H}_q^n$ belongs to the diagram $A^*.A.\mathcal{A}_{n-2}$ of order $q$
\[\begin{array}{cccccccc}
A^* & \cdots & A^* & \cdots & A^* \n q & q-1 & q & q & q & q
\end{array}\]

and satisfies (PL$_1$).

Also the converse is true, namely, we have the following corollary.

**Corollary 5.5.** An $A^*.A.\mathcal{A}_{n-2}$ geometry
\[\begin{array}{cccccccc}
A^* & \cdots & A^* & \cdots & A^* \n q & q-1 & q & q & q & q
\end{array}\]
of order $q$ satisfies (PL$_1$) if and only if it is isomorphic to the rank $n$ geometry $\mathcal{H}_q^n$ defined in Example 5.4.

**Proof.** Let $\mathcal{G}$ be an $A^*.A.\mathcal{A}_{n-2}$ geometry of order $q$ with (PL$_1$). Clearly, the $\{0,1,2\}$-truncation of $\mathcal{G}$ is an $A^*.L$ geometry with (PL$_1$) of order $(q,q-1,u)$. Thus, it is isomorphic to the rank 3 $\mathcal{H}_q^2$ by Corollary 5.3.

It is not difficult to see that every $i$-space of $\mathcal{G}$ is completely determined from its $j$-spaces ($j \leq i-1$, $i=4,\ldots,n-1$), so that $\mathcal{G}$ is isomorphic to the rank $n$ $\mathcal{H}_q^n$. $\square$

As we have already noted in Section 1, we deduce from Corollaries 5.3 and 5.5 the following consequence.

**Corollary 5.6.** Every $A^*.L$ geometry with (PL$_1$) is a $\{0,1,2\}$-truncation of an $A^*.A.\mathcal{A}_{n-2}$ geometry of order $q$ with (PL$_1$).

The result in Corollary 5.6 is not trivial, as the following example shows.

**Example 5.7.** Let $Sp_{2n-1}(q)$ be the symplectic variety in $PG(2n-1,q)$.

Take as points the points of $Sp_{2n-1}(q)$, as lines the nonisotropic lines of $PG(2n-1,q)$, and as planes the nontotally isotropic planes of $PG(2n-1,q)$. The incidence $*$ is the same as in Example 2.4 (which is the particular case $n=2$), namely

$P*\pi \iff P\in\pi$, $P\perp \neq \pi$.

This geometry, like the rank 3 $\mathcal{H}_q^n$, is an $AF^*.L$ geometry for the diagram
\[\begin{array}{cccc}
A^* & \cdots & A^* \n q & q-1 & q^{2n-3} + q^{2n-4} + \cdots + q
\end{array}\]

but, like in the Example 2.4 (which is the particular case $n=3$), (PL$_1$) does not hold. Here the linear space residue of a point is a truncation of $AG(n-1,q)$ as before, but it
does not seem that the whole geometry is a truncation of some \(Af^*AfA_{2n-3}\) geometry.

Actually, this geometry can indeed be obtained by truncation from the geometry of rank \(2n-1\) which arises adding as new elements all proper subspaces \(X\) of \(PG(2n-1,q)\) of dimension 3, 4, ... such that \(|X \cap X^\perp| \leq 1\) (whence \(X \cap X^\perp = \emptyset\), if \(X\) has odd dimension, while \(X \cap X^\perp\) is a singleton if \(X\) has even dimension). However, this larger geometry has the following diagram:

\[
\begin{array}{cccccccc}
Af^* & Af & Af^* & Af & \cdots & Af^* & Af \\
q & q-1 & q & q-1 & q & q-1 & q \\
\end{array}
\]

as the reader may check. We observe that the diagram \(Af_Af^*\) appearing above as \(\{i, i+1, i+2\}\)-truncation when \(i \equiv 1 \pmod{2}\) has been classified in [18].

### 6. \(N^*.L\) geometries

In this section \(\Gamma\) represents a geometry satisfying (PL1) belonging to the diagram \(N^*.L\)

\[
\begin{array}{ccc}
N^* & L & \\
\alpha & s & t \\
\end{array}
\]

Since \(\Gamma\) is a \(pG.L\) geometry of order \((\alpha, s, t)\), the point–line system \(S(\Gamma)\) is a

\[
pG_{\alpha}(\alpha, s(t+1)),
\]

(i.e. a dual net of order \(\alpha+1\) and deficiency \(s(t+1)-\alpha+1\)). In this case \(r=\alpha\), Lemma 1.2 implies the following:

(i) Every plane contains at most \(m=1+s\) noncollinear points.

(ii) On two noncollinear points \(P\) and \(Q\) of \(\Gamma\) there are exactly \(1+st/(s+1)\) planes. In particular, \(s+1\) divides \(t\), hence, \(s+1 \leq t\).

(iii) If two planes of \(\Gamma\) do not meet in a line, then they meet in either 0, 1 or \(k\leq s+1\) pairwise noncollinear points.

Therefore, (see Definition 1.3) \(\Gamma\) is said to be regular if two planes \(\pi\) and \(\pi'\) not meeting in a line meet in either 0, 1 or \(s+1\) pairwise noncollinear points.

We note that in this case property (R) implies property (WLC), since in every plane the noncollinearity is an equivalence relation.

**Theorem 6.1.** An \(N^*.L\) geometry with (PL1) of order \((\alpha, s, t)\) is regular and does not contain planes meeting in just one point if and only if \(t=s+1\).

**Proof.** Let \(\pi\) be a plane of \(\Gamma\). Since the residue \(\mathcal{I}_\pi\) is a \(pG_{\alpha}(\alpha, s)\) (i.e. a dual net of order \(\alpha+1\) and deficiency \(s-\alpha+1\)), the relation of noncollinearity is transitive. Thus, the \((\alpha+1)(s+1)\) points of \(\pi\) are partitioned into \(\alpha+1\) noncollinearity classes consisting of
s + 1 points each. Furthermore, we have on every line of π one and only one point of each noncollinearity class.

If P is a point of π, the number of planes ≠ π, and meeting π in one of the s + 1 lines of π through P, is (s + 1)t (since on any such line we have in I_p exactly t planes different from π). Therefore, the number of planes not meeting P in a line is

\[
\frac{s(t + 1) + 1}{s + 1}(t + 1) - 1 -(s + 1)t = \frac{s(t - s)}{s + 1}.
\]

This number is greater than or equal to the number \(st/(s + 1)\) (see (ii)) of planes ≠ π on P and on another point Q ≠ P of π. Thus,

\[
st/(s + 1) \geq \frac{s(t - s)}{s + 1}.
\]

or equivalently

\[
t \geq s + 1.
\]

Clearly, equality holds in (24) if and only if \(t = s + 1\) or, equivalently, if and only if every plane ≠ π on P not meeting π in a line meets π in the s + 1 points of the noncollinearity class of P. Therefore, \(t = s + 1\) if and only if we have \(|\pi \cap \pi'| = s + 1\) for every pair of distinct planes π, π' such that π \(\cap\) π' ≠ \(\emptyset\), and is not a line.

We note that when \(t = s + 1 = q\) our geometry has the diagram

\[
\begin{array}{ccccccccc}
N^* & A_f & & & & & & & N \\
\alpha & & q - 1 & q & & & & & & \end{array}
\]

Remark 6.2. Obviously, when \(\alpha = s + 1 = q\), we obtain that every \(A_f^* - A_f\) geometry with \((PL_1)\) is regular (furthermore, no pair of planes meets in exactly one point). This completes the direct proof of Corollary 2.8 (see Remark 2.9).

We now consider the particular case \(\alpha = s = q - 1 = t - 1\).

Example 6.3. Let \(\pi_\infty = PG(2, q)\) be embedded as a plane in \(P = PG(3, q)\), and let \(l_\infty\) be a line of \(\pi_\infty\). If \(K = AG(2, q) = \pi_\infty - l_\infty\), we define a rank 3 geometry \(F_2^* (K)\) as follows.

Points of \(F_2^* (K)\) are the points of \(AG(3, q) = P - \pi_\infty\); lines of \(F_2^* (K)\) are all lines of \(AG(3, q)\) with ideal point in \(K\); planes of \(F_2^* (K)\) are all planes of \(AG(3, q)\) with ideal line different from \(l_\infty\). Incidence is inherited from \(AG(3, q)\).

Clearly, \(F_2^* (K)\) belongs to the diagram

\[
\begin{array}{ccccccccc}
N^* & A_f & & & & & & & N \\
q - 1 & & q - 1 & q & & & & & & \end{array}
\]
The point–line system $S(\mathcal{T}(K))$ is the well-known partial geometry $T(K)$ constructed by Thas ([24, 25]).

We shall call pseudo-lines (pseudo-planes) of $\mathcal{T}(K)$ the lines (planes) of $AG(3, q)$ with ideal point in $l_\infty$ (ideal line $l_\infty$).

$\mathcal{T}(K)$ is the only $N^*Af = N.Af$ geometry of order $(q-1, q-1, q)$, with $(PL_1)$, as the following theorem shows.

**Theorem 6.4.** If $\Gamma$ is an $N^*.Af$ geometry of order $(q-1, q-1, q)$

$$
\begin{array}{ccc}
N^* & Af & q \\
q-1 & q-1 & q
\end{array}
$$

satisfying $(PL_1)$, then $\Gamma$ is isomorphic to the geometry $\mathcal{T}(K)$, where $K = AG(2, q)$, of Example 6.3.

**Proof.** $\Gamma$ is regular and every two planes $\pi$ and $\pi'$ with $\pi \cap \pi' \neq \emptyset$ meet either in a line or in $s + 1 = q$ pairwise noncollinear points (briefly a pseudo-line of $\Gamma$), by Theorem 6.1.

On every point $P$ of $\Gamma$ we have $q^2$ lines and $q^2 + q$ planes (since $G_\pi$ is an affine plane). On every plane on $P$ we have exactly one pseudo-line (consisting of all the points of the plane noncollinear with $P$). Since the same pseudo-line on $P$ is in exactly $1 + st/(s-1)q/q = q$ planes on $P$, we have on $P$ exactly $(q^2 + q)/q = q + 1$ pseudo-lines. Therefore, if we consider the structure with points the points of $\Gamma$ and with blocks the lines and the pseudo-lines, we have a design with the same parameters as the point–lines system of $AG(3, q)$.

If $\pi$ is a plane of $\Gamma$, then $\Gamma_\pi$ is an $N^* = N$ of order $(q-1, q-1)$. Hence, adding to $\pi$ the pseudo-lines of $\Gamma$ contained in $\pi$, we obtain an affine plane of order $q$. Counting the incident point–plane pairs $(P, \pi)$ of $\Gamma$ in two ways gives that the number of planes of $\Gamma$ is $q^3(q^2 + q)/q^2 = q^3 + q^2$.

We shall call pseudo-plane of $\Gamma$ the structure whose points and lines are the points of a noncollinearity class of $\Gamma$ and the pseudo-lines of $\Gamma$ on these points, respectively. Since $S(\Gamma)$ is $pG_{q-1}(q-1, q^2-1))$ (see (23)), the noncollinearity relation $\not\in$ partitions the $q^3$ points of $S(\Gamma)$ into $q$ classes consisting of $q^2$ points each. Clearly, each of the $q$ pseudo-planes of $\Gamma$ is an affine plane of order $q$.

The rank 3 geometry $\Gamma^+$ obtained from $\Gamma$ adding as new lines and planes the pseudo-lines and the pseudo-planes of $\Gamma$ is obviously $AG(3, q)$. Now it is not difficult to see the isomorphism between $\Gamma$ and the geometry $\mathcal{T}(K)$ of Example 6.3.

**Acknowledgment**

We would like to express our gratitude especially to Antonio Pasini, who has initiated much of this and other work recently carried out by us. He has taught us — mathematically and otherwise — more than we can properly thank him for.
References