Askey–Wilson Polynomials by Means of a $q$-Selberg Type Integral

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An integral representation of the Askey–Wilson polynomials is presented in terms of a $q$-Selberg type integral. Our motivation consists in the study of $q$-Selberg type integrals from the viewpoint of de Rham theory or holonomic systems.

1. INTRODUCTION

$q$-Selberg type integrals [4, 8–10] are important examples in the study of $q$-de Rham theory [2, 3, 11, 16] and are used to construct solutions of the quantum Knizhnik–Zamolodchikov equation [15, 17, 21, 23, 24]. It is also known [17] that the Macdonald polynomials [13, 14] associated with the root system of type $A_n$ are represented by a $q$-Selberg type integral.

On the other hand, the Askey–Wilson polynomials are known to be the most general orthogonal polynomials in single variable, in the sense that various sets of classical orthogonal polynomials—Wilson polynomials, continuous Hahn polynomials, Jacobi polynomials, Laguerre polynomials, Hermite polynomials, etc.—can be derived as limiting cases from the Askey–Wilson polynomials [5]. The present paper is devoted to making a bridge between the $q$-Selberg type integrals and the Askey–Wilson polynomials; the Askey–Wilson polynomials are represented by means of a $q$-Selberg type integral with the continuous measure.

The following is known as related works: an integral representation of the Jacobi polynomials by means of a $q$-Selberg type integral is presented in [1], an integral representation of the little $q$-Jacobi polynomials by means of a $q$-Selberg type integral with the Jackson measure is presented in [18], and integral representations of the Wilson polynomials and the continuous dual Hahn polynomials by means of multidimensional generalizations of...
the Barnes type integral are presented in [19]. The present paper can be also considered as a continuation of them.

Throughout this paper, we fix $q$ to be a real number such that $0 < q < 1$.

2. ASKEY–WILSON POLYNOMIALS

The Askey–Wilson polynomials [5] are orthogonal polynomials defined by

$$p_n(z; a, b, c, d | q) = a^{-n} (a b, a c, a d; q)_n 4\varphi_3 \left[ q^{-n}, q^{n-1} a b c d, a z^{-1}; q, q \right]$$

(2.1)

where $(a_1, \ldots, a_m; q)_n = (a_1; q)_n \cdots (a_m; q)_n$ with $(a; q)_n = \prod_{j=0}^{n-1} (1 - a q^j)$, and $4\varphi_3$ is a basic hypergeometric series

$$4\varphi_3 \left[ a_1, a_2, a_3, a_4; b_1, b_2, b_3; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3, a_4; q)_n}{(b_1, b_2, b_3, q; q)_n} z^n.$$

Their orthogonality relation is

$$\frac{1}{2\pi i} \oint_T p_m(z) p_n(z) \frac{(z^2, z^{-2}; q)_\infty}{(a z, b z, c z, d z, a z^{-1}, b z^{-1}, c z^{-1}, d z^{-1}; q)_\infty} \frac{dz}{z} = \delta_{m,n},$$

$$= \frac{(abcd q^2, abcd q^{-n+1}; q)_\infty}{(abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n, q^{n+1}; q)_\infty}, \quad m = n,$$

for max($|a|, |b|, |c|, |d|) < 1$. Here the contour $T$ is the unit circle with the counterclockwise direction and $(a_1, \ldots, a_m; q)_\infty = (a_1; q)_\infty \cdots (a_m; q)_\infty$ with $(a; q)_\infty = \prod_{j=0}^{n-1} (1 - a q^j)$.

The $q$-difference equation satisfied by the Askey–Wilson polynomials is

$$\begin{cases}
[D_z, q - E_d(q)] p_n(z) = 0, \\
D_{z, q} = \left(1 - a z\right)\left(1 - b z\right)\left(1 - c z\right)\left(1 - d z\right) \left(T_{z, q} - 1\right) \\
\quad + \frac{(1 - a z)^{-1}\left(1 - b z^{-1}\right)\left(1 - c z^{-1}\right)\left(1 - d z^{-1}\right)}{(1 - z^{-2})(1 - q z^{-2})} \left(T_{z, q}^{-1} - 1\right), \\
E_d(q) = -(1 - q^{-n})(1 - q^{n-1}abcd),
\end{cases}$$

(2.2)

where $(T_{z, q})_n(z) = f(qz)$.
It is known that various sets of classical orthogonal polynomials are obtained by some limit process from the Askey-Wilson polynomials. We refer the reader to [5] for the hierarchy of orthogonal polynomials in single variable.

3. MAIN RESULT

We introduce the function

$$
\Phi(x) = \Phi(x^+) \Phi(x^{-1}),
$$

where

$$
\Phi(x^+) = \prod_{1 \leq k < n} \frac{(x_k^2; q)_\infty}{(ax_k, bx_k, cx_k, dx_k; q)_\infty} \prod_{1 \leq k < l \leq n} \frac{(x_k/x_l, x_k x_l; q)_\infty}{(tx_k/x_l, tx_k x_l; q)_\infty}.
$$

For $\max(|a|, |b|, |c|, |d|) < 1$ and $t > 0$. The function $\Phi(x)$ respects the permutation $x_j \mapsto x_k$ and the transposition $x_j \mapsto x_j^{-1}$. This invariance corresponds to the Weyl group of type $B_n$, and this symmetry plays a crucial role in the present paper.

It is known [6, 7] that

$$
\frac{1}{(2\pi \sqrt{-1})^n} \int_{T^n} \Phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}
$$

is equal to

$$
2^n n! \prod_{1 \leq j \leq n} \frac{(t, t^{n+j-2} abcd, q)_\infty}{(t', q, t^{n'+1} ab, t^{n'+1} ac, t^{n'+1} ad, t^{n'+1} bd, t^{n'+1} cd, q)_\infty},
$$

where $T^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n; |x_j| = 1 (1 \leq j \leq n)\}$ with the counterclockwise direction.

From the viewpoint of de Rham theory, our interest is in the study of functions in $z$ given by

$$
\int_{T^n} f(z, x) \Phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n},
$$

where $f(z, x)$ is invariant with respect to the permutation of $x_1, \ldots, x_n$ and the transposition of $x_j$ and $x_j^{-1}$.

In the present paper, we consider the special case $f(z, x) = \prod_{1 \leq k \leq n} (z + x_k - z_k^{-1})$, in which case we obtain the following.
THEOREM.

\[
\int_{T^n} \prod_{1 \leq k \leq n} (z + \frac{1}{x_k} - x_k) \Phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \int_{T^n} \Phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \left( \frac{1}{(t^{n-1}abcd; t)_n} \right)^4 \phi_h \left( \frac{t^{-n}, t^{n-1}abcd, az, az^{-1}; t, t}{ab, ac, ad} \right) \]

Here \( p_a(z; a, b, c, d | q) \) is the Askey–Wilson polynomial defined by (2.1).

We remark that, while the base of each integrand on the left-hand side is \( q \), the base of the polynomial on the right-hand side is \( t \). This suggests a duality behind our theorem, which would be clarified in our future research.

4. PROOF OF THEOREM

In this section, we use the symbol \( \langle \rangle \) defined by

\[
\langle \varphi(x) \rangle = \int_{T^n} \varphi(x) \Phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n},
\]

and \( z' \) or \( x_k' \) designates \( z + z^{-1} \) or \( x_k + x_k^{-1} \) for the sake of brevity. Here, if it is needed, we deform the contour \( T^n \) appropriately.

To reach our Theorem, it is enough to show the following \( q \)-difference equation

\[
\begin{cases}
\left[ D_{z, t} - E_{d}(t) \right] \left( \prod_{1 \leq k \leq n} (z' - x_k) \right) = 0, \\
D_{z, t} = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - tz^2)(1 - z^{-2})(1 - tz^{-2})} (T_{z, t} - 1), \\
E_{d}(t) = -(1 - t^{-n})(1 - t^{n-1}abcd),
\end{cases}
\]

(4.1)
which implies, by noting (2.2), that
\[
\left( \prod_{1 \leq k \leq n} (z' - x'_k) \right)
\]
is a constant-multiple of
\[
\phi_3 \left[ t^{-n}, t^{n-1}abcd, az, az^{-1} \middle| ab, ac, ad ; t, t \right].
\]
The constant \( C \) of the equality
\[
\left( \prod_{1 \leq k \leq n} (z' - x'_k) \right) = C \cdot \phi_3 \left[ t^{-n}, t^{n-1}abcd, az, az^{-1} \middle| ab, ac, ad ; t, t \right]
\]
is determined to be
\[
\langle 1 \rangle \times a^{-n}(ab, ac, ad; t)_n \frac{(t^n-abcd, t)_n}{(1-t^n)}
\]
by comparing the coefficients of \( z^n \).

In what follows, we concentrate ourselves on deriving the \( q \)-difference equation (4.1).

Since
\[
T^{-1}_{x_i, q} \phi(x) = \frac{(1 - q^{-2}x_i^2)(1 - q^{-1}x_i^2)(1 - ax_i^{-1}) \cdots (1 - dx_i^{-1})}{(1 - x_i^{-2})(1 - qx_i^{-2})(1 - q^{-1}ax_i) \cdots (1 - q^{-1}dx_i)}
\]
\[
\times \prod_{j=2}^n (1 - q^{-1}tx_jx_i^{-1})(1 - q^{-1}tx_jx_i^{-1})(1 - q^{-1}tx_jx_i^{-1})
\]
\[
\times (1 - tx_i^{-1}x_j)(1 - tx_i^{-1}x_j)(1 - tx_i^{-1}x_j) \phi(x),
\]
we have
\[
T^{-1}_{x_i, q} \left\{ \frac{x_i^{-2}(1 - ax_i) \cdots (1 - dx_i)}{x_i^{-1} - x_i} \prod_{j=2}^n \frac{1 - tx_i x_j^{-1}}{1 - x_i x_j} \frac{1 - tx_i x_j^{-1}}{1 - x_i x_j} \phi(x) \right\}
\]
\[
= -x_i^{-2}(1 - ax_i^{-1}) \cdots (1 - dx_i^{-1}) \prod_{j=2}^n \frac{1 - tx_i^{-1}x_j}{1 - x_i^{-1}x_j} \frac{1 - tx_i^{-1}x_j}{1 - x_i^{-1}x_j} \phi(x),
\]
The invariance of the integral with respect to the \( q \)-shift \( T_{x_j, q} \) guarantees

\[
\left\langle \prod_{k=2}^n (z' - x_k') \cdot \frac{x_1^{-2}(1-ax_1) \cdots (1-dx_1)}{x_1 - x_1} \prod_{j=2}^n \frac{1-tx_1x_j^{-1}}{1-x_1x_j} \right\rangle
\]

\[
= \left\langle \prod_{k=2}^n (z' - x_k') \cdot \frac{x_1^2(1-ax_1^{-1}) \cdots (1-dx_1^{-1})}{x_1^{-1} - x_1} \times \prod_{j=2}^n \frac{1-tx_1^{-1}x_j 1-tx_1^{-1}x_j^{-1}}{1-x_1^{-1}x_j^{-1}} \right\rangle,
\]

which implies, by considering the transposition \( x_1 \mapsto x_1^{-1} \),

\[
0 = \left\langle \prod_{k=2}^n (z' - x_k') \cdot \frac{x_1^{-2}(1-ax_1) \cdots (1-dx_1)}{x_1^{-1} - x_1} \prod_{j=2}^n \frac{1-tx_1x_j^{-1}}{1-x_1x_j} \right\rangle.
\]

This can be written as

\[
0 = \sum_{l=0}^{n-1} \left\langle \prod_{k=2}^n (z' - x_k') \cdot \frac{x_1^{-2}(1-ax_1) \cdots (1-dx_1)}{x_1^{-1} - x_1} \times \gamma(x_1, t)^l \sum_{1 < k_1 < \ldots < k_l \leq n} \frac{1}{(x_1' - x_{k_1}')(x_1' - x_{k_2}')(\ldots)(x_1' - x_{k_l}')} \right\rangle,
\]

where \( \gamma(x, t) = (t^{-1} - 1)(z^{-1} - tz) \), by using the expansion

\[
\prod_{j=2}^n \frac{1-tx_1x_j^{-1}}{1-x_1x_j}
\]

\[
= \prod_{j=2}^n \frac{t^2 x_1 + x_1^{-1} - tx_j - tx_j^{-1}}{x_1 + x_1^{-1} - x_j - x_j^{-1}}
\]

\[
= \prod_{j=2}^n \left\{ t + \frac{\gamma(x_1, t)}{x_1' - x_j'} \right\}
\]

\[
= t^{n-1} \prod_{j=2}^n \left\{ 1 + \sum_{l=1}^{n-1} \gamma(x_1, t)^l \sum_{1 < k_1 < \ldots < k_l \leq n} \frac{1}{(x_1' - x_{k_1}')(x_1' - x_{k_2}')(\ldots)(x_1' - x_{k_l}')} \right\}.
\]

On one hand, if we set

\[
\left[ \frac{x^{-2}(1-ax) \cdots (1-dx)}{x^{-1} - x} \gamma(x, t)^l \right]_x = \sum_{0 \leq m \leq l+1} C_m^{(l)}(z' - x')^m
\]
with

\[
C_0^{(l)} = \left[ z^{-2(1 - az) \cdots (1 - dz)} \frac{1}{z - t} z^{t-1} \right]_z,
\]

\[
C_{l+1}^{(l)} = (-1)^{l+1} \frac{1}{l!} (t^{-1} - t)^l (1 - (-t)^l \cdot abcd)
\]

and

\[
[f(z)]_z = \frac{1}{2} \{ f(z) + f(z^{-1}) \},
\]

we have

\[
\left\langle \prod_{2 \leq k \leq n} (z' - x_k') \frac{x_1'^{-2}(1 - ax_1) \cdots (1 - dx_1)}{x_1'^{-1} - x_1} \right. \\
\times \gamma(x_1, t)^l \sum_{1 < k_1 < \ldots < k_n \leq n} \frac{1}{(x_1' - x_{k_1}') \cdots (x_1' - x_{k_n}')}
\]

\[
= \left\langle \prod_{2 \leq k \leq n} (z' - x_k') \sum_{1 < k_1 < \ldots < k_n \leq n} \frac{1}{(x_1' - x_{k_1}') \cdots (x_1' - x_{k_n}')} \right. \\
\times \sum_{1 \leq m \leq l+1} C_m^{(l)} \left( \prod_{1 \leq k \leq n} (z' - x_k') \right)
\]

\[
+ \frac{1}{n} C_0^{(l)} \left\langle e_{n-l-1}(z' - x') \right. \\
+ \frac{(-1)^l}{l+1} \binom{n-1}{l} C_{l+1}^{(l)} \left( e_{l}(z' - x') \right), \tag{4.3}
\]

where \( e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k} \) is the \( k \)th elementary symmetric polynomial, and \( e_k(z' - x') \) expresses \( e_k(z' - x_1', \ldots, z' - x_n') \) for brevity.

The last equality of (4.3) follows from the relations

\[
\left\langle \prod_{2 \leq k \leq n} (z' - x_k') \sum_{1 < k_1 < \ldots < k_n \leq n} \frac{1}{(x_1' - x_{k_1}') \cdots (x_1' - x_{k_n}')} \right. \\
= \frac{1}{n} e_{n-l-1}(z' - x') \tag{4.4}
\]
and

\[
\left( \prod_{1 \leq k \leq n} (z' - x_k') \right) \sum_{1 < k_1 < \cdots < k_l \leq n} \frac{(z' - x_1')^{m-1}}{(x_1' - x_2') \cdots (x_l' - x_{l+1}')}
\]

\[
= \binom{n - 1}{l} \left( \prod_{1 \leq k \leq n} (z' - x_k') \frac{(z' - x_1')^{m-1}}{(x_1' - x_2') \cdots (x_l' - x_{l+1}')}
\right)
\]

\[
= \begin{cases} 
0, & \text{if } 1 \leq m \leq l, \\
\frac{(-1)^l}{l + 1} \binom{n - 1}{l}, & \text{if } m = l + 1.
\end{cases}
\]

Equations (4.4) and (4.5) are derived by

\[
\sum_{1 \leq j \leq l+1} \sum_{1 < k_1 < \cdots < k_l \leq j} \frac{(z' - x_j) \prod_{x_j \neq x_i} (x_j' - x_i)}{(x_j - x_i) \cdots (x_j' - x_i)} = e_{n-l-1}(z' - x')
\]

and

\[
\sum_{1 \leq j \leq l+1} (z' - x_j) \prod_{x_j \neq x_i} (x_j' - x_i) = \begin{cases} 
0, & \text{if } 0 \leq p \leq l - 1, \\
1, & \text{if } p = l.
\end{cases}
\]

which follow from substituting \(z' - x_j\) into \(y_j\) of the equalities of Lemma 1 and Lemma 2 in the next section.

Combination of (4.2) and (4.3) leads to

\[
0 = \sum_{l=0}^{n-1} \left\{ \frac{1}{n} \binom{n}{l} \left( e_{n-l-1}(z' - x') \right) + \frac{(-1)^l}{l + 1} \binom{n - 1}{l} C_{l+1}^{(l)} e_{l+1}(z' - x') \right\}.
\]

(4.6)

Here we have

\[
\sum_{l=0}^{n-1} \frac{(-1)^l}{l + 1} \binom{n}{l} C_{l+1}^{(l)} = \frac{-1}{2n} \sum_{l=1}^{n} \binom{n}{l} (t^{-1} - 1)^{l-1} (1 - (-t)^{l-1})abcd
\]

\[
= \frac{-1}{2n} (t^{-1} - 1)^{-1} \left\{ \sum_{l=1}^{n} \binom{n}{l} (t^{-1} - 1)^{l} + t^{-1}abcd \right\}.
\]

(4.7)
The binomial theorem shows
\[
\sum_{l=0}^{n} \binom{n}{l} (t-1)^l = \sum_{l=0}^{n} \binom{n}{l} (t-1)^l - 1 = t^n - 1
\]
and
\[
\sum_{l=1}^{n} (t^{-1} - 1)^l \binom{n}{l} = t^{-n} - 1,
\]
which imply
\[
\sum_{l=1}^{n} \binom{n}{l} (t^{-1} - 1)^l + t^{-1}abcd \sum_{l=1}^{n} \binom{n}{l} (t-1)^l
\]
\[
= t^{-n} - 1 + t^{-1}abcd(t^n - 1)
\]
\[
= (t^{-n} - 1)(1 - t^{-1}abcd). \tag{4.8}
\]
As a consequence, by (4.7) and (4.8), (4.6) is written as
\[
0 = \sum_{l=0}^{n-1} C_{n-l}^l \langle e_{n-l}(z'-x') \rangle - \frac{1}{2}(t^{-1} - 1)^{-1} (t^{-n} - 1)
\]
\[
\times (1 - abcd(t^{-n} - 1)) \langle e_n(z'-x') \rangle. \tag{4.9}
\]
Finally we make a bridge between (4.1) and (4.9). We have
\[
\gamma(z^{-1}, t) = \gamma(z, t^{-1}) \text{ and }
\]
\[
T_{z, t^n} \prod_{s=1}^{n} (z'-x'_s) = \prod_{s=1}^{n} (\gamma(z, t) + z'-x'_s) = \sum_{0 \leq l \leq n} \gamma(z, t)^l e_{n-l}(z'-x').
\]
Hence we obtain
\[
D_{z, t^n} \prod_{s=1}^{n} (z'-x'_s)
\]
\[
= \frac{(1 - az) \cdots (1 - dz)}{(1 - z^n)(1 - tz^n)} \sum_{0 \leq l \leq n} \gamma(z, t)^l e_{n-l}(z'-x')
\]
\[
+ \frac{(1 - az^{-1}) \cdots (1 - dz^{-1})}{(1 - z^{-n})(1 - tz^{-n})} \sum_{0 \leq l \leq n} \gamma(z^{-1}, t)^l e_{n-l}(z'-x').
\]
Thus, (4.1) is equivalent to (4.9). What remains is to demonstrate Lemma 1 and Lemma 2.

5. LEMMAS

Lemma 1.

\[
\sum_{j=1}^{n} \frac{y_j^l}{(y_j - y_k)} = \begin{cases} 
0, & 0 \leq l \leq n - 2, \\
1, & l = n - 1.
\end{cases} \tag{5.1}
\]

Proof. The left-hand side of (5.1) can be expressed by means of the integral

\[
\sum_{j=1}^{n} \text{Res}_{x = y_j} \left( \prod_{1 \leq k < n, \ k \neq j} \frac{x^l}{(x - y_k)} \right) dx = \frac{1}{2\pi i} \oint \left( \prod_{1 \leq k \leq n} \frac{x^l}{(x - y_k)} \right) dx,
\]

where the contour \( C \) circles the origin in the counterclockwise direction so that all poles \( y_1, \ldots, y_n \) are inside the contour. Moreover, by using power series expansions, we have

\[
\frac{1}{2\pi i} \oint \left( \prod_{1 \leq k \leq n} \frac{x^l}{(x - y_k)} \right) dx = \sum_{m_1, \ldots, m_n \geq 0} \frac{1}{2\pi i} \oint \left( \prod_{1 \leq k \leq n} \frac{x^{l-n}}{1 - \frac{y_k}{x}} \right) dy
\]

\[
= \begin{cases} 
0, & 0 \leq l \leq n - 2, \\
\sum_{m_1, \ldots, m_n \geq 0} y_1^{m_1} \cdots y_n^{m_n}, & l \geq n - 1.
\end{cases}
\]

Hence we reach the desired relations. \( \blacksquare \)
Lemma 2. For \( l = 0, \ldots, n-1 \), we have
\[
\sum_{j=1}^{n} \frac{y_1 \cdots y_j \cdots y_n}{y_{k_1} y_{k_2} \cdots y_{k_j} \cdots y_{k_l}} = e_{n-l-1}(y_1, \ldots, y_n),
\] (5.2)
where \( y_k \) designates \( y_k - y_j \).

**Proof.** The left-hand side of (5.2) is equal to
\[
\sum_{1 \leq k_1 < k_2 < \cdots < k_{l+1} \leq n} \left\{ \frac{y_1 \cdots y_{k_1} \cdots y_{k_{l+1}}}{y_{k_1} y_{k_2} \cdots y_{k_{l+1}}}, \frac{y_1 \cdots y_{k_1} \cdots y_{k_{l+1}}}{y_{k_1} y_{k_2} \cdots y_{k_{l+1}}} \right\}
\]
\[
+ \cdots + \frac{y_1 \cdots y_n}{y_{k_1} y_{k_2} \cdots y_{k_{l+1}}}
\]
\[
= \sum_{1 \leq k_1 < k_2 < \cdots < k_{l+1} \leq n} \frac{y_1 \cdots y_{k_1} \cdots y_{k_{l+1}}}{y_{k_1} y_{k_2} \cdots y_{k_{l+1}}} \left\{ \frac{y_{k_1} \cdots y_{k_{l+1}}}{y_{k_1} y_{k_2} \cdots y_{k_{l+1}}} + \cdots \right\}
\]
\[
= \sum_{1 \leq k_1 < k_2 < \cdots < k_{l+1} \leq n} \frac{y_1 \cdots y_{k_1} \cdots y_{k_{l+1}}}{y_{k_1} y_{k_2} \cdots y_{k_{l+1}}} = e_{n-l-1}(y_1, \ldots, y_n).
\] (5.3)

The second equality in (5.3) is obtained by
\[
\sum_{1 \leq r < l+1} \frac{y_{k_r} \cdots y_{k_{l+1}}}{y_{k_r} y_{k_1} \cdots y_{k_{l+1}}} = \sum_{1 \leq r < l+1} \frac{y_{k_r}^{-1}}{\prod_{1 \leq p < r} (y_{k_p}^{-1} - y_{k_r}^{-1})} = 1,
\]
which is from Lemma 1. This completes the proof of Lemma 2.

This completes the proof of our Theorem.

6. FINAL COMMENT

Our work should be generalized to the case of multivariable polynomials: the Macdonald–Koornwinder polynomials [12, 13]. By considering the integral representation of the Jack polynomials given by [20], it is conjectured that when
\[
f(z, t) = \prod_{1 \leq l \leq m} \prod_{1 \leq k \leq n} (z_l - x_k),
\]
the integral

\[ \int_{\Gamma_n} f(z, x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \]

represents the Macdonald–Koornwinder polynomial parametrized by the rectangular Young diagram \((n^m)\). In our future work, we would like to solve it.

REFERENCES


