Characterizations for restricted graphs of NLC-width 2

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Abstract

In this paper we give a finite forbidden subgraph characterization of graphs defined by NLC-width 2-expressions, by NLCT-width 2-expressions, or by linear NLC-width 2-expressions that have tree-width 1.

Keywords: NLC-width; NLCT-width; Linear NLC-width; Graph characterizations

1. Introduction

The NLC-width of a graph is defined by a composition mechanism for vertex-labeled graphs [11]. The operations are the unions of two graphs in which edges can be inserted specified by a set of label pairs, and the relabeling of vertices. The NLC-width of a graph $G$ is the minimum number of labels needed to define it. A similar concept which is called clique-width was defined by Courcelle and Olariu [2]. NLC-width and clique-width bounded graphs are particularly interesting from an algorithmic point of view, since a lot of NP-complete graph problems can be solved in polynomial time for graphs of bounded NLC-width [1,4,6].

The computation of the NLC-width of a given graph has shown to be NP-complete [7]. The recognition problem for graphs of NLC-width at most $k$ is still open for any fixed $k \geq 3$. NLC-width of at most 2 is decidable in polynomial time [9]. Graphs of NLC-width 1 are co-graphs, i.e. $P_4$-free, and thus recognizable in linear time [3,11].

The following two restrictions of NLC-width have been defined. A graph has linear NLC-width at most $k$ if it can be defined by an NLC-width $k$-expression in that at least one argument of every union operation defines a single labeled vertex [8]. An extended form of linear NLC-width is the NLCT-width [11,8], where additionally the disjoint union of two defined graphs is permitted as an operation. The set of all graphs of NLCT-width 1 is exactly the set of all $(C_4, P_4)$-free graphs and thus equal to the set of trivially perfect graphs, and further the set of all graphs of linear NLC-width 1 is exactly the set of all $(C_4, P_4, 2K_2)$-free graphs and thus equal to the set of threshold graphs; see [5].

In this paper we give a forbidden subgraph characterization for graphs of NLC-width at most 2, graphs of NLCT-width at most 2, and graphs of linear NLC-width at most 2, for the case where they have tree-width $^1$ 1. This is the first characterization of graphs defined by NLC-width 2-expressions.

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1 See [10] for a definition of tree-width.
2. Preliminaries

Let \([k] := \{1, \ldots, k\}\) be the set of all integers between 1 and \(k\). We work with finite undirected labeled graphs \(G = (V_G, E_G, \text{lab}_G)\), where \(V_G\) is a finite set of vertices labeled by some mapping \(\text{lab}_G : V_G \to [k]\) and \(E_G \subseteq \{(u, v) \mid u, v \in V_G, u \neq v\}\) is a finite set of edges. A labeled graph \(J = (V_J, E_J, \text{lab}_J)\) is a subgraph of \(G\) if \(V_J \subseteq V_G, E_J \subseteq E_G\) and \(\text{lab}_J(u) = \text{lab}_G(u)\) for all \(u \in V_J\). \(J\) is an induced subgraph of \(G\) if additionally \(E_J = \{(u, v) \in E_G \mid u, v \in V_J\}\). The labeled graph consisting of a single vertex labeled by \(a \in [k]\) is denoted by \(\bullet_a\).

The notion of NLCT-width of labeled graphs is defined by Wanke in [11].

**Definition 1 (NLCT\(_k\), NLCT-width [11]).** The graph class NLCT\(_k\) of labeled graphs is recursively defined as follows.

1. The single vertex graph \(\bullet_a\) for some \(a \in [k]\) is in NLCT\(_k\).
2. Let \(G = (V_G, E_G, \text{lab}_G) \in\) NLCT\(_k\) and \(J = (V_J, E_J, \text{lab}_J) \in\) NLCT\(_k\) be two vertex disjoint labeled graphs and \(S \subseteq [k]^2\) be a relation; then \(G \times_S J := (V', E', \text{lab}'\)) defined by \(V' := V_G \cup V_J\), \(E' := E_G \cup E_J \cup \{(u, v) \mid u \in V_G, v \in V_J, (\text{lab}_G(u), \text{lab}_J(v)) \in S\}\), and

\[
\text{lab}'(u) := \begin{cases} 
\text{lab}_G(u) & \text{if } u \in V_G \\
\text{lab}_J(u) & \text{if } u \in V_J, \forall u \in V'
\end{cases}
\]

is in NLCT\(_k\).
3. Let \(G = (V_G, E_G, \text{lab}_G) \in\) NLCT\(_k\) be a labeled graph and \(R : [k] \to [k]\) be a function; then \(\circ_R(G) := (V_G, E_G, \text{lab}_G)\) defined by \(\text{lab}'(u) := R(\text{lab}_G(u)), \forall u \in V_G\) is in NLCT\(_k\).

The **NLCT-width** of a graph \(G\) is the least integer \(k\) such that \(G \in\) NLCT\(_k\).

The operations of NLCT-width\(^2\) are defined in [11,8] as a restriction of the operations of NLCT-width.

**Definition 2 (NLCT\(_k\), NLCT-width [8]).** The graph class NLCT\(_k\) of labeled graphs is recursively defined as follows.

1. The single vertex graph \(\bullet_a\) for some \(a \in [k]\) is in NLCT\(_k\).
2. Let \(G = (V_G, E_G, \text{lab}_G) \in\) NLCT\(_k\) and \(J = (V_J, E_J, \text{lab}_J) \in\) NLCT\(_k\) be two vertex disjoint labeled graphs, \(S \subseteq [k]^2\), and \(a \in [k]\); then
   (a) \(G \times_S J\) is in NLCT\(_k\)
   (b) \(G \times_S \bullet_a\) is in NLCT\(_k\).
3. Let \(G \in\) NLCT\(_k\) and \(R : [k] \to [k]\); then \(\circ_R(G)\) is in NLCT\(_k\).

The **NLCT-width** of a graph \(G\) is the least integer \(k\) such that \(G \in\) NLCT\(_k\).

A further restriction of NLCT-width and NLCT-width operations yields to the definition of linear NLCT-width.

**Definition 3 (lin-NLCT\(_k\), linear NLCT-width [8]).** The graph class lin-NLCT\(_k\) of labeled graphs is recursively defined as follows.

1. The single vertex graph \(\bullet_a\) for some \(a \in [k]\) is in lin-NLCT\(_k\).
2. Let \(G = (V_G, E_G, \text{lab}_G) \in\) lin-NLCT\(_k\) be a labeled graph, \(S \subseteq [k]^2\), and \(a \in [k]\); then \(G \times_S \bullet_a\) is in lin-NLCT\(_k\).
3. Let \(G \in\) lin-NLCT\(_k\) and \(R : [k] \to [k]\); then \(\circ_R(G)\) is in lin-NLCT\(_k\).

The **linear NLCT-width** of a graph \(G\) is the least integer \(k\) such that \(G \in\) lin-NLCT\(_k\).

An expression \(X\) built with the operations \(\bullet_a, \times_S, \circ_R\) for \(a \in [k]\), \(S \subseteq [k]^2\), and \(R : [k] \to [k]\) according to Definition 1, Definition 2, or Definition 3 is called an NLCT-width \(k\)-expression, NLCT-width \(k\) expression, or linear NLCT-width \(k\) expression, respectively. The NLCT-width (NLCT-width, linear NLCT-width) of an unlabeled graph \(G = (V, E)\) is the smallest integer \(k\), such that there is some mapping \(\text{lab} : V \to [k]\) such that the labeled graph \((V, E, \text{lab})\) has NLCT-width (linear NLCT-width, linear NLCT-width) at most \(k\). The graph defined by expression \(X\) is denoted

\(^2\) The abbreviation NLCT indicates those restrictions of NLCT-width operations which are powerful enough to define graphs of bounded tree-width.
by \( \text{val}(X) \). By the definition of \( k \)-expressions it is easy to verify that graphs of bounded NLC-width, graphs of bounded NLCT-width, and graphs of bounded linear NLC-width are closed under taking induced subgraphs.

For example any path \( P_n = (\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\}) \), denoted by \( v_1v_2v_3 \ldots v_n \), has linear NLC-width (and thus NLC-width and NLCT-width) at most 3; this can easily be shown with the following expressions for \( X_{P_n} \):

\[
X_{P_3} = (\bullet \times \{(1,2)\}) \times \{(2,3)\} \bullet 3
\]

\[
X_{P_n} = \circ_{\{(1,1),(2,1),(3,2)\}}(X_{P_{n-1}}) \times \{(2,3)\} \bullet 3, \quad n \geq 4.
\]

The length of a path is the number of its edges. The distance between two vertices \( u \) and \( w \) of some graph \( G \) is the length of a shortest path between \( u \) and \( w \) in \( G \). The diameter \( d(G) \) of a graph \( G \) is the greatest distance between two vertices of \( G \).

Further results on graph classes of bounded linear NLC-width or bounded NLCT-width, their relations, and corresponding restrictions for the operations of clique-width can be found in [8].

The concept of NLC-width generalizes the well-known concept of tree-width defined in [10] with the existence of a tree-decomposition. The set of graphs of tree-width at most \( k \) is denoted by \( \text{TW}_k \). In this paper we consider graphs in \( \text{TW}_1 \), which are also denoted as forests.

3. Characterizations for graphs in \( \text{TW}_1 \cap \text{NLC}_1 \)

Graphs of NLC-width 1 are known to be co-graphs, i.e. \( P_4 \)-free graphs [11]. The set of all graphs of NLCT-width 1 is exactly the set of all \((C_4, P_4)\)-free graphs and thus equal to the set of trivially perfect graphs; further the set of all graphs of linear NLC-width 1 is exactly the set of all \((C_4, P_4, 2K_2)\)-free graphs and thus equal to the set of threshold graphs; see [5].

If we just consider graphs of tree-width 1, it is easy to obtain the following characterization using the three graphs shown in Fig. 1.

**Corollary 4.** For every \( G \in \text{TW}_1 \) the following statements are equivalent.

1. \( G \) has NLC-width 1.
2. \( G \) has NLCT-width 1.
3. \( G \) contains no \( P_4 \) as an induced subgraph.
4. \( G \) is the disjoint union of some \( K_{1,n} \).

**Corollary 5.** For every \( G \in \text{TW}_1 \) the following statements are equivalent.

1. \( G \) has linear NLC-width 1.
2. \( G \) contains no \( 2K_2 \) and no \( P_4 \) as an induced subgraph.
3. \( G \) is the subgraph of some \( K_{1,n} \).

4. Characterizations for graphs in \( \text{TW}_1 \cap \text{NLCT}_2 \) and \( \text{TW}_1 \cap \text{NLC}_2 \)

In order to find forbidden induced trees for graphs of tree-width 1 and NLC-width at most 2, we first want to notice that we do not have to take care with the relabeling operation, except for a final relabeling.

**Lemma 6.** Let \( G \in \text{NLC}_2, \ G \in \text{NLCT}_2, \) or \( G \in \text{lin-NLC}_2 \) be a tree; then there exists an NLC-width 2-expression, NLCT-width 2-expression, or linear NLC-width 2-expression, respectively, for \( G \) that does not use any relabeling operation.
By a case distinction on the possible 2-expressions and $G$ contains no graph of $X$. Easy by the definition of NLC-width.

(1) to characterize graphs of NLC-width at most 2 and NLCT-width at most 2.

For every $G \in \text{TW}_1$ the following statements are equivalent.

(1) $G$ has NLCT-width at most 2.

(2) $G$ has NLC-width at most 2.

(3) $G$ contains no graph of $X_1 = \{P_{10}, G_1, G_2, G_3, G_4, G_5\}$ as an induced subgraph.

Proof. (1) $\Rightarrow$ (2) Easy by the definition of NLC-width.

(2) $\Rightarrow$ (3) By a case distinction on the possible 2-expressions and Lemma 6, it is easy to show that graphs in $X_1$ have NLC-width $\geq 2$. Since graphs of bounded NLC-width are closed under taking induced subgraphs, graphs of NLC-width at most 2 cannot contain any graph of $X_1$ as an induced subgraph.

(3) $\Rightarrow$ (1) Let $G$ be a tree\footnote{Since the disjoint union is a feasible operation for NLC-width and NLCT-width, we can assume without loss of generality that we have given a tree.} that satisfies condition (3) of the theorem. Since $G$ does not contain a $P_{10}$ as a subgraph we can show our statement by giving an NLC-width 2-expression $X$ for $G$ by a case distinction on its diameter $d(G)$. For any $4 \leq d \leq 8$ we show how to construct a complete tree $\text{val}(X)$ of diameter $d$ by constructing a path $p = v_1 \ldots v_{d+1}$ of length $d$ and all possible subtrees adjacent to the vertices of $p$. These subtrees are shown in Fig. 3 and can be defined by the following NLC-width 2-expressions: $X_{B_h}, 0 \leq h \leq 3$.

- $X_{B_0} = \bullet_2$
- $X_{B_1} = (\bullet_2 \times (2,1)\bullet_1) \ldots \times (2,1)\bullet_1$
- $X_{B_2} = (((\bullet_2 \times \bullet_1) \ldots \times \bullet_1) \times (1,1),(2,1)\bullet_1) \ldots \times (2,1)\bullet_1$
- $X_{B_3} = (((((\bullet_1 \times (1,2))\bullet_2) \times \bullet_1) \ldots \times \bullet_1) \times (1,1),(2,1)\bullet_1) \ldots \times (2,1)\bullet_1$

If $d(G) \leq 4$, then $G$ is an induced subgraph of graph $\text{val}(X)$ which contains of a path $p = v_1v_2v_3v_4v_5$ of length 4 such that vertices $v_2$ and $v_4$ are adjacent to a number of trees $B_0$ and vertex $v_3$ is adjacent to a number of trees $B_1$, which obviously also can be defined by the following NLCT-width 2-expression: $X = ((X_{B_1} \times \emptyset X_{B_1}) \ldots \times \emptyset X_{B_1}) \times (2,2)\bullet_2$. \hfill \Box
For every $G$ vertex if vertex $v$ is adjacent to subtrees of type $B_1$; otherwise $G_5$ would be an induced subgraph, and we assume $v_3$ to have this property. Then vertices $v_2$, $v_4$, and $v_5$ are adjacent to a number of trees $B_0$ and $v_3$ is adjacent to a number of trees $B_1$, which obviously also can be defined by the following NLCT-width 2-expression: $X = ((X_{B_0} \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1}) \times_{\{i\leq 2\}} \bullet_2$.

- If $d(G) = 6$, then $G$ is an induced subgraph of graph $\text{val}(X)$ which contains a path $p = v_1v_2v_3v_4v_5v_6$ of length 6, where only vertex $v_1$ or $v_4$ may be adjacent to subtrees of type $B_1$; otherwise $G_5$ would be an induced subgraph, and we assume $v_3$ to have this property. Then vertices $v_2$, $v_4$, and $v_5$ are adjacent to a number of trees $B_0$ and $v_3$ is adjacent to a number of trees $B_1$, which obviously also can be defined by the following NLCT-width 2-expression: $X = ((X_{B_0} \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1}) \times_{\{i\leq 2\}} \bullet_2$.

- If $d(G) = 7$, then $G$ is an induced subgraph of graph $\text{val}(X)$ which contains a path $p = v_1v_2v_3v_4v_5v_6v_7v_8$ of length 7. Either vertex $v_3$ or vertex $v_6$ does not have any further adjacent vertices besides those of $p$; otherwise $G_3$ would be an induced subgraph, and we assume vertex $v_6$ to have this property. Vertex $v_3$ may only be adjacent to subtrees of type $B_0$; otherwise $G_2$ would be an induced subgraph. Either vertex $v_4$ or vertex $v_5$ does not have any further adjacent trees $B_1$ or $B_2$; otherwise $G_5$ would be an induced subgraph,

$$
\text{if vertex } v_4 \text{ does so, then } \text{val}(X) \text{ can be defined by the following NLCT-width 2-expression: } X = (((X_{B_1} \times_{\emptyset} X_{B_1}) \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1}) \times_{\{i\leq 2\}} \bullet_2.
$$

- $G_3$ would be an induced subgraph, and $\text{val}(X)$ can be defined by the following NLCT-width 2-expression: $X = (((X_{B_1} \times_{\emptyset} X_{B_1}) \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1}) \times_{\{i\leq 2\}} \bullet_2$.

- If $d(G) = 8$, then $G$ is an induced subgraph of graph $\text{val}(X)$ which contains a path $p = v_1v_2v_3v_4v_5v_6v_7v_8$ of length 8. Vertices $v_4$ and $v_6$ may only be adjacent to subtrees of type $B_0$; otherwise $G_2$ would be an induced subgraph. Further vertices $v_3$ and $v_7$ have no further neighbors outside $p$; otherwise $G_1$ would be an induced subgraph. Finally, vertex $v_5$ can be adjacent to a number of trees of type $B_0$, $B_1$, $B_2$, and $B_3$ and thus $\text{val}(X)$ can be defined by the following NLCT-width 2-expression: $X = (((X_{B_1} \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1}) \ldots \times_{\emptyset} X_{B_1} \ldots \times_{\emptyset} X_{B_1} \times_{\{i\leq 2\}} \bullet_2$.

5. Characterizations for graphs in $TW_1 \cap \text{lin-NLC}_2$

If we consider graphs defined by linear NLC-width 2-expressions we conclude that $P_6$ is the longest path which can be defined, and thus each graph in $\text{lin-NLC}_2$ has diameter at most 5. We next use the four graphs shown in Fig. 4 to characterize graphs of linear NLC-width at most 2 for the case of forests.

Theorem 8. For every $G \in TW_1$ the following statements are equivalent.
G has linear NLC-width at most 2.
(2) G contains no graph of $X_2 = \{2P_4, P_7, G_6, G_7\}$ as an induced subgraph.

Proof. (1) $\Rightarrow$ (2) By a case distinction on the possible expressions it is easy to show that graphs in $X_2$ have linear NLC-width > 2.
(2) $\Rightarrow$ (1) First we want to note that the expressions $G$ contains no graph of $B$. By a case distinction on the possible expressions it is easy to show that graphs in $D$. G. Corneil, Y. Perl, L.K. Stewart, A linear recognition algorithm for cographs, SIAM Journal on Computing 14 (4) (1985) 926–934.

Theorem.

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6. Conclusions

In this paper we found first forbidden subgraph characterizations for restricted graphs of NLC-width 2.

If we want to characterize general graphs of NLC-width at most 2, we have to add further forbidden graphs (e.g. cycles $C_7, C_8, C_9, C_{10}$) to set $X_1$ which are more difficult to find, since for general graphs the relabeling operation cannot be omitted. But since also for general graphs of NLC-width at most 2 the diameter is bounded by 9, this problem seems to be solvable. If we even consider graphs defined by 3-expressions, we know that all trees are definable by NLCT-width 3-expressions [8], while there exists no integer $k$ such that all trees are definable by linear NLC-width $k$-expressions [8].

In order to characterize further sets of graphs of bounded NLC-width we suggest following two strategies. The first method was used in [5] where subclasses of co-graphs defined by restricted NLC-width operations are characterized by adding additional forbidden subgraphs to the known forbidden induced subgraph ($P_4$) of co-graphs. Further we suggest following the constructive strategy of this paper at least for $k$-connected graphs of bounded NLC-width.

References


