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Sets of Periods for Maps on Connected Graphs with Zero Euler Characteristic Having All Branching Points Fixed

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We characterize all possible sets for all continuous self-maps on a connected topological graph with zero Euler characteristic having all branching points fixed.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let f: $X \to X$ be a continuous map on the topological space X. A point x of X will be called *periodic with respect to* f (or just *periodic*, if f is obvious from the context) if $f^n(x) = x$ for some integer $n > 0$, where f^n is f composed with itself n times. The least n satisfying the above equality is called the *period* of x. The *orbit* of x is the set $\{f^n(x): n \geq 0\}$, where f^0 is the identity map. We denote by $\text{Per}(f)$ the set $\{n : f$ has a point of period $n\}$.

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In the 1960s Sarkovskii [12] proved a remarkable theorem about the interrelationships of periodic points of continuous maps on the closed unit interval. Let $\leftarrow$ (the *Sarkovskii ordering*) be the following linear ordering of the positive integers (a more precise definition will be given below):

$$1 \leftarrow 2 \leftarrow 2^2 \leftarrow 2^3 \leftarrow \cdots \leftarrow 7 \cdot 2^2 \leftarrow 5 \cdot 2^2 \leftarrow 3 \cdot 2^2 \leftarrow \cdots$$
$$\leftarrow 7 \cdot 2 \leftarrow 5 \cdot 2 \leftarrow 3 \cdot 2 \leftarrow \cdots \leftarrow 7 \leftarrow 5 \leftarrow 3.$$

INTERVAL THEOREM (Sarkovskii [12]). *Let I be the unit interval.*

(1) *For every continuous map $f\colon I \to I$, if $k \in \mathrm{Per}(f)$, then $m \in \mathrm{Per}(f)$ for every $m \leftarrow k$.*

(2) *Conversely, if S is any initial segment of the Sarkovskii ordering (i.e., a set of positive integers which is closed under $\leftarrow$-predecessors), then there is a continuous map $f\colon I \to I$ such that* $\mathrm{Per}(f) = S$.

Attempts to generalize Sarkovskii's theorem have proceeded in several different directions. The formulation we shall use first appeared in [1], where Sarkovskii's theorem was viewed as a characterization of all sets $\mathrm{Per}(f) = S$ (as in the statement of Sarkovskii's theorem above). In [1], Alsedà, Llibre, and Misiurewicz gave a characterization of all sets $\mathrm{Per}(f)$ for continuous maps f on $Y = \{z \in C : z^3 \in [0,1]\}$ having the branching point 0 fixed. This was extended by Baldwin [3] to all continuous maps on the n-od, as described below.

The n-od X_n is the subspace of the plane which is most easily described as the set of all complex numbers z such that z^n is in the unit interval I, i.e., a central point (the origin 0) with n copies of I attached. Notice that the 1-od and 2-od are homeomorphic. In order to study the structure of the set of periodics of the continuous maps $f\colon X_n \to X_n$ we need to define partial orderings $\leq_t$ for all positive integers t.

The ordering $\leq_1$ is defined by

$$2^1 \leq_1 2^{i+1} \leq_1 2^{j+1}(2m+1) \leq_1 2^j(2k+3) \leq_1 2^j(2k+1)$$

for all integers $i, j \geq 0$ and $k, m > 0$.

If $n > 1$ then the ordering $\leq_n$ is defined as follows. Let m, k be positive integers.

Case 1. $k = 1$. Then $m \leq_n k$ if and only if $m = 1$.

Case 2. k is divisible by n. Then $m \leq_n k$ if and only if either $m = 1$ or m is divisible by n and $(m/n) \leq_1 (k/n)$.

Case 3. $k > 1$, k not divisible by n. Then $m \leq_n k$ if and only if either $m = 1$, $m =$ k, or $m = ik + jn$ for some integers $i \geq 0$, $j \geq 1$.

In [3] some diagrams illuminating these partial orderings are given. From the definition we have that $\leq_1$ and $\leq_2$ are just the Sarkovskii ordering. A set S of positive integers is an *initial segment* of $\leq_n$ if whenever k is an element of S and $m \leq_n k$, then m is also an element of S; i.e., S is closed under $\leq_n$ -predecessors.

n-od THEOREM (Baldwin [3]). *Let X_n be the n-od.*

(1) *Let f: $X_n \to X_n$ be a continuous map. Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq n\}$.

(2) *Conversely, if S is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq n\}$, *then there is a continuous map f: $X_n \to X_n$ such that* $f(0) = 0$ *and* $\mathrm{Per}(f) = S$.

A *tree* is any space which is uniquely arcwise connected and homeomorphic to the union of finitely many copies of the unit interval. If T is a tree and $x \in T$, the *valence* of x is the number of components of $T \setminus \{x\}$. A point of valence 1 is called an *endpoint* and a point of valence ≥ 3 is called a *branching point*. Given a tree T, let $e(T)$ be the number of its endpoints.

The n-od theorem has been extended by Baldwin and Llibre [6] to continuous maps on a tree having all its branching points fixed. In the case of the n-od, the result is the same, regardless of whether the branching point is required to be fixed. This is not true for trees in general. For example, let H be the tree which consists of two 3-ods attached to one endpoint (i.e., the tree shaped like the letter H). Then there is a continuous map f on H such that $\mathrm{Per}(f) = \{1, 2, 6\}$ (which switches the two branching points; see [4] for this and other examples), but there is no continuous map g: $H \to H$ fixing the branching points such that $\mathrm{Per}(g) = \{1, 2, 6\}$, as will follow from the tree theorem below. Thus, it is of interest to ask what $\mathrm{Per}(f)$ can be if all branching points of T are fixed.

TREE THEOREM (Baldwin and Llibre [6]). *Let T be a tree.*

(1) *Let f: $T \to T$ be a continuous map with all the branching points fixed. Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq e(T)\}$.

(2) *Conversely, if S is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq e(T)\}$, *then there is a continuous map f: $T \to T$ with all the branching points fixed such that* $\mathrm{Per}(f) = S$.

The result on the n-od has been recently extended to all trees by Baldwin (without assumptions on the branching points; see [5]), but these results do not tell which sets $\mathrm{Per}(f)$ are possible if the branching points remain fixed. For similar results on graphs, which characterize sets of periods without specifying which sets of periods correspond to which graphs, see Blokh [9].

A *finite graph* (simply a *graph*) G is a Hausdorff space which has a finite subspace V (points of V are called *vertices*) such that $G \setminus V$ is the disjoint union of a finite number of open subsets $e_1, \dots, e_k$ called *edges*, each e_i is homeomorphic to an open interval of the real line, and at the boundary of each edge are attached one or two vertices.

Note that a graph is compact, since it is the union of a finite number of compact subsets (the closed edges $\bar{e}_i$ and the vertices). It may be connected or disconnected, and it may have isolated vertices.

The number of edges having a vertex as an endpoint (with the closed edges homeomorphic to a circle counted twice) will be called the *valence* of this vertex. As for trees, a point of valence 1 is called an *endpoint* and a point of valence ≥ 3 is called a *branching point*. Given a graph G, let $e(G)$ and $b(G)$ be the number of its endpoints and branching points, respectively.

The rational homology group of G are well known: $H_0(G; Q) \approx Q^c$ and $H_1(G; Q) \approx Q^d$, where c and d are the number of connected components of G and the number of independent circuits of G, respectively. A *circuit* is a subset of G homeomorphic to a circle. The *Euler characteristic* $\chi(G)$ of G is $c - d$. If v and e are the number of vertices and edges of G, respectively, then $\chi(G) = v - e$. For more details on graphs see [11].

The following open question was stated in [6]: What can be said about the sets of periodics of continuous self-maps on connected graphs having all branching points fixed? In [6] this question was solved for connected graphs with Euler characteristic equal to 1, i.e., trees. The main result of this paper is the answer to the above question for connected graphs with zero Euler characteristic. The ordering $\leq_0$ denotes the converse ordering of the usual ordering in the set of natural numbers with one exception, the one which is the smallest element in this ordering, i.e., $\cdots \leq_0 7 \leq_0 6 \leq_0 5 \leq_0 4 \leq_0 3 \leq_0 2$.

GRAPH THEOREM. *Let G be a connected graph such that $\chi(G) = 0$ and $b(G) \neq 0$.*

(1) *Let $f\colon G \to G$ be a continuous map with all branching points fixed. Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 0 \leq p \leq e(G) + 2\}$.

(2) *Conversely, if S is a nonempty finite union of initial segments of* $\{\leq_p : 0 \leq p \leq e(G) + 2\}$, *then there is a continuous map $f\colon G \to G$ with all the branching points fixed such that* $\mathrm{Per}(f) = S$.

The proof of the graph theorem will be given in Section 2.

If G is a connected graph such that $\chi(G) = 0$ and $b(G) = 0$, then G is homeomorphic to the circle S^1. The sets of periods of continuous self-maps on the circle have been characterized by Block et al. [8], Block [7], and Misiurewicz (see also [2]). A particular case of a connected graph σ with $\chi(\sigma) = 0$ and $b(\sigma) \neq 0$, where σ is the graph shaped like the letter sigma, was studied by the authors in [10].

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2. PROOF OF THE GRAPH THEOREM

Let G be a connected graph such that $\chi(G) = 0$ and $b(G) \neq 0$. Since G is connected and its Euler characteristic is zero, G has a unique circuit C.

We say that $I \subset G$ is a *closed interval* if there exists a homeomorphism h between the closed interval $[0, 1]$ and I. Set $h(0) = a$ and $h(1) = b$. Then we write $I = [a, b] = [b, a]$ and we say that a and b are the *endpoints* of I. We define the *open interval* (a, b) by $[a, b] \setminus \{a, b\}$. Notice that in this notation the "open" interval (a, b) is not always an open set in the topology of the graph.

Let I be an open interval of G such that $G \setminus I$ is a tree with $e(G) + 2$ endpoints. If S is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(G) + 2\}$, then from statement (2) of the tree theorem there is a continuous map f: $G \to G$ with all branching points fixed such that $\text{Per}(f|_{G \setminus I}) = S$, $f(I) \subset f(G \setminus I)$, and consequently $\text{Per}(f) = \text{Per}(f|_{G \setminus I}) = S$. Hence the orderings $\leq_p$ with $1 \leq p \leq e(G) + 2$ are necessary in order to describe the sets of periods of all continuous self-maps on G. The necessity of the ordering $\leq_0$ will become clear later in Theorem 2.

Let f: $G \to G$ *be a continuous map having all branching points of G fixed.* Taking into account the previous paragraph we prove statement (1) of the graph theorem by showing that if $k \in \text{Per}(f)$ then there exists an ordering $\leq_p$ with $0 \leq p \leq e(G) + 2$ such that $m \in \text{Per}(f)$ for all $m \leq_p k$. Therefore, we assume that $k \in \text{Per}(f)$ and that P is a periodic orbit of f with period k.

After removing 1 from the set of natural numbers (due to the fact that f has fixed branching points), the smallest natural numbers in the orderings $\leq_p$ for $p = 1, 2, \ldots, e(G) + 2$ are $2, 2, 3, 4, \ldots, e(G) + 2$, respectively; therefore we can assume that $e(G) + 2 < k$.

If $P \cap C \neq \emptyset$ then we denote by $[P]$ the smallest connected subgraph of G containing $P \cup C$. Now we assume that $P \cap C = \emptyset$. First we shrink C to a point z and we denote by G' the tree obtained. Then we choose $[P]$ as the smallest connected subtree of G' containing P. We note that $[P]$

can either contain or not contain the point z, and that all the endpoints of $[P]$ are points of P. Moreover, since $e([P]) + 2 < k$ and $e([P]) < e(G)$, we get that $e([P]) < k - 2$.

A *retraction* r is a continuous map $r: G \to [P]$ such that $r|_{[P]}$ is the identity map if $P \cap C \neq \emptyset$ or $P \cap C = \emptyset$ and $z \notin [P]$, and $r|_{[P]\setminus\{z\}}$ is the identity map if $P \cap C = \emptyset$ and $z \in [P]$. A retraction is called a *boundary retraction* if r maps every point of $G\setminus[P]$ to the boundary of $[P]$ if $P \cap C \neq \emptyset$ or $P \cap C = \emptyset$ and $z \notin [P]$, and r maps every point of $G\setminus(C \cup ([P]\setminus\{z\}))$ to the boundary of $[P]$ and C to $\{z\}$ if $P \cap C = \emptyset$ and $z \in [P]$. Notice that there is exactly one boundary retraction from G to $[P]$, which is called the *natural retraction*.

LEMMA 1 (First Reduction Lemma). *Let* $r: G \to [P]$ *be the natural retraction. Then* $\mathrm{Per}(rf) \subseteq \mathrm{Per}(f)$.

Proof. By the construction of $[P]$ and the definition of r, all the periodic points of rf with period larger than 1 are also periodic points of f with the same period. So the lemma follows. ∎

In what follows, we will assume that the above reduction has been done and we will write f instead of rf and G instead of $[P]$.

THEOREM 2. *The following two statements hold.*

(1) (*Circle theorem*) *Suppose that* $G = [P] = C$ (*i.e.*, $P \subset C$). *Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 0 \leq p \leq 1\}$.

(2) (*Tree theorem*) *Suppose that* $G = [P]$ *is a tree* (*i.e.*, $P \cap C = \emptyset$). *Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq e(G)\}$.

Proof. Under the assumptions of statement (1), f is a continuous self-map of the circle having some fixed point. So statement (1) follows from the main result of [7]. Notice that the old branching points of f on C are now fixed points.

Statement (2) follows from the tree theorem. ∎

From Theorem 2 we can assume that $C \subset G$, $G \neq C$, and $P \cap C \neq \emptyset$.

Let $B(G)$ denote the set of all branching points of G. We say that J is a *basic interval* if J is the closure of a component of $G\setminus\{B(G) \cup P\}$. Notice that a basic interval is a closed interval and that the interior of any basic interval does not contain branching points.

Define a relation $\to$ on the set of closed intervals as follows. If I and J are closed intervals, then we write $I \to J$ and we say that I *f-covers* (or simply *covers*) J if and only if there is a closed subinterval K of I such that $f(K) = J$. Usually we will use the relation $\to$ on the set of basic intervals.

A *loop of length* n is a sequence $I_0 \to I_1 \to \cdots \to I_n$ such that each I_j is a basic interval and $I_0 = I_n$. Such a loop is called *nonrepetitive* if there is no integer r dividing n such that $I_j = I_{j+r}$ for all j such that $0 \le j \le n - r$.

The following lemma will be necessary later.

LEMMA 3. *Let J be a basic interval which is not contained in the unique circuit C of G. Then there is a loop of length k (which may or may not be repetitive) containing J.*

Proof. The proof is the same as in [3, Lemma 2.6]. ∎

A basic tool for obtaining periodic orbits is the following result.

LEMMA 4 (Loop Lemma). *Let $I_0 \to I_1 \to \cdots \to I_m = I_0$ be a nonrepetitive loop of length $m \ne k$ such that $\bigcap_{j=1}^{m} I_j \subset P$. Then f has a point x of period m such that $f^j(x) \in I_j$ for all j, $1 \le j \le m$.*

Proof. The proof is an easy modification of the proof appearing in many other places, e.g., [13] or [8]. ∎

It is easy to see that any tree T has a metric μ such that if $x, y \in T$ and $z \in [x, y]$, then $\mu(x, y) = \mu(x, z) + \mu(z, y)$ (the *taxicab* metric).

If I is a basic interval of G, we denote by $G_{I,f}$ the maximal subgraph (formed by the union of basic intervals) contained in $f(I) \subset G$. Then we say that $f|_I$ is *linear* if $f(I) = G_{I,f}$, $G_{I,f}$ is an interval, and $f|_I$ is linear with respect to the taxicab metrics of I and $G_{I,f}$. We say that f is *piecewise linear* if $f|_I$ is linear for each basic interval I of G. We note that due to the fact that $G = [P]$, G is the union of all the basic intervals.

A *B-interval* is a basic interval $[x, y]$ where x, y are branching points. Given f and P as above, a *piecewise linearization* of f is the map $g\colon G \to G$ such that $g|_{P \cup B(G)} = f|_{P \cup B(G)}$, for each basic interval I $g|_I$ is linear, and $G_{I,g} \subseteq G_{I,f}$ if I is not a B-interval. If I is a B-interval then we choose $g|_I$ as the identity. Notice that there is always a linearization g of f, but it is not necessarily unique.

The following result shows that for a piecewise linear map each periodic orbit of period different from 1 and k has an associated nonrepetitive loop.

LEMMA 5. *If g has a point x of period m, where m is neither 1 nor k, then there is a nonrepetitive loop $I_0 \to I_1 \to \cdots \to I_m = I_0$ of length m such that $f^j(x) \in I_j$ for all j, $1 \le j \le m$. Furthermore, if $e(G) + 2 < m$ then $\bigcap_{j=1}^{m} I_j = \emptyset$.*

Proof. The first part of the lemma follows as in Lemma 2.4 of [3]. Suppose that $e(G) + 2 < m$. Since the valence of any branching point of G is less than or equal to $e(G) + 2$, it follows that $\bigcap_{j=1}^{m} I_j = \emptyset$. ∎

LEMMA 6 (Piecewise Linearization Lemma). *If g is the piecewise linearization of f, then* $\mathrm{Per}(g) \subseteq \mathrm{Per}(f)$.

Proof. See [3, Corollary 2.5 and Lemma 5]. ∎

In what follows, we will assume that the above piecewise linearization has been made and we write f instead of g; i.e., f is piecewise linear.

Since f is piecewise linear, if I is a B-interval, then $f|_I =$ identity. The next result will show that we can omit the B-intervals.

LEMMA 7 (Shrinking Lemma). *Suppose $X \subseteq G$ is a tree different from a point and that $f|_X =$ identity. Let G' be the graph obtained by shrinking X to a point, let $h: G \to G'$ be the natural quotient map, and define $g: G' \to G'$ by $gh = hf$ (such a g clearly exists since $f|_X =$ identity). Then* $\mathrm{Per}(g) = \mathrm{Per}(f)$.

Proof. Trivial. ∎

In what follows, we will assume that the above shrinking process has been performed several times if necessary until all the B-intervals have been shrunken and, by using the notation of Lemma 7, we write f instead of g and G instead of G'. So, from now on if I is a basic interval, then $P \cap I \neq \emptyset$. In particular it follows that at least one of the two endpoints of any basic interval belongs to P. Moreover, if $G = G'$ is a tree, then the tree theorem implies statement (1) of the graph theorem. So in what follows G is not a tree.

PROPOSITION 8. *Let I be a basic interval which is not covered by any basic interval. Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq e(G) + 2\}$.

Proof. Since $G = [P]$ the basic interval I must be contained in the circuit C of G. Therefore $T = G \setminus \mathrm{Int}(I)$ is a tree with at most $e(G) + 2$ endpoints (here $\mathrm{Int}(I)$ denotes the interior of I). Since f is piecewise linear and I is not covered by any basic interval, the continuous map $f|_T: T \to T$ is well defined and $\mathrm{Per}(f|_T) = \mathrm{Per}(f)$. Then the proposition follows from the tree theorem. ∎

In what follows we will assume that any basic interval is covered by some other basic interval.

PROPOSITION 9. *Let I be a basic interval which is only covered by itself. Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq e(G) + 2\}$.

Proof. We claim that I is contained in C. To prove the claim we assume that I is not contained in C. Then I is contained in a subtree T of G such that $T \cap C$ reduces to a point x (a branching point of G). Let J

be the basic interval contained in T having x as an endpoint. Clearly, the other endpoint of J belongs to P. First, we assume that $J \neq I$. Therefore there is some positive integer $n < k$ such that $f^n(J) \supset I$. Let m be the smallest of such n's. Then there exists a basic interval K contained in $f^{m-1}(J)$ such that $K \to I$. By the definition of m, $K \neq i$, which contradicts the assumption. Hence the claim is proved if $J \neq I$.

If $J = I$ then the image of the endpoint of J different from x belongs to T because by the assumption I covers itself. Since T contains some point of P which is mapped by f in $G \backslash T$, it follows that there is a basic interval contained in T and different from $J = I$ covering I, again in contradiction with the assumption, so the claim is proved.

Since $I \subset C$ we have that $T = G \backslash \mathrm{Int}(I)$ is a tree with at most $e(G) + 2$ endpoints. Since f is piecewise linear and I is not covered by any basic interval different from itself, the continuous map $f|_T \colon T \to T$ is well defined and $\mathrm{Per}(f|_T) = \mathrm{Per}(f)$ (because the unique periodic points in $\mathrm{Int}(I)$ are fixed points). Then, from the tree theorem the proposition follows. ∎

In what follows we will assume that any basic interval is covered by some other basic interval different from itself.

LEMMA 10. *Let I be a basic interval. If $I \to I$ then for every basic interval J there is a path $I = I_0 \to I_1 \to \cdots \to I_r = J$ with $r \leq k$.*

Proof. Let B_j be the set of basic intervals K for which there is a path of length j from I to K, i.e., there exist $I_1, \dots, I_{j-1}$ with $I \to I_1 \to \cdots \to I_{j-1} \to k$. We define $B_0 = \{I\}$. Let D_j be the union of all the intervals of B_j for $j = 0, 1, 2, \dots$. Since f is continuous, it follows easily that D_j is connected for all $j \geq 0$. Since $I \to I$, clearly $B_j \subseteq B_{j+1}$, and if $B_q = B_{q+1}$, then $B_j = B_q$ for all $j \geq q$. Since at least one of the two endpoints of any basic interval belongs to P, all the endpoints of G are points of P, $P \subset D_q$, and D_q is connected, it follows that all the basic intervals not contained in the circuit C are contained in D_q, and consequently all the branching points of G are in D_q. Then, all the basic intervals of G except perhaps a unique basic interval are contained in D_q. But the closure of $G \backslash D_q$ cannot be a unique basic interval because each basic interval is covered by some other basic interval different from itself. Hence $D_q = G$.

Let s be the smallest integer such that $D_s = D_j = G$ for all $j \geq s$. Since I at least contains one point of P and $D_1 \subset D_2 \subset \cdots \subset D_i \subset \cdots$, it follows that D_i contains at least $i + 1$ points of P for $i = 0, 1, 2, \dots, k - 1$. So, by the above arguments, $s \leq k$. Therefore, the lemma follows. ∎

PROPOSITION 11. *If there exists a basic interval I such that $I \to I$, then $m \in \mathrm{Per}(f)$ for all $m \leq_0 k$.*

Proof. Since any basic interval is covered by some other basic interval different from itself, there is a basic interval $J \neq I$ such that $J \to I$. By Lemma 10, there is a path $I = I_0 \to I_1 \to \cdots \to I_r = J$ with $r \leq k$. Assume that r is the shortest length of such paths. Therefore, we have a nonrepetitive loop from I into itself of length $r + 1$ with $1 < r + 1 \leq k + 1$. If $\bigcap_{j=1}^{r} I_j \subset P$, then for any $m \geq r + 1$ we consider the nonrepetitive loop $I = I_0 \to I_1 \to \cdots \to I_r = J \to I \to I \to \cdots \to I$ of length m, and from the loop lemma we get that $m \in \mathrm{Per}(f)$ for all $m \leq_0 k$. So the proposition is proved. ▮

PROPOSITION 12. *If G has some branching point outside the circuit G, then $m \in \mathrm{Per}(f)$ for all $m \leq_0 k$.*

Proof. By the assumptions there exists a closed interval $[x, y]$ such that x and y are branching points and at least one of them does not belong to C. Since $[x, y]$ is not a B-interval, $P' = [x, y] \cap P \neq \emptyset$. Suppose that the cardinality of P' is 1, i.e., $P' = \{p\}$. Since f is piecewise linear, it follows that either $[x, p] \to [x, p]$ or $[p, y] \to [p, y]$. Therefore, by Proposition 11 we are done.

Now we assume that the cardinality of P' is greater than 1. Then we can write $P' = \{p_1, \dots, p_s\}$ with $s > 1$, $(x, p_1) \cap P = \emptyset$, $(p_i, p_{i+1}) \cap P = \emptyset$ for $i = 1, \dots, s - 1$, and $(p_s, y) \cap P = \emptyset$. We denote by G_x and G_y the two components of $G \setminus (x, y)$ such that $x \in G_x$ and $y \in G_y$. Then, since f is piecewise linear either $[x, p_1] \to [x, p_1]$ or $f(p_1) \in G_x$. Similarly, either $[p_s, y] \to [p_s, y]$ or $f(p_s) \in G_y$. By Proposition 11 we can assume that $f(p_1) \in G_x$ and $f(p_s) \in G_y$. Hence, there is a fixed point on $[p_1, p_s]$. By the piecewise linearity, there exists $i \in \{1, \dots, s - 1\}$ such that $[p_i, p_{i+1}] \to [p_i, p_{i+1}]$. So, again from Proposition 11 we are done. ▮

By Propositions 11 and 12, in what follows we will assume that there is no basic interval covered by itself and that all the branching points of G belong to the circuit C.

A *branching interval* is a basic interval such that one of its endpoints is a branching point (of course, its other endpoint belongs to P since we assume that there are no B-intervals).

Let z be a branching point with valence n. Let A be the set of all branching intervals having z as endpoint. Define φ: $A \to A$ by $\varphi(I) = J$ if and only if $I \to J$. We say that z has *type* t if φ has an element of period t. Note that z must have at least one type $t \leq n$, but may have more than one type. We remark that the type of z depends only on $f|_P$.

Since there is no basic interval covered by itself, the type of any branching point is at least 2.

PROPOSITION 13. *If G has a branching point of type t greater than 2, then $m \in \mathrm{Per}(f)$ for all $m \leq_t k$.*

Proof. Let $z \in C$ be a branching point of type $t \geq 3$. Then, from the definition of type, there are $p_1, \ldots, p_t$ points of P such that $[z, p_1] \to \cdots \to [z, p_t] \to [z, p_1]$ and $(z, p_i) \cap P = \emptyset$ for $i = 1, \ldots, t$. Let C_i be the component of $G \setminus \{z\}$ containing p_i for $i = 1, \ldots, t$. Notice that if two distinct basic intervals $[z, p_i]$ and $[z, p_j]$ are contained in the circuit C, then the components C_i and C_j coincide, but we denote them in the two manners. Since $t \geq 3$ we can assume that at least C_1 satisfies $C_1 \cap C = \emptyset$. By Lemma 3, there is a loop L of length k containing $[z, p_1]$. Since f is piecewise linear and $[z, p_1] \cap C = \emptyset$, we can assume that the loop L is nonrepetitive and that the intersection of all the basic intervals of L is contained in P because $e(G) + 2 < k$.

First, suppose that k is not divisible by t. Then $m \leq_t k$ if and only if $m = 1$, $m = k$, or $m = ik + jt$ for some integers $i \geq 0$, $j \geq 1$. So, we can assume that $m = ik + jt$ with $i \geq 0$ and $j \geq 1$. If $i \geq 1$ then the concatenation of i times the loop L with j times the loop $[z, p_1] \to \cdots \to [z, p_t] \to [z, p_1]$ shows that $m \in \mathrm{Per}(f)$ (here we use the loop lemma and the facts that t does not divide k and that the intersection of all the basic intervals of the loop of length m is contained in P because the intersection of all the basic intervals of L is contained in P). Now, we suppose that $m = jt$ with $j \geq 1$.

Since k is not divisible by t, there exists $p \in P$ such that $p \in C_j$, but $f(p) \notin C_{j+1}$ ($j + 1$ is considered modulus t). Using the fact that z has type t, note that $f(z, p_j]$ contains $(z, p_{j+1}]$ for $0 < j < t$ and $f(z, p_t]$ contains $(z, p_1]$, and thus $f^{t+j}(z, p_1]$ contains $f^j(z, p_1]$ for all j. There must be an integer i so that $f^i(z, p_1]$ contains z, for otherwise it is easy to see that we are in contradiction with the existence of the point $p \in P$. Fix the least such i.

Assume $i > t$. Let r be the largest positive integer such that $i - rt > 0$. Since C_1 is an interval we can write $f^{(r-1)t}(z, p_1] = (z, u]$ and $f^{rt}(z, p_1] = f^t(z, u] = (z, v]$ with $u \in (z, v)$. Then there exists $a \in (z, u]$ such that $f^t(a) = v$. Since $z \in f^i(z, p_1[$, by the minimality of i there exists $b \in (u, v]$ such that $f^{i-rt}(b) = z$. Consequently $f^t(b) = z$. Since the closure of C_1 is a closed interval, $f^t[z, a] \supset [z, a] \cup [a, b]$, $f^t[a, b] \supset [z, a] \cup [a, b]$, and $f^t(a) \neq a$. By well-known results for interval maps f^t has points of all periods in $[z, a] \cup [a, b]$, and it is easy to check that this leads to points of period any multiple of t for f.

Suppose $i \le t$. Since $f^t(z, p_1] \supset [z, p_1]$, there exists $a \in (z, p_1)$ such that $f^t(a) = p_1$ and $f^t(y) \neq p_1$ for any $y \in (z, a)$. Then, by the piecewise linearity of f and since $f^t[z, a] = [z, p_1]$, it follows that $f^t|_{[z,a]}$ is linear. Since $z \in f^i(z, p_1]$ and $i \le t$, there exists $b \in (z, p_1)$ such that $f^t(b) = z$. Notice that $b \in (a, p_1)$; otherwise we get a contradiction with the fact that $f^t|_{[z,a]}$ is linear, $f^t(z) = z$ and $f^t(a) = p_1$. Hence $f^t[z, a] \supset [z, a] \cup [a, b]$ and $f^t[a, b] \supset [z, a) \cup [a, b]$, and the proof ends as in the case $i > t$.

Now, assume that k is divisible by t. Then $m \le_t k$ if and only if $m = 1$ or $t|m$ and $(m/t) \le_1 (k/t)$. We consider two cases.

Case 1. There exists $p \in P$ such that $p \in C_j$, but $f(p) \notin C_{j+1}$ (where $j + 1$ means $j + 1$ modulus t). The proof follows as above.

Case 2. Assume that the assumption of Case 1 is not satisfied. Then if $p \in P \cap C_j$, $g(p) \in P \cap C_{j+1}$ for all $p \in P$. Then, from the piecewise linearity of f and by applying the interval theorem to $f^t|_{C_1}$, we obtain that f has points of all periods $m \le_t k$. ∎

In what follows we will assume that the unique type of any branching point of G is 2.

COROLLARY 14. *Let z be a branching point of G and let $[z, p_1] \to [z, p_2] \to [z, p_1]$ be as in the definition of type (p_1, p_2 distinct points of P and $(z, p_i) \cap P = \emptyset$ for $i = 1, 2$). If (z, p_i) is not contained in the circuit C for some $i \in \{1, 2\}$, then $m \in$* Per(f) *for all $m \le_t k$.*

Proof. The corollary follows by repeating the arguments of the proof of Proposition 13. ∎

In what follows we will assume that the unique loop of length 2 $[z, p_1] \to [z, p_2] \to [z, p_1]$ (p_1, p_2 distinct points of P and $(z, p_i) \cap P = \emptyset$ for $i = 1, 2$) associated to the type of any branching point z of G satisfies that $[z, p_i] \subset C$ for $i = 1, 2$.

We need some new notation. In what follows we think of the graph G as a subset of $\mathbb{R}^2$. Thus we can consider on the circuit C the counterclockwise orientation induced from $\mathbb{R}^2$. If $x, y \in C$ we denote by $[x, y]$ the closed interval on C starting at x, ending at y, and going from x to y counterclockwise, and we denote by (x, y) the open interval $[x, y] \setminus \{x, y\}$.

PROPOSITION 15. *If G has more than two branching points, then* Per(f) $\supset \mathbb{N} \setminus \{2\}$. *Consequently $m \in$* Per(f) *for all $m \le_0 k$.*

Proof. Assume that x and y are two branching points of G such that (x, y) does not contain any branching point. We claim that $f([x, y]) \supset [y, x]$. To prove the claim suppose that it fails. Then it is clear that $[x, y] \to [x, y]$ (notice that since $[x, y] \subset C$ and x and y are two consecutive branching points in C, either $[x, y] \to [x, y]$ or $f([x, y]) \supset [y, x]$).

Then there exists a closed subinterval H of $[x, y]$ such that $f(H) = [x, y]$. We denote by I_x and I_y the branching intervals contained in $[x, y]$ with endpoints x and y, respectively. Since f is piecewise linear and by the assumption I_x does not cover itself, we see that its image is not contained in $[x, y]$. Therefore, I_x and H are disjoint, and so are I_y and H. Therefore, $H \subset [x, y] \setminus (I_x \cup I_y)$. Hence, since $f(H) = [x, y]$ and f is piecewise linear, there exists a basic interval I contained in $[x, y]$ and different from I_x and I_y such that $I \to I$, in contradiction with the assumptions. Consequently, the claim is proved.

Now we assume that there are $n \geq 3$ branching points in G. These n points divide the circuit C into n closed intervals. By the claim each one of these intervals covers the other $n - 1$. By well-known results for interval and circle maps it follows that f has points of all periods except perhaps period 2 when $n = 3$, because in this case all the loops of length 2 are formed by two basic intervals having in common a fixed branching point. ∎

In what follows we will assume that G has at most two branching points in C. Of course, G has at least one branching point in C.

If G has exactly two branching points x and y, then denote by G_x the closure of the component of $G \setminus \{x\}$ which does not contain $C \setminus \{x\}$. In a similar way we define G_y. Without loss of generality we may assume that there exists $p \in P \cap (x, y)$ such that $f(p) \in G_x$ and $f(P \cap (x, p)) \subset C$. We separate the set of such maps into the following three cases:

(1) $f(P \cap (y, x)) \subset C$;

(2) there exists $q \in P \cap (y, x)$ such that $f(q) \in G_x$ and $f(P \cap (q, x)) \subset C$;

(3) there exists $q \in P \cap (y, x)$ such that $f(q) \in G_y$ and $f(P \cap (q, x)) \subset C$.

LEMMA 16. *Under all the previous assumptions* $\mathrm{Per}(f)$ *contains the set of all even natural numbers.*

Proof. From the assumptions on the type of a branching point x, there exist p_1 and p_2 points of P such that $\{p_1, p_2\} \subset C$, $(x, p_1) \cap P = \emptyset$, $(p_2, x) \cap P = \emptyset$, and $[x, p_1] \to [p_2, x] \to [x, p_1]$. Clearly we have

$$[x, p_1] \subset f^2[x, p_1] \subset f^4[x, p_1] \subset \cdots,$$

$$[p_2, x] \subset f[x, p_1] \subset f^3[x, p_1] \subset \cdots.$$

Since there are points of P in G_x, there must be an integer i so that $f^i(x, p_1]$ contains x. Fix the least such i. Since f is piecewise linear and $[x, p_1] \to [p_2, x]$, we get that $i \geq 2$. We separate the proof into two cases.

Case 1. Assume that G has a unique branching point or it has exactly two branching points x and y and f satisfies (1) or (2). Then, by the piecewise linearity of f, $f^j[x, p_1]$ is a closed interval contained in C with endpoints x and some $q_j \in P$ for $j = 0, 1, \dots, i-2$, and $f^{i-1}[x, p_1]$ is either a closed interval contained in C with endpoints x and $q_{i-1} \in P$, or it is C. Of course, we denote by f^0 the identity map. In what follows we assume that i is odd; if i is even the proof follows in a similar way.

From the definition of i, we choose $x' \in (x, q_{i-1})$ to be the closest to the x f^i-preimage of x. Then $f(x') = x$. Since $[x, q_{i-1}] = f[q_{i-2}, x]$ there exists $x'' \in (q_{i-2}, x)$ such that $f(x'') = x'$. We claim that there exists $z \in (x'', x)$ such that $f^2(z) = q_{i-2}$. Now we prove the claim. Since $f[x, q_{i-3}] = [q_{i-2}, x]$, there exists $z' \in (x, q_{i-3}]$ such that $f(z') = q_{i-2}$. Due to the minimality of i the point x' is not contained in $[x, q_{i-3}]$. Therefore, $z' \in [x, x'] \subset f([x'', x])$ and so there exists $z \in (x'', x')$ with $f^2(z) = f(z') = q_{i-2}$. Hence, the claim is proved.

From the loop lemma and the two loops $[z, x] \to [x, z'] \to [x'', z] \to [z', x'] \to [z, x]$ and $[z, x] \to [x, z'] \to [z, x]$, it follows that $\mathrm{Per}(f)$ contains the set of all even natural numbers.

Case 2. Assume that G has exactly two branching points x and y and that f satisfies (3). If $q \notin f^j[x, p_1]$ for $j = 1, 2, \dots, i-1$, the proof of Case 2 follows as in Case 1. So assume that $q \notin f^j[x, p_1]$ for $j = 0, 1, \dots, l-1$ and $q \in f^l[x, p_1]$ with $l < i$. Notice that $l \geq 1$, l is odd, and $f^j[x, p_1]$ is a closed interval contained in C with endpoints x and some $q_j \in P$ for $j = 0, 1, \dots, l-1$.

From the definition of l, there exists a point $y' \in (q, x)$ such that $f(y') = y$. Since $y' \in f[x, q_{l-1}]$, there exists $y'' \in (x, q_{l-1})$ such that $f(y'') = y'$. Since there exists $p \in P \cap (x, y)$ such that $f(p) \in G_x$ and $l < i$, then there exists $x' \in (y'', y)$ such that $f(x') = x$. From the loop lemma and the two loops $[x, y'] \to [x, y''] \to [x, y']$ and $[x, y'] \to [y'', x'] \to [x, y']$, it follows that $\mathrm{Per}(f)$ contains the set of all even natural numbers. ∎

We note that the next proposition completes the proof of statement (1) of the graph theorem.

PROPOSITION 17. *Under all the previous assumptions* $m \in \mathrm{Per}(f)$ *for all* $m \leq_1 k$.

Proof. If k is even, the proposition follows immediately from Lemma 16. So assume that k is odd.

From the assumptions on the type of a branching point x, there exist $p_1 \in P$ and $p_2 \in P$ such that $\{p_1, p_2\} \subset C$, $(x, p_1) \cap P = \emptyset$, $(p_2, x) \cap P = \emptyset$, and $[x, p_1] \to [p_2, x] \to [x, p_1]$. Regardless of whether G has one or two branching points, without loss of generality we can assume that there exists $p \in P \cap C$ such that $f(p)$ belongs to the component of $G \setminus \{x\}$ which does not contain $C \setminus \{x\}$. Therefore there exists a basic interval $I \subset C$ such that either $I \to [x, p_1]$ or $I \to [p_2, x]$. Again, without loss of generality, we can suppose that $I \to [x, p_1]$.

Since $f^k(x) = x$ and $f^k(p_2) = p_2$, it follows that $f^k(J) \supset J$ or $f^k(J) \supset C \setminus \mathrm{Int}(J)$, where $J = [p_2, x]$. Therefore J f^k-covers or J f^k-covers I.

If J f^k-covers J, then by using concatenations with repetitions of the loop $[x, p_1] \to J \to [x, p_1]$ with the corresponding loop of length k from J to J, since k is odd, from the loop lemma we get that all the odd numbers of the form $k + 2j$ with $j = 1, 2, \ldots$ are periods of f. If J f^k-covers I, then by using concatenations with repetitions of the loop $[x, p_1] \to J \to [x, p_1]$ with the corresponding loop of length $k + 2$ from J to J (where we use that $I \to [x, p_1] \to J$), since $k + 2$ is odd, from the loop lemma we get that all the odd numbers of the form $k + 2 + 2j$ with $j = 0, 1, 2, \ldots,$ are periods of f. Hence, in any case, from Lemma 16 the proposition follows. ∎

Proof of Statement (2) *of the Graph Theorem.* Let G be a given graph with zero Euler characteristic and having branching points, and let S be a nonempty finite union of initial segments of $\{\leq_p : 0 \leq p \leq e(G) + 2\}$. We choose a closed interval $[a, b] \subset C$ such that $[a, b]$ does not contain any branching point of G. We denote by T the tree $G \setminus (a, b)$. Clearly $e(T) = e(G) + 2$ because a and b are also new endpoints of T.

Let S' be the finite union of all the initial segments of S restricted to the orderings $\leq_p$ with $p \geq 1$. Since we are looking for a continuous map $f\colon G \to G$ with all the branching points fixed, $1 \in S$ and $1 \in S'$. Now, from statement (2) of the tree theorem, there is a continuous map $g\colon T \to T$ with all the branching points fixed such that $\mathrm{Per}(g) = S'$.

Let S'' be the finite union of all the initial segments of S restricted to the ordering $\leq_0$. Since $\leq_0$ is a total ordering, S'' can be reduced to a unique initial segment. If $S'' = \emptyset$, then we extend $g\colon T \to T$ to a continuous map $f\colon G \to G$ defined by $f|_T = g$ and $f|_{[a,b]}\colon [a, b] \to \langle g(a), g(b) \rangle$ is a homeomorphism, where $\langle g(a), g(b) \rangle$ denotes the unique closed interval with endpoints $g(a)$ and $g(b)$ contained in T.

Assume that $S'' \neq \emptyset$. Then $S'' = \{m : m \leq_0 n\} = \{n, n + 1, n + 2, \ldots\}$ for some natural number $n \geq 2$. In order to extend g from T to G we choose points $a', p_1, \ldots, p_n, b'$ in (a, b) as follows. If $Q = \{a', p_1, \ldots, p_n, b'\}$ then $(a, a') \cap Q = \emptyset$, $(a, p_1) \cap Q = \emptyset$, $(p_i, p_{i+1}) \cap Q = \emptyset$ for $i = 1, \ldots, n - 1$, $(p_n, b') \cap Q = \emptyset$, and $(b', b) \cap Q = \emptyset$. We construct f in $[a, b]$ as follows. Let $f(p_i) = p_{i+1}$ for $i = 1, \ldots, n - 1$ and $f(p_n) = p_1$. Let $f(a') = a'$ and $f(b') = b'$. For $i = 1, \ldots, n - 2$, let f map the interval $[p_i, p_{i+1}]$ homeomorphically onto $[p_{i+1}, p_{i+2}]$. Let f map $[p_{n-1}, p_n]$ homeomorphically onto $[p_n, p_1]$. Also let f map $[p_n, b']$ homeomorphically onto $[b', p_1]$ and let f map $[a', p_1]$ homeomorphically onto $[a', p_2]$. Finally, it remains to define f on $[b', b]$ and $[a', a]$. Let f map $[b', b]$ homeomorphically on the closed interval I with endpoints b' and $f(b) = g(b)$ satisfying $I \cap (a', b') = \emptyset$, and let f map $[a, a']$ homeomorphically onto the closed interval J with endpoints $f(a) = g(a)$ and a' satisfying $J \cap (a', b') = \emptyset$.

By construction a' and b' are fixed points of f and $\{p_1, \ldots, p_n\}$ is a periodic orbit of period n. It follows easily from the loops $[a', p_1] \to [p_1, p_2] \to [p_2, p_3] \to \cdots \to [p_{n-1}, p_n] \to [a', p_1] \to \cdots \to [a', p_1]$ that $m \in \text{Per}(f)$ for all $m \leq_0 n$ and that f has no other periods of periodic points in $[a', b']$ (for more details see the proof of Theorem C and Theorem A_1 of [7]). Now, since by construction there are no periodic points in $[a, a']$ and $[b', b]$ different from a' and b', statement (2) of the graph theorem follows. ∎

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