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Kazhdan and Haagerup properties from the median viewpoint

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Abstract

We prove the existence of a close connection between spaces with measured walls and median metric spaces.

We then relate properties (T) and Haagerup (a-T-menability) to actions on median spaces and on spaces with measured walls. This allows us to explore the relationship between the classical properties (T) and Haagerup and their versions using affine isometric actions on L^p-spaces. It also allows us to answer an open problem on a dynamical characterization of property (T), generalizing results of Robertson and Steger.
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Contents

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1. Introduction

1.1. Median spaces and spaces with measured walls

The theory of median spaces involves a blend of geometry, graph theory and functional analysis. A *median space* is a metric space for which, given any triple of points, there exists a unique *median point*, that is a point which is simultaneously between any two points in that triple. A point x is said to be *between* two other points a, b in a metric space (X, dist) if $\text{dist}(a, x) + \text{dist}(x, b) = \text{dist}(a, b)$.

Examples of median spaces are real trees, sets of vertices of simplicial trees, $\mathbb{R}^n$, $n \geqslant 1$, with the ℓ^1-metric, CAT(0) cube complexes with the cubes endowed with the ℓ^1-metric, 0-skeleta of such CAT(0) cube complexes. According to Chepoi [23] the class of 1-skeleta of CAT(0) cube complexes coincides with the class of median graphs (simplicial graphs whose combinatorial distance is median). See also [32] and [33] for an equivalence in the same spirit. Median graphs are much studied in graph theory and in computer science [9] and are relevant in optimization theory (see for instance [45,72] and references therein for recent applications).

Median metric spaces can thus be seen as non-discrete generalizations of 0-skeleta of CAT(0) cube complexes (and geodesic median spaces can be seen as non-discrete generalizations of 1-skeleta of CAT(0) cube complexes), same as real trees are non-discrete generalizations of simplicial trees.

The ternary algebra naturally associated with a median space is called a *median algebra*. There is an important literature studying median algebras. Without attempting to give an exhaustive list, we refer the reader to [65,66,53,41,10,68,11]. More geometrical studies of median spaces were

started in [60] or [51]. In this article, we emphasize that a median space has a richer structure than the algebraic one, and we use this structure to study groups.

Nica in [52], and Niblo and the first author in [21] proved simultaneously and independently an equivalence of category between CAT(0) cube complexes, possibly infinite-dimensional, and *discrete spaces with walls* (notion introduced by F. Paulin and the third author in [39]). Discrete spaces with walls were generalized by Cherix, Martin and Valette in [25] to *spaces with measured walls*. Spaces with measured walls are naturally endowed with a (pseudo-)metric. It turns out (Corollary 5.4) that a (pseudo-)metric on a space is induced by a structure of measured walls if and only if it is induced by an embedding of the space into a median space (i.e. it is submedian, in the sense of Definition 2.7). This is a consequence of the following results:

Theorem 1.1.

(1) *Any space with measured walls X embeds isometrically in a canonically associated median space $\mathcal{M}(X)$. Moreover, any homomorphism between two spaces with measured walls induces an isometry between the associated median spaces.*

(2) *Any median space (X, dist) has a canonical structure of space with measured walls, and the wall metric coincides with the original metric. Moreover, any isometry between median spaces induces an isomorphism between the structures of measured walls.*

(3) *Any median space (X, dist) embeds isometrically in $L^1(\mathcal{W}, \mu)$, for some measured space $(\mathcal{W}, \mu)$.*

The fact that a median space embeds into an L^1-space was known previously, though the embedding was not explicitly constructed, but obtained via a result of Assouad that a metric space embeds into an L^1-space if and only if any finite subspace of it embeds [6,5,4,69]. It is moreover known that complete median normed spaces are linearly isometric to L^1-spaces [69, Theorem III.4.13].

We note here that there is no hope of defining a median space containing a space with measured walls and having the universality property with respect to embeddings into median spaces (see Remark 2.10). Nevertheless, the medianization $\mathcal{M}(X)$ of a space with measured walls X appearing in Theorem 1.1(1) is canonically defined and it is minimal in some sense. This is emphasized for instance by the fact that, under some extra assumptions, a space with measured walls X is at finite Hausdorff distance from $\mathcal{M}(X)$ [22]. In particular, it is the case when X is the n-dimensional real hyperbolic space with the natural structure of space with measured walls (see Example 3.7).

1.2. Properties (T) and Haagerup, actions on L^p-spaces

Topological groups with Kazhdan's property (T) act on real trees with bounded orbits, moreover with global fixed point if the tree is complete ([18, Corollary 5.2], see also [2,70] and [13, Theorems 2.10.4 and 2.12.4]). The converse implication however does not hold in general. Coxeter groups with every pair of generators satisfying a non-trivial relation act on any real tree with fixed point (an application of Helly's theorem in real trees); on the other hand these groups are known to have the Haagerup property, also called a-T-menability [19]. Nevertheless, if one extends the bounded orbits property from actions on real trees to actions on median spaces, the equivalence with property (T) does hold, for locally compact second countable groups. More

precisely, Theorem 1.1 allows us to prove the following median characterizations of property (T) and Haagerup property:

Theorem 1.2. *Let G be a locally compact second countable group.*

(1) *The group G has property (T) if and only if any continuous action by isometries on a median space has bounded orbits.*
(2) *The group G has the Haagerup property if and only if it admits a proper continuous action by isometries on a median space.*

Note that the direct implication in (1) and the converse implication in (2) follow immediately from known results of Delorme–Guichardet, respectively Akemann–Walter (see Theorem 6.20 in this paper), and from the fact that median spaces embed into L^1-spaces (see for instance [69, Theorem V.2.4]). For discrete countable groups geometric proofs of the same implications are provided implicitly in [48,49] (see also [60]) and explicitly in [51]. Nica conjectured [51] that the converse implication in (1.2) and the direct implication in (2) hold for discrete countable groups. This is answered in the affirmative by Theorem 1.2.

Theorem 1.2 can be reformulated in terms of spaces with measured walls as follows:

Theorem 1.3. *Let G be a locally compact second countable group.*

(1) *The group G has property (T) if and only if any continuous action by automorphisms on a space with measured walls has bounded orbits (with respect to the wall pseudo-metric).*
(2) *The group G has the Haagerup property if and only if it admits a proper continuous action by automorphisms on a space with measured walls.*

The equivalence in Theorem 1.3 improves the result of Cherix, Martin and Valette [25], who showed the same equivalence for discrete groups.

Using Theorem 1.3, the classical properties (T) and Haagerup can be related to their versions for affine actions on L^p-spaces.

Definition 1.4. Let $p > 0$, and let G be a topological group.

(1) The group G *has property* FL^p if any affine isometric continuous action of G on a space $L^p(X, \mu)$ has bounded orbits (equivalently, for $p > 1$, it has a fixed point).
(2) The group G is *a-FL^p-menable* if it has a proper affine isometric continuous action on some space $L^p(X, \mu)$.

Property FL^2 is equivalent to property FH, i.e. the fixed point property for affine continuous actions on Hilbert spaces, and the latter is equivalent to property (T) for σ-compact groups, in particular for second countable locally compact groups, as proved in [35] and [29]. Likewise a-FL^2-menability is equivalent to a-T-menability (or Haagerup property).

Theorem 1.3 and a construction from [25] and [27] associating to every action on a space with measured walls an affine isometric action on an L^p-space implies the following.

Corollary 1.5. *Let G be a second countable locally compact group.*

(1) *If G has property FL^p for some $p > 0$ then G has property (T).*
(2) *If G has the Haagerup property then for every $p > 0$ the group G is a-FL^p-menable.*

Cornulier, Tessera and Valette proved the implication in (2) for countable discrete groups [27, Proposition 3.1].

The implication in (2) with $p \in (1, 2)$ and a proper action on $L^p([0, 1])$ has been announced in [54], and a complete proof has been provided in [55] (see Remark 6.24).

Remark 1.6. The converse statements in Corollary 1.5 hold for $p \in (0, 2]$, in the following strengthened version:

(1) If G has property (T) then it has property FL^p for every $p \in (0, 2]$.
(2) The group G has the Haagerup property if it is a-FL^p-menable for some $p \in (0, 2]$.

These statements (even slightly generalized, see Corollary 6.23) follow from results of Delorme–Guichardet [35,29] and Akemann–Walter [1], combined with a classical Functional Analysis result [71, Theorem 4.10].

Thus, property (T) is equivalent to all properties FL^p with $p \in (0, 2]$. Likewise, the Haagerup property is equivalent to a-FL^p-menability for every $p \in (0, 2]$. For a discussion of the cases when $p > 2$ see Section 1.3.

We prove Theorems 1.2 and 1.3 using median definite kernels, that turn out to coincide with the Robertson–Steger measure definite kernels [59] and are very natural settings for these notions. Along the way this allows us to answer Robertson–Steger question [59, Question (i)] whether measure definite kernels can be given an intrinsic characterization among the conditionally negative definite kernels (Corollary 6.17).

We also generalize to locally compact second countable groups Robertson–Steger's dynamical characterization of property (T) [59]. This answers Open Problem 7 in [14]. Moreover, we give the a-T-menability a dynamical characterization as well.

Theorem 1.7. *Let G be a locally compact second countable group.*

(1) *The group G has property (T) if and only if for every measure-preserving action of G on a measure space $(X, \mathcal{B}, \mu)$ and every set $S \subset X$ such that for all $g \in G$, $\mu(S \triangle gS) < \infty$ and $\lim_{g \to 1} \mu(S \triangle gS) = 0$, the supremum $\sup_{g \in G} \mu(S \triangle gS)$ is finite.*
(2) *The group G is a-T-menable if and only if there exists a measure-preserving action of G on a measure space $(X, \mathcal{B}, \mu)$ and there exists a set $S \subset \mathcal{B}$ such that for all $g \in G$, $\mu(S \triangle gS) < \infty$ and $\lim_{g \to 1} \mu(S \triangle gS) = 0$, but $\mu(S \triangle gS) \to \infty$ when $g \to \infty$.*

1.3. Current developments and open questions

It is natural to ask whether the equivalence between properties (T) and FL^p (respectively between a-T-menability and a-FL^p-menability) can be extended to $p > 2$. In [7, §3.c], by an argument attributed to D. Fisher and G. Margulis, it is proved that for every group G with property (T) there exists $\varepsilon = \varepsilon(G)$ such that the group has property FL^p for every $p \in [1, 2 + \varepsilon)$. Nevertheless, no positive uniform lower bound for $\varepsilon(G)$ is known.

For $p \gg 2$ the statements in Corollary 1.5 cannot be turned into equivalences. Indeed, it follows from results of P. Pansu [57] that the group $G = Sp(n, 1)$ does not have property FL^p for $p > 4n + 2$. More recently, Y. de Cornulier, R. Tessera and A. Valette proved in [27] that any

simple algebraic group of rank one over a local field is a-FL^p-menable for p large enough. In particular, $G = Sp(n, 1)$ is a-FL^p-menable for $p > 4n + 2$.

Also, results of M. Bourdon and H. Pajot [17] imply that non-elementary hyperbolic groups have fixed-point-free isometric actions on $\ell^p(G)$ for p large enough, hence do not have property FL^p. G. Yu later proved [73] that every discrete hyperbolic group G is a-FL^p-menable for p large enough. In particular this holds for hyperbolic groups with property (T).

The above quoted results of Y. de Cornulier, R. Tessera and A. Valette, and of G. Yu, illustrate that neither of the two converse implications in Corollary 1.5 hold for $p \gg 2$. This shows that for every $p > 2$ property FL^p is *a priori* stronger than property (T). Also, the property of a-FL^p-menability is a weaker version of a-T-menability/Haagerup property.

Question 1.8. Can Corollary 1.5 be generalized to: "for every $p \geqslant q \geqslant 2$ property FL^p implies property FL^q and a-FL^q-menability implies a-FL^p-menability"?

Question 1.9. Are different properties FL^p and FL^q with $p, q > 2$ large enough, equivalent? Is it on the contrary true that for any $p_0 \geqslant 2$ there exist groups that have property FL^p for $p \leqslant p_0$ and are a-FL^p-menable for $p > p_0$?

Question 1.10. What is the relation between FL^p with $p \gg 2$ and other strong versions of property (T) defined in terms of uniformly convex Banach spaces, like for instance the one defined in [44]?

Note that, like other strong versions of property (T), the family of properties FL^p separates the semisimple Lie groups of rank one from the semisimple Lie groups with all factors of rank at least 2 (and their respective lattices). By the results of G. Yu [73] all cocompact rank one lattices are a-FL^p-menable for p large enough. On the other hand, lattices in semisimple Lie groups of higher rank have property FL^p for all $p \geqslant 1$, by results of Bader, Furman, Gelander and Monod [7].

Note also that the other possible version of property (T) in terms of L^p-spaces, namely that "almost invariant vectors imply invariant vectors for linear isometric actions", behaves quite differently with respect to the standard property (T); namely the standard property (T) is equivalent to this L^p version of it, for $1 < p < \infty$ [7, Theorem A]. This shows in particular that the two definitions of property (T) (i.e. the fixed point definition and the almost invariant implies invariant definition) are no longer equivalent in the setting of L^p spaces.

According to Bass–Serre theory, a group splits if and only if it acts non-trivially on a simplicial tree. This implies that amalgamated products do not have property (T). Splittings were later extended to semisplittings, using CAT(0) cube complexes. M. Sageev showed that if G is a finitely generated group acting on a finite-dimensional CAT(0) cube complex without a fixed point then there exists a stabilizer H of some convex wall such that $e(G, H) > 1$ (see [61]). Then, in [32,33], V. Gerasimov removed the finite dimension assumption. (Here $e(G, H)$ stands for the number of ends of the group G with respect to the subgroup H, in the sense of [40].) Conversely, V. Gerasimov showed that any group G that has a subgroup H with $e(G, H) > 1$ acts on a CAT(0) cube complex without a fixed point, so that H is a finite index subgroup in the stabilizer of a convex wall (see [32,33] and also [61,50]).

Under certain stability assumptions, a non-trivial action of a group on a real tree leads to a non-trivial action on a simplicial tree and to a splitting of the group (according to Rips, Bestvina and Feighn [15, Theorem 9.5], Sela [64, Section 3], Guirardel [37]).

Question 1.11. A group G acts non-trivially on a median space (equivalently, G does not have property (T)). Under what assumptions is there a non-trivial action of G on a CAT(0) cube complex (hence a semisplitting of G in the sense of Gerasimov–Sageev)? On a finite-dimensional CAT(0) cube complex?

Via Question 1.11, Theorem 1.2 relates to one implication in M. Cowling's conjecture (stating that a countable discrete group is a-T-menable if and only if it is weakly amenable with Cowling–Haagerup constant 1 [24, §1.3.1]). Indeed, Guentner and Higson [34] showed that a countable discrete group acting properly on a finite-dimensional CAT(0) cubical complex is weakly amenable with Cowling–Haagerup constant 1. If for a discrete countable a-T-menable group it would be possible (under extra-hypotheses) to extract from its proper action on a median space a proper action on the 1-skeleton of a finite-dimensional CAT(0) cubical complex, then by Guentner and Higson weak amenability would follow. Extra-hypotheses are needed: recent results show that the implication "a-T-menable $\Rightarrow$ weakly amenable with Cowling–Haagerup constant 1" does not hold in full generality. More precisely, a wreath product $H \wr \mathbb{F}_2$, where H is finite and $\mathbb{F}_2$ is the free group on two generators, is a-T-menable according to Cornulier, Stalder and Valette [26], but cannot be weakly amenable with Cowling–Haagerup constant 1 according to Ozawa and Popa [56, Corollary 2.11].

1.4. Plan of the paper

The paper is organized as follows. Section 2 gives a general introduction of median spaces (further geometric considerations on such spaces will be found in [22]) and proves some general results used in the sequel. In Section 3 we recall the notion of measured wall spaces and show how those embed isometrically in a median space, proving Theorem 1.1 part (1). In Section 4 we outline a few known results on median algebras. We emphasize on the results needed for Section 5, which explains a structure of measured wall spaces hidden in a median space. Section 6 is devoted to the study of several types of kernels, leading to the proof of Theorem 1.2 and its consequences.

2. Median spaces

2.1. Definitions and examples

A *pseudo-metric* or *pseudo-distance* on a set X is a symmetric map $\text{pdist} : X \times X \to \mathbb{R}_+$ satisfying the triangle inequality. Distinct points $x \neq y$ with $\text{pdist}(x, y) = 0$ are allowed.

A map $f : (X_1, \text{pdist}_1) \to (X_2, \text{pdist}_2)$ between two pseudo-metric spaces is an *isometry* if $\text{pdist}_2(f(x), f(y)) = \text{pdist}_1(x, y)$. Note that f is not necessarily injective.

A space X with a pseudo-metric pdist has a canonical metric quotient $\widetilde{X} = X/\sim$ composed of the equivalence classes for the equivalence relation $x \sim y \Leftrightarrow \text{pdist}(x, y) = 0$, endowed with the metric $\text{dist}(\tilde{x}, \tilde{y}) = \text{pdist}(x, y)$. We call $\widetilde{X}$ the *metric quotient* of X. The natural projection map $X \to \widetilde{X}$ is an isometry.

Notation 2.1. If x is a point in X and $r \geqslant 0$ then $B(x, r)$ denotes the closed ball of radius r around x, that is the set $\{y \in X; \ \text{pdist}(y, x) \leqslant r\}$.

For every $Y \subseteq X$ and $r \geqslant 0$, we denote by $\mathcal{N}_r(Y)$ the closed r-tubular neighborhood of Y in X, $\{y \in X; \ \text{pdist}(y, Y) \leqslant r\}$.

Definition 2.2 *(Intervals and geodesic sequences).* Let (X, pdist) be a pseudo-metric space. A point b is *between a and c* if $\text{pdist}(a, b) + \text{pdist}(b, c) = \text{pdist}(a, c)$. We denote by $I(a, c)$ the set of points that are between a and c, and we call $I(a, c)$ the *interval between a and c*. A *path* is a finite sequence of points $(a_1, a_2, \ldots, a_n)$. It is called a *geodesic sequence* if and only if

$$\text{pdist}(a_1, a_n) = \text{pdist}(a_1, a_2) + \text{pdist}(a_2, a_3) + \cdots + \text{pdist}(a_{n-1}, a_n).$$

So (a, b, c) is a geodesic sequence if and only if $b \in I(a, c)$.

Definition 2.3 *(Median point).* Let a, b, c be three points of a pseudo-metric space (X, dist). We denote the intersection $I(a, b) \cap I(b, c) \cap I(a, c)$ by $M(a, b, c)$, and we call any point in $M(a, b, c)$ a *median point for a, b, c*. We note that $I(a, b) = \{x \in X, \; x \in M(a, x, b)\}$.

Definition 2.4 *(Median spaces).* A *median (pseudo-)metric space* is a (pseudo-)metric space in which for any three points x, y, z the set $M(x, y, z)$ is non-empty and of diameter zero (any two median points are at pseudo-distance 0). In particular a metric space is median if any three points x, y, z have one and only one median point, which we will denote by $m(x, y, z)$.

Note that a pseudo-metric space is median if and only if its metric quotient is median.

A *strict median subspace* of a median pseudo-metric space (X, dist) is a subset Y of X such that for any three points x, y, z in Y, the set $M(x, y, z)$ is contained in Y.

A subset $Y \subset X$ is a *median subspace* if for any three points x, y, z in Y, we have $M(x, y, z) \cap Y \neq \emptyset$. Note that Y is then median for the induced pseudo-metric. An intersection of strict median subspaces is obviously a strict median subspace, thus any subset $Y \subset X$ is contained in a smallest strict median subspace, which we call the *strict median hull* of Y. When X is a metric space median subspaces are strict, thus we simplify the terminology to *median hull*.

A *homomorphism of median pseudo-metric spaces* is a map $f : X_1 \to X_2$ between two median pseudo-metric spaces such that for any three points x, y, $z \in X_1$ we have $f(M_{X_1}(x, y, z)) \subset M_{X_2}(f(x), f(y), f(z))$. This is equivalent to asking that f preserves the betweenness relation, that is $f(I(a, b)) \subset I(f(a), f(b))$.

Remark 2.5. A median metric space together with the ternary operation $(x, y, z) \mapsto m(x, y, z)$ is a particular instance of what is called a *median algebra* (see Example 4.5 in Section 4). We will use freely some classical results in the theory of abstract median algebras – although it is not difficult to prove them directly in our geometric context.

Convention 2.6. Throughout the paper, we will call median metric spaces simply *median spaces*.

Definition 2.7. We say that a metric space (X, dist) is *submedian* if it admits an isometric embedding into a median space.

Here are the main examples we have in mind.

Examples 2.8.

(1) On the real line $\mathbb{R}$, the median function is just taking the middle point of a triple, that is
    $m_{\mathbb{R}}(a, b, c) = a + b + c - [\max(a, b, c) + \min(a, b, c)]$. More generally, $\mathbb{R}^n$ with the ℓ_1

norm is a median space and

$$m(\vec{x}, \vec{y}, \vec{z}) = \big(m_{\mathbb{R}}(x_1, y_1, z_1), \ldots, m_{\mathbb{R}}(x_n, y_n, z_n)\big).$$

The interval between two points $\vec{x}, \vec{y} \in \mathbb{R}^n$ is the right-angled n-parallelepiped with opposite corners $\vec{x}$ and $\vec{y}$ and edges parallel to the coordinate axes.

(2) The ℓ_1-product of two pseudo-metric spaces (X_1, pdist_1) and (X_2, pdist_2) is the set $X_1 \times X_2$, endowed with the pseudo-metric

$$\mathrm{pdist}\big((x_1, x_2), (y_1, y_2)\big) = \mathrm{pdist}_1(x_1, y_1) + \mathrm{pdist}_2(x_2, y_2).$$

Then $(X_1 \times X_2, \mathrm{pdist})$ is median if and only if (X_1, pdist_1) and (X_2, pdist_2) are median (the components of a median point in $X_1 \times X_2$ are median points of the components).

(3) (*trees*) Every $\mathbb{R}$-tree is a median space.

(4) (*motivating example:* CAT(0) *cube complexes*) The 1-skeleton of a CAT(0) cube complex is a (discrete) median space. In fact, according to [23, Theorem 6.1] a simplicial graph is median if and only if it is the 1-skeleton of a CAT(0) cube complex.

(5) A discrete space with walls (in the sense of [39]) is submedian by [21] and [52]. We shall prove further in this paper that actually a space is submedian if and only if it is a space with measured walls (see Corollary 5.4).

(6) Various examples of submedian spaces can also be deduced from Remark 6.10. For instance, Remark 6.10 combined with results in [19] and with Proposition 6.4 implies that if (W, S) is a Coxeter system and dist_S is the word distance on W with respect to S then $(W, \mathrm{dist}_S^{1/2})$ is submedian.

Likewise, from [8] can be deduced that if X is a polygonal complex locally finite, simply connected and of type either $(4, 4)$ or $(6, 3)$ and dist is its geodesic distance then $(X, \mathrm{dist}^{1/2})$ is submedian.

(7) (L^1-*spaces*) Given a measured space $(X, \mathcal{B}, \mu)$, the metric space $L^1(X, \mu)$ is median. Indeed, it is enough to see that the real vector space $\mathcal{L}^1(X, \mu)$ of measurable functions $f : X \to \mathbb{R}$ with finite L^1-norm is a median pseudo-metric space. Define on $\mathcal{L}^1(X, \mu)$ a ternary operation $(f, g, h) \mapsto m(f, g, h)$ by

$$m(f, g, h)(x) = m_{\mathbb{R}}\big(f(x), g(x), h(x)\big).$$

Clearly $m = m(f, g, h)$ is measurable and since it is pointwise between f and g, it satisfies $\|f - g\|_1 = \|f - m\|_1 + \|m - g\|_1$. In particular $m \in \mathcal{L}^1(X, \mu)$ and $m \in I(f, g)$, where the interval is defined with respect to the pseudo-distance $\mathrm{pdist}(f, g) = \|f - g\|_1$. Similarly we have $m \in I(g, h)$ and $m \in I(f, h)$, so that $m(f, g, h)$ is a median point for f, g, h.

It is easy to see that a function $p \in \mathcal{L}^1(X, \mu)$ belongs to $I(f, g)$ if and only if the set of points x such that $p(x)$ is not between $f(x)$ and $g(x)$ has measure 0. It follows that $M(f, g, h)$ is the set of functions that are almost everywhere equal to $m(f, g, h)$, so that $\mathcal{L}^1(X, \mu)$ is a median pseudo-metric space. We conclude that $L^1(X, \mu)$ is median because it is the metric quotient of $\mathcal{L}^1(X, \mu)$.

(8) (*symmetric differences*) Let $(X, \mathcal{B}, \mu)$ still denote a measured space. For any subset $A \subset X$, we define

$$\mathcal{B}_A = \big\{ B \subseteq X \mid A \triangle B \in \mathcal{B}, \ \mu(A \triangle B) < +\infty \big\}.$$

Notice that we don't require the sets in $\mathcal{B}_A$ to be measurable, only their symmetric difference with A should be. Denote as usual by χ_C the characteristic function of a set C. Then the map $\chi^A : \mathcal{B}_A \to \mathcal{L}^1(X, \mu)$ defined by $B \mapsto \chi_{A \triangle B}$ is injective. The range of χ^A consists in the class $\mathcal{S}^1(X, \mu)$ of all characteristic functions of measurable subsets with finite measure. Indeed the preimage of $\chi_{B'}$ (with $B' \in \mathcal{B}$, $\mu(B') < +\infty$) is the subset $B := A \triangle B'$. Observe that the L^1-pseudo-distance between two functions $\chi_{B'}$ and $\chi_{C'}$ in $\mathcal{S}^1(X, \mu)$ is equal to $\mu(B' \triangle C')$. Since we have

$$(A \triangle B) \triangle (A \triangle C) = B \triangle C,$$

it follows that for any two elements $B_1, B_2 \in \mathcal{B}_A$ the symmetric difference $B_1 \triangle B_2$ is measurable with finite measure, and the pull-back of the L^1-pseudo-distance by the bijection $\mathcal{B}_A \to \mathcal{S}^1(X, \mu)$ is the pseudo-metric pdist_μ defined by $\mathrm{pdist}_\mu(B_1, B_2) = \mu(B_1 \triangle B_2)$. We claim that $(\mathcal{B}_A, \mathrm{pdist}_\mu)$ is a median pseudo-metric space, or equivalently that $\mathcal{S}^1(X, \mu)$ is a median subspace of $\mathcal{L}^1(X, \mu)$. This follows easily from the explicit formula:

$$m(\chi_A, \chi_B, \chi_C) = \chi_{(A \cup B) \cap (A \cup C) \cap (B \cup C)} = \chi_{(A \cap B) \cup (A \cap C) \cup (B \cap C)}.$$

Note that $I(\chi_A, \chi_B) \cap \mathcal{S}^1(X, \mu)$ is composed of the characteristic functions χ_C such that there exists $C' \in \mathcal{B}$ satisfying $\mu(C' \triangle C) = 0$ and $A \cap B \subset C' \subset A \cup B$. Later we will prove that any median space embeds isometrically as a median subspace of some space $\mathcal{S}^1(X, \mu)$ (compare with the similar result in the context of median algebras appearing in Corollary 4.11).

Remark 2.9. In view of Lemma 3.12 and Example 2.8(7), a metric space (X, dist) is submedian if and only if it embeds isometrically in a space $L^1(\mathcal{W}, \mu)$, for some measured space $(\mathcal{W}, \mu)$. Thus, the notion of submedian space coincides with the notion of metric space of type 1 as defined in [20, Troisième partie, §2]. Similarly, submedian metric is the same thing as metric of type 1.

Remark 2.10. (1) It is not possible in general to define for every submedian space Y a median completion, that is a median space containing an isometric copy of Y, and such that any isometric embedding of Y into a median space extends to it. This can be seen in the following example.

Let $E = \mathbb{R}^7$ endowed with the ℓ_1 norm, and let $\{e_i; \ i = 1, 2, \ldots, 7\}$ be the canonical basis. Let Y_x be the set composed of the four points A, B, C, D in E defined by $A = \frac{x}{2}(e_1 + e_2 + e_3) + (1 - x)e_4$, $B = \frac{x}{2}(-e_1 - e_2 + e_3) + (1 - x)e_5$, $C = \frac{x}{2}(e_1 - e_2 - e_3) + (1 - x)e_6$, $D = \frac{x}{2}(-e_1 + e_2 - e_3) + (1 - x)e_7$, where $x \in [0, 1]$.

Any two distinct points in Y_x are at ℓ_1-distance 2. Thus all Y_x with the ℓ_1-distance are pairwise isometric. The median hull of Y_x is composed of Y_x itself and of the eight vertices of a cube of edge length x defined by $\frac{x}{2}(\pm e_1 \pm e_2 \pm e_3)$. Thus, for two distinct values $x \neq x'$ the median hulls of Y_x and of $Y_{x'}$ are not isometric.

Note that the median hull of Y_0 is the simplicial tree with five vertices, four of which are endpoints. The median hull of Y_1 is the set of eight vertices of the unit cube. Consequently, it cannot even be guaranteed that two median hulls of two isometric submedian spaces are isomorphic as median algebras.

(2) Given a subspace Y of a median space X, it is in general not possible to extend an isometry of Y to an isometry (or at least an isomorphisms of median algebras) of the median hull of Y.

With the same notations as in (1), the isometry $Y_0 \times Y_1 \to Y_0 \times Y_1$ switching the points of Y_0 with the points of Y_1 cannot be extended to the median hull of $Y_0 \times Y_1$ in $E \times E$.

2.2. Convexity and gate property in median spaces

Definition 2.11. Let (X, pdist) denote some pseudo-metric space. A subset $Y \subset X$ is said to be *convex* if for any $a, b \in Y$ the set $I(a, b)$ is contained in Y. It is *quasi-convex* if for any $a, b \in Y$ the set $I(a, b)$ is contained in $\mathcal{N}_M(Y)$ for some M uniform in $a, b \in Y$. The *convex hull* of a subset $Y \subset X$ is the intersection of all convex subsets containing Y.

Note that any convex subspace of a median space is median but not the converse, as for instance any subset of cardinality two is a median subspace, while it might not be convex. The median hull of a subset is contained in the convex hull, and as the example above shows the inclusion may be strict.

We now introduce a notion which is related to convexity in median spaces, and which is commonly used in the theory of Tits buildings (see for example [62]) and in graph theory [46,68].

Definition 2.12 *(Gate).* Let (X, dist) be a metric space, let Y be a subset of X, and x some point in X.

We say that a point $p \in X$ is *between x and Y* if it is between x and any $y \in Y$. When a point $p \in Y$ is between x and Y, we say that p is a *gate between x and Y*. Note that there is always at most one gate p between x and Y, and that $\text{dist}(x, p) = \text{dist}(x, Y)$.

We say that Y is *gate-convex* if for any point $x \in X$ there exists a gate (in Y) between x and Y. We then denote by $\pi_Y(x)$ this gate, and call the map π_Y the *projection map onto Y*.

Lemma 2.13 *(Gate-convex subsets).*

(1) *The projection map onto a gate-convex subset is* 1*-Lipschitz.*
(2) *Any gate-convex subset is closed and convex.*
(3) *In a complete median space, any closed convex subset is gate-convex.*

In other words, for closed subsets of a complete median space, convexity is equivalent to gate-convexity.

Proof. (1) Let x, x' be two points in a metric space X, and let p, p' be the respective gates between x, x' and a gate-convex subset Y. Since (x, p, p') and (x', p', p) are geodesic sequences, we have that

$$\text{dist}(x, p) + \text{dist}(p, p') \leqslant \text{dist}(x, x') + \text{dist}(x', p'),$$
$$\text{dist}(x', p') + \text{dist}(p', p) \leqslant \text{dist}(x', x) + \text{dist}(x, p).$$

By summing up the two inequalities, we conclude that $\text{dist}(p, p') \leqslant \text{dist}(x, x')$.

(2) Assume that Y is gate-convex and that (x, y, z) is a geodesic sequence with $x, z \in Y$. Let p be the gate between y and Y, so that (y, p, x) and (y, p, z) are geodesic sequences. Hence (x, p, y, p, z) is a geodesic sequence, which forces $y = p \in Y$.

Any point x in the closure of Y satisfies $\text{dist}(x, Y) = 0$. Thus if p is the gate between x and Y we have $\text{dist}(x, p) = 0$, hence $x \in Y$. We conclude that Y is closed.

(3) Let Y be a closed convex subset of a complete median space X. For any $x \in X$ choose a sequence $(y_k)_{k \geqslant 0}$ of points in Y such that $\text{dist}(y_k, x)$ tends to $\text{dist}(x, Y)$. First observe that $(y_k)_{k \geqslant 0}$ is a Cauchy sequence. Indeed, denote by $\epsilon_k = \text{dist}(y_k, x) - \text{dist}(Y, x)$, which clearly is a sequence of positive numbers converging to zero. Let $m_{k,\ell}$ be the median point of (x, y_k, y_ℓ). Then $\text{dist}(x, y_k) + \text{dist}(x, y_\ell) = 2 \text{dist}(x, m_{k,\ell}) + \text{dist}(y_k, y_\ell)$ and so by convexity of Y we have $\text{dist}(x, y_k) + \text{dist}(x, y_\ell) \geqslant 2 \text{dist}(x, Y) + \text{dist}(y_k, y_\ell)$. It follows that $\text{dist}(y_k, y_\ell) \leqslant \epsilon_k + \epsilon_\ell$. Since X is complete the sequence $(y_k)_{k \geqslant 0}$ has a limit p in X. Since Y is closed, the point p is in Y. Note that $\text{dist}(x, p) = \text{dist}(x, Y)$. It remains to check that p is between x and Y.

Let y be some point in Y, and let m be the median point of x, p, y. By convexity of Y we have $m \in Y$, so that $\text{dist}(x, m) \geqslant \text{dist}(x, Y)$. We also have $\text{dist}(x, p) = \text{dist}(x, m) + \text{dist}(m, p)$. Since $\text{dist}(x, p) = \text{dist}(x, Y)$ we get $\text{dist}(m, p) = 0$ as desired. $\quad\square$

We now prove that in a median space the metric intervals are gate-convex.

Lemma 2.14. *In a median metric space any interval $I(a, b)$ is gate-convex, and the gate between an arbitrary point x and $I(a, b)$ is $m(x, a, b)$.*

Proof. Consider an arbitrary point x in the ambient median metric space X, p the median point $m(x, a, b)$ and y an arbitrary point in $I(a, b)$. We will show that (x, p, y) is a geodesic sequence.

We consider the median points $a' = m(x, a, y)$, $b' = m(x, b, y)$ and $p' = m(x, a', b')$. Note that $p' \in I(x, a') \subset I(x, a)$ and similarly $p' \in I(x, b)$.

Since (a, y, b), (a, a', y) and (y, b', b) are geodesic sequences, the sequence (a, a', y, b', b) is geodesic as well. So $I(a', b') \subset I(a, b)$, hence $p' \in I(a, b)$.

We proved that $p' \in I(x, a) \cap I(x, b) \cap I(a, b)$, which by the uniqueness of the median point implies $p' = p$. It follows that $p \in I(x, a') \subset I(x, y)$. $\quad\square$

We can now deduce that the median map is 1-Lipschitz, in each variable and on $X \times X \times X$ endowed with the ℓ_1-metric.

Corollary 2.15. *Let X be a median space.*

(1) *For any two fixed points $a, b \in X$ the interval $I(a, b)$ is closed and convex, and the map $x \mapsto m(x, a, b)$ is 1-Lipschitz.*

(2) *The median map $m : X \times X \times X \to X$ is 1-Lipschitz (here $X \times X \times X$ is endowed with the ℓ_1-product metric as defined in Example 2.8(1)).*

Proof. Combine Lemmas 2.14 and 2.13, and use the fact that, given six points a, b, c, a', $b', c' \in X$, the distance between the median points $m(a, b, c)$ and $m(a', b', c')$ is at most

$$\text{dist}\big(m(a, b, c), m(a', b, c)\big) + \text{dist}\big(m(a', b, c), m(a', b', c)\big) + \text{dist}\big(m(a', b', c), m(a', b', c')\big).$$

$\quad\square$

2.3. Approximate geodesics and medians; completions of median spaces

We prove that the median property is preserved under metric completion. In order to do it, we need an intermediate result stating that in a median space, approximate geodesics are close

to geodesics, and approximate medians are close to medians. We begin by defining approximate geodesics and medians.

Definition 2.16. Let (X, dist) be a metric space and let δ be a non-negative real number. We say that *z is between x and y up to δ* provided

$$\text{dist}(x, z) + \text{dist}(z, y) \leqslant \text{dist}(x, y) + \delta.$$

We say that $(a_1, a_2, \ldots, a_n)$ is a *δ-geodesic sequence* if

$$\text{dist}(a_1, a_2) + \text{dist}(a_2, a_3) + \cdots + \text{dist}(a_{n-1}, a_n) \leqslant \text{dist}(a_1, a_n) + \delta.$$

Notation 2.17. Let x, y be two points of X. We denote by $I_\delta(a, b)$ the set of points that are between a and b up to δ.

Let x, y, z be three points of X. We denote by $M_\delta(a, b, c)$ the intersection

$$I_{2\delta}(a, b) \cap I_{2\delta}(b, c) \cap I_{2\delta}(a, c).$$

In accordance with the previous notation, whenever $\delta = 0$ the index is dropped.

Lemma 2.18. *Given $\delta, \delta' \geqslant 0$, for every $c \in I_\delta(a, b)$ the set $I_{\delta'}(a, c)$ is contained in $I_{\delta+\delta'}(a, b)$.*

Definition 2.19. Let x, y, z be three points in a metric space. If $M_\delta(x, y, z)$ is non-empty then any point in it is called a *δ-median point for x, y, z.*

Lemma 2.20. *Let (X, dist) be a median space, and a, b, c three arbitrary points in it.*

(i) *The set $I_{2\delta}(a, b)$ coincides with $\mathcal{N}_\delta(I(a, b))$.*
(ii) *The following sequence of inclusions holds:*

$$B\big(m(a, b, c), \delta\big) \subseteq M_\delta(a, b, c) \subseteq B\big(m(a, b, c), 3\delta\big). \tag{1}$$

Proof. Statement (i) immediately follows from Lemma 2.14.

The first inclusion in (1) is obvious. We prove the second inclusion. To do it, we consider the median points $p_1 = m(p, a, b)$, $p_2 = m(p, b, c)$, $p_3 = m(p, a, c)$, $q = m(p_1, b, c)$, $r = m(q, a, c)$.

First we show that $r = m(a, b, c)$. Indeed $r \in I(a, c)$ by definition. We also have $r \in I(q, c)$, and since $q \in I(c, b)$ it follows that $r \in I(b, c)$. Finally we have $r \in I(a, q)$. Now $q \in I(p_1, b)$ and $p_1 \in I(a, b)$, so $q \in I(a, b)$. It follows that $r \in I(a, b)$.

It remains to estimate the distance between p and r. According to (i) and Lemma 2.14 the point p is at distance at most δ from p_1, p_2 and p_3 respectively.

By Corollary 2.15 we have $\text{dist}(p_2, q) \leqslant \text{dist}(p, p_1) \leqslant \delta$. Hence $\text{dist}(p, q) \leqslant 2\delta$. Applying Corollary 2.15 again we get $\text{dist}(p_3, r) \leqslant \text{dist}(p, q) \leqslant 2\delta$, consequently $\text{dist}(p, r) \leqslant 3\delta$. □

The following result is also proved in [69, Corollary II.3.5]. For completeness we give another proof here.

Proposition 2.21. *The metric completion of a median space is a median space as well.*

Proof. Let (X, dist) be a median space, and let $X \to \hat{X}$ be the metric completion. For simplicity we denote the distance on $\hat{X}$ also by dist.

The median map $m : X \times X \times X \to X \subset \hat{X}$ is 1-Lipschitz by Corollary 2.15. Thus it extends to a 1-Lipschitz map $\hat{X} \times \hat{X} \times \hat{X} \to \hat{X}$, also denoted by m.

Clearly for any three points a, b, c in $\hat{X}$, the point $m(a, b, c)$ is median for a, b, c. We now prove that $m(a, b, c)$ is the unique median point for a, b, c. Let p be another median point for a, b, c. The points a, b, c are limits of sequences (a_n), (b_n), (c_n) of points in X. Let m_n be the median point of a_n, b_n, c_n. Set $\delta_n = \text{dist}(a, a_n) + \text{dist}(b, b_n) + \text{dist}(c, c_n)$.

We show that p is a δ_n-median point for a_n, b_n, c_n. Indeed we have that $\text{dist}(a_n, p) + \text{dist}(p, b_n)$ is at most $\text{dist}(a_n, a) + \text{dist}(a, p) + \text{dist}(p, b) + \text{dist}(b, b_n) = \text{dist}(a_n, a) + \text{dist}(a, b) + \text{dist}(b, b_n) \leqslant 2\,\text{dist}(a, a_n) + \text{dist}(a_n, b_n) + 2\,\text{dist}(b, b_n) \leqslant \text{dist}(a_n, b_n) + 2\delta_n$. The other inequalities are proved similarly.

The point p is also the limit of a sequence of points p_n in X, such that $\text{dist}(p, p_n) \leqslant \delta_n$. It follows that p_n is a $2\delta_n$-median point for a_n, b_n, c_n. By Lemma 2.20 we then have that $\text{dist}(p_n, m_n) \leqslant 6\delta_n$. Since $\delta_n \to 0$ we get $p = m(a, b, c)$. $\square$

2.4. Rectangles and parallel pairs

In a median space X, the following notion of rectangle will allow us to treat median spaces as a continuous version of the 1-skeleton of a CAT(0) cube complex.

Definition 2.22. A *quadrilateral* in a metric space (X, dist) is a closed path (a, b, c, d, a), which we rather denote by $[a, b, c, d]$. A quadrilateral $[a, b, c, d]$ is a *rectangle* if the four sequences (a, b, c), (b, c, d), (c, d, a) and (d, a, b) are geodesic.

Remark 2.23.

(1) By the triangular inequality, in a rectangle $[a, b, c, d]$ the following equalities hold: $\text{dist}(a, b) = \text{dist}(c, d)$, $\text{dist}(a, d) = \text{dist}(b, c)$ and $\text{dist}(a, c) = \text{dist}(b, d)$.
(2) (*rectangles in intervals*) If $x, y \in I(a, b)$ then $[x, m(x, y, a), y, m(x, y, b)]$ is a rectangle.
(3) (*subdivision of rectangles*) Let $[a, b, c, d]$ be a rectangle. Let $e \in I(a, d)$ and $f = m(e, b, c)$. Then $[a, b, f, e]$ and $[c, d, e, f]$ are rectangles.

Definition 2.24 *(Parallelism on pairs).* Two pairs (a, b) and (d, c) are *parallel* if $[a, b, c, d]$ is a rectangle.

We mention without proof the following remarkable fact that confirms the analogy with CAT(0) cube complexes:

Proposition 2.25. *In a median space the parallelism on pairs is an equivalence relation.*

We now explain how to any 4-tuple of points one can associate a rectangle.

Lemma 2.26. *Let $[x, a, y, b]$ be any quadrilateral in a median space. Then there exists a unique rectangle $[x', a', y', b']$ satisfying the following properties:*
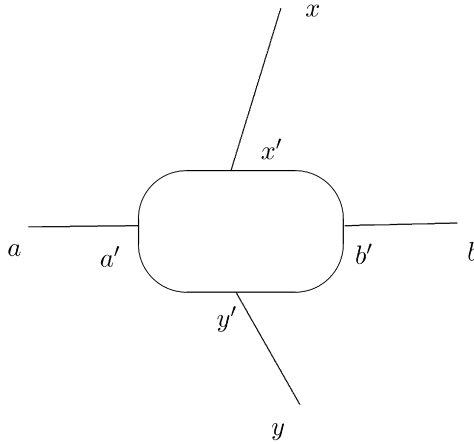
Fig. 1. Central rectangle.

(1) *the following sequences are geodesic:*

$$\big(x, x', a', a\big), \big(a, a', y', y\big), \big(y, y', b', b\big), \big(b, b', x', x\big);$$

(2) (a, a', b', b) *is a geodesic sequence;*
(3) (x, x', y') *and* (y, y', x') *are geodesic sequences.*

Proof. *Existence.* Let $x' = m(x, a, b)$ and $y' = m(y, a, b)$, and let $a' = m(a, x', y')$ and $b' = m(b, x', y')$ (see Fig. 1). Then $[x', a', y', b']$ is a rectangle by Remark 2.23(3). Properties (1) and (2) follow immediately from the construction, property (3) follows from Lemma 2.14 applied to x and $y' \in I(a, b)$, respectively to y and $x' \in I(a, b)$.

Uniqueness. Let $[x', a', y', b']$ be a rectangle satisfying the three required properties. Properties (1), (2) and the fact that $[x', a', y', b']$ is a rectangle imply that $x' = m(x, a, b)$ and $y' = m(y, a, b)$. Again property (2) and the fact that $[x', a', y', b']$ is a rectangle imply that $a' = m(a, x', y')$ and $b' = m(b, x', y')$. $\square$

Definition 2.27. We call the rectangle $[x', a', y', b']$ described in Lemma 2.26 the *central rectangle* associated with the quadrilateral $[x, a, y, b]$.

Remark 2.28. Property (3) cannot be improved to "(x, x', y', y) is a geodesic sequence", as shown by the example of a unit cube in $\mathbb{R}^3$, with a, b two opposite vertices of the lower horizontal face, and x, y the two opposite vertices of the upper horizontal face that are not above b or d (see Fig. 2).

Note also that in general the central rectangle associated with $[x, a, y, b]$ is distinct from the central rectangle associated with $[a, x, b, y]$ (again see Fig. 2).

Property (3) in Lemma 2.26 can be slightly improved as follows.

Lemma 2.29. *Let* x, y, p, q *be four points such that* (x, p, q) *and* (p, q, y) *are geodesic sequences. Then there exists a geodesic sequence* (x, x', y', y) *such that* (x', y') *and* (p, q) *are parallel.*
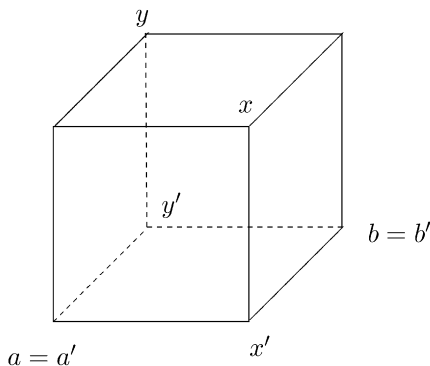
Fig. 2. Example of central rectangle.

Proof. Applying Lemma 2.26 to the quadrilateral $[p, q, y, x]$, we note that the resulting central rectangle $[p', q', y', x']$ satisfies $p' = p, q' = q$.   □

3. Space with measured walls, median space associated to it

3.1. Preliminaries on measures

We recall the relevant definitions on measured spaces. A reference is [12], whose terminology we adopt here. Let Y be a non-empty set and let $\mathcal{P}(Y)$ be the power set of Y. A *ring* is a subset of $\mathcal{P}(Y)$ containing the empty set, closed with respect to finite unions and differences. A σ-*algebra* is a subset in $\mathcal{P}(Y)$ containing the empty set, closed with respect to countable unions and containing together with any subset its complementary.

Given a ring $\mathcal{R}$, a *premeasure* on it is a function $\mu : \mathcal{R} \to [0, +\infty]$ such that

(M_0) $\mu(\emptyset) = 0$;
(M_1) For any sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{R}$ such that $\bigsqcup_{n \in \mathbb{N}} A_n \in \mathcal{R}$,

$$\mu\left(\bigsqcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n).$$

Property (M_1) is equivalent to

(M_1') $\mu(A \sqcup B) = \mu(A) + \mu(B)$;
(M_1'') If $(A_n)_{n \in \mathbb{N}}$ is a non-increasing sequence of sets in $\mathcal{R}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$, then $\lim_{n \to \infty} \mu(A_n) = 0$.

A premeasure is called σ-*finite* if there exists a sequence (A_n) in $\mathcal{R}$ such that $\mu(A_n) < +\infty$ for every n, and $\bigcup_n A_n = Y$.

A premeasure defined on a σ-algebra is called a *measure*.

An *additive function* on a ring $\mathcal{R}$ is a map $\mu : \mathcal{R} \to [0, +\infty]$ satisfying properties (M_0) and (M_1).

We need a precise version of Carathéodory's theorem on the extension of any premeasure μ to a measure, therefore we recall here the notion of outer measure. For every $Q \subset Y$ let $\mathcal{U}(Q)$

designate the set of all sequences (A_n) in $\mathcal{R}$ such that $Q \subset \bigcup_n A_n$. Define $\mu^*(Q) = +\infty$ if $\mathcal{U}(Q) = \emptyset$; if $\mathcal{U}(Q) \neq \emptyset$ then

$$\mu^*(Q) = \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n); \; (A_n) \in \mathcal{U}(Q) \right\}.$$

The function μ^* is an *outer measure* on the set Y.

A subset A of Y is called μ^*-*measurable* if for every $Q \in \mathcal{P}(Y)$,

$$\mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \cap A^c).$$

Theorem 3.1 *(Carathéodory). (See [12, §I.5].)*

(1) *The collection $\mathcal{A}^*$ of μ^*-measurable sets is a σ-algebra containing $\mathcal{R}$, and the restriction of μ^* to $\mathcal{A}^*$ is a measure, while the restriction of μ^* to $\mathcal{R}$ coincides with μ.*
(2) *If μ is σ-finite, then it has a unique extension to a measure on the σ-algebra generated by $\mathcal{R}$.*

3.2. Spaces with measured walls

From [39], we recall that a *wall* of a set X is a partition $X = h \sqcup h^c$ (where h is possibly empty or the whole X). A collection $\mathcal{H}$ of subsets of X is called a *collection of half-spaces* if for every $h \in \mathcal{H}$ the complementary subset h^c is also in $\mathcal{H}$. We call *collection of walls* on X the collection $\mathcal{W}_\mathcal{H}$ of pairs $w = \{h, h^c\}$ with $h \in \mathcal{H}$. For a wall $w = \{h, h^c\}$ we call h and h^c the two *half-spaces bounding* w.

We say that a wall $w = \{h, h^c\}$ *separates* two disjoint subsets A, B in X if $A \subset h$ and $B \subset h^c$ or vice-versa and denote by $\mathcal{W}(A|B)$ the set of walls separating A and B. In particular $\mathcal{W}(A|\emptyset)$ is the set of walls $w = \{h, h^c\}$ such that $A \subset h$ or $A \subset h^c$; hence $\mathcal{W}(\emptyset|\emptyset) = \mathcal{W}$.

When $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_m\}$ we write

$$\mathcal{W}(A|B) = \mathcal{W}(x_1, \ldots, x_n|y_1, \ldots, y_m).$$

We use the notation $\mathcal{W}(x|y)$ to designate $\mathcal{W}(\{x\}|\{y\})$. We call any set of walls of the form $\mathcal{W}(x|y)$ a *wall-interval*. By convention $\mathcal{W}(A|A) = \emptyset$ for every non-empty set A.

Definition 3.2 *(Space with measured walls)*. (See [25].) A *space with measured walls* is a 4-uple $(X, \mathcal{W}, \mathcal{B}, \mu)$, where $\mathcal{W}$ is a collection of walls, $\mathcal{B}$ is a σ-algebra of subsets in $\mathcal{W}$ and μ is a measure on $\mathcal{B}$, such that for every two points $x, y \in X$ the set of separating walls $\mathcal{W}(x|y)$ is in $\mathcal{B}$ and it has finite measure. We denote by pdist_μ the pseudo-metric on X defined by $\mathrm{pdist}_\mu(x, y) = \mu(\mathcal{W}(x|y))$, and we call it the *wall pseudo-metric*.

Lemma 3.3. *The collection $\mathcal{R}$ of disjoint unions $\bigsqcup_{i=1}^{n} \mathcal{W}(F_i|G_i)$, where $n \in \mathbb{N}^*$, and F_i, G_i are finite non-empty sets for every $i = 1, 2, \ldots, n$, is a ring.*

Proof. Property (1) is obviously satisfied.

We first note that $\mathcal{W}(F|G) \cap \mathcal{W}(F'|G') = \mathcal{W}(F \cup F'|G \cup G') \sqcup \mathcal{W}(F \cup G'|G \cup F')$.

Let now F, G be two finite.

$$\mathcal{W}(F|G)^c = \bigsqcup_{S \sqcup T = F \cup G, \{S,T\} \neq \{F,G\}} \mathcal{W}(S|T).$$

From the two statements above it follows that $\mathcal{R}$ satisfies property (3), i.e. it is closed with respect to the operation $\backslash$. But $\mathcal{R}$ is closed with respect to intersection, thus $\mathcal{R}$ is also closed with respect to union. $\square$

Theorem 3.1 and Lemma 3.3 imply the following.

Proposition 3.4 (*Minimal data required for a structure of measured walls*). *Let X be a space and let $\mathcal{W}$ be a collection of walls on it. A structure of measured walls can be defined on $(X, \mathcal{W})$ if and only if on the ring $\mathcal{R}$ composed of disjoint unions $\bigsqcup_{i=1}^{n} \mathcal{W}(F_i|G_i)$, where $n \in \mathbb{N}^*$, and F_i, G_i, $i = 1, 2, \ldots, n$, are finite non-empty sets, can be defined a premeasure μ such that for every $x, y \in X$, $\mu(\mathcal{W}(x|y))$ is finite.*

Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ and $(X', \mathcal{W}', \mathcal{B}', \mu')$ be two spaces with measured walls, and let $\phi : X \to X'$ be a map.

Definition 3.5. The map ϕ is a *homomorphism between spaces with measured walls* provided that:

- for any $w' = \{h', h'^c\} \in \mathcal{W}'$ we have $\{\phi^{-1}(h'), \phi^{-1}(h'^c)\} \in \mathcal{W}$ – this latter wall we denote by $\phi^*(w')$;
- the map $\phi^* : \mathcal{W}' \to \mathcal{W}$ is surjective and for every $B \in \mathcal{B}$, $(\phi^*)^{-1}(B) \in \mathcal{B}'$ and $\mu'((\phi^*)^{-1}(B)) = \mu(B)$.

Note that ϕ induces an isometry of the spaces equipped with the wall pseudo-distances.

Consider the set $\mathcal{H}$ of half-spaces determined by $\mathcal{W}$, and the natural projection map $\mathfrak{p} : \mathcal{H} \to \mathcal{W}$, $h \mapsto \{h, h^c\}$. The preimages of the sets in $\mathcal{B}$ define a σ-algebra on $\mathcal{H}$, which we denote by $\mathcal{B}^{\mathcal{H}}$; hence on $\mathcal{H}$ can be defined a pull-back measure that we also denote by μ. This allows us to work either in $\mathcal{H}$ or in $\mathcal{W}$. Notice that the σ-algebra $\mathcal{B}^{\mathcal{H}}$ does not separate points in $\mathcal{H}$, as sets in $\mathcal{B}^{\mathcal{H}}$ are unions of fibers of $\mathfrak{p}$.

Definition 3.6. (See [21,52].) A section $\mathfrak{s}$ for $\mathfrak{p}$ is called *admissible* if its image contains together with a half-space h all the half-spaces h' containing h.

Throughout the paper we identify an admissible section $\mathfrak{s}$ with its image $\sigma = \mathfrak{s}(\mathcal{W})$; with this identification, an admissible section becomes a collection of half-spaces, σ, such that:

- for every wall $w = \{h, h^c\}$ either h or h^c is in σ, but never both;
- if $h \subset h'$ and $h \in \sigma$ then $h' \in \sigma$.

For any $x \in X$ we denote by $\mathfrak{s}_x$ the section of $\mathfrak{p}$ associating to each wall the half-space bounding it and containing x. Obviously it is an admissible section. We denote by σ_x its image, that is the set of half-spaces $h \in \mathcal{H}$ such that $x \in h$. Observe that σ_x is not necessarily in $\mathcal{B}^{\mathcal{H}}$.

Note that $\mathfrak{p}(\sigma_x \bigtriangleup \sigma_y) = \mathcal{W}(x|y)$.

Example 3.7 *(Real hyperbolic space).* For all the discussion below, see [58].

Define the half-spaces of the real hyperbolic space $\mathbb{H}^n$ to be closed or open geometric half-spaces, with boundary an isometric copy of $\mathbb{H}^{n-1}$, so that a wall consists of one closed half-space and its (open) complement, as in Section 3 of [25]. Recall that the full group of direct isometries of $\mathbb{H}^n$ is $SO_0(n, 1)$. The associated set of walls $\mathcal{W}_{\mathbb{H}^n}$ is naturally identified with the homogeneous space $SO_0(n, 1)/SO_0(n - 1, 1)$; as $SO_0(n - 1, 1)$ is unimodular, there is a $SO_0(n, 1)$-invariant borelian measure $\mu_{\mathbb{H}^n}$ on the set of walls [47, Chapter 3, Corollary 4]. The set of walls separating two points has compact closure and finite measure. Thus $(\mathbb{H}^n, \mathcal{W}_{\mathbb{H}^n}, \mathcal{B}, \mu_{\mathbb{H}^n})$ is a space with measured walls. By Crofton's formula [58, Proposition 2.1] up to multiplying the measure $\mu_{\mathbb{H}^n}$ by some positive constant the wall pseudo-metric on $\mathbb{H}^n$ is just the usual hyperbolic metric.

Definition 3.8. The action by automorphisms of a topological group G on a space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$ is called *continuous* if for every $x \in X$ the map $G \to X$, $g \mapsto gx$ is continuous, where X is endowed with the topology defined by the pseudo-distance pdist_μ.

The following result allows to produce many examples of spaces with measured walls.

Lemma 3.9 *(Pull back of a space with measured walls). Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a space with measured walls, let S be a set and $f : S \to X$ a map. There exists a pull back structure of space with measured walls $(S, \mathcal{W}_S, \mathcal{B}_S, \mu_S)$ turning f into a homomorphism. Moreover:*

 (i) *if S is endowed with a pseudo-metric pdist and f is an isometry between (S, pdist) and (X, pdist_μ), then the wall pseudo-metric pdist_{μ_S} coincides with the initial pseudo-metric pdist;*
(ii) *if a group G acts on S by bijective transformations and on X by automorphisms of space with measured walls, and if f is G-equivariant, then G acts on $(S, \mathcal{W}_S, \mathcal{B}_S, \mu_S)$ by automorphisms of space with measured walls. Moreover, if the action on X is continuous, the action on S is.*

Proof. Define the set of walls $\mathcal{W}_S$ on S as the set of walls $\{f^{-1}(h), f^{-1}(h^c)\}$, where $\{h, h^c\}$ is a wall in X. This defines a surjective map $f^* : \mathcal{W} \to \mathcal{W}_S$. We then consider the push-forward structure of measured space on $\mathcal{W}_S$. This defines a structure of measured space with walls on S such that f is a homomorphism of spaces with measured walls.

 (i) It is easily seen that for every $x, y \in S$, $(f^*)^{-1}(\mathcal{W}_S(x|y)) = \mathcal{W}(f(x), f(y))$, hence $\mathrm{pdist}_{\mu_S}(x, y) = \mathrm{pdist}_\mu(f(x), f(y)) = \mathrm{pdist}(x, y)$.

 (ii) If f is G-equivariant then the whole structure of space with measured walls $(S, \mathcal{W}_S, \mathcal{B}_S, \mu_S)$ is G-equivariant. $\square$

One of the main interests in actions of groups on spaces with measured walls is given by the following result.

Lemma 3.10. *(See [25,27].) Let G be a group acting (continuously) by automorphisms on a space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$. Let $p > 0$ and let π_p be the representation of G on $L^p(\mathcal{H}, \mu_\mathcal{H})$.*

Then for every $x \in X$, the map $b : G \to L^p(\mathcal{H}, \mu_{\mathcal{H}})$ defined by $b(g) = \chi_{\sigma_{gx}} - \chi_{\sigma_x}$ is a (continuous) 1-cocycle in $Z^1(G, \pi_p)$. In other words, a (continuous) action of G on $L^p(\mathcal{H}, \mu_{\mathcal{H}})$ by affine isometries can be defined by:

$$g \cdot f = \pi_p(g)f + b(g).$$

Remark 3.11. Recall that for a space $L^p(X, \mu)$ with $p \in (0, 1)$, $\|f\|_p = (\int |f|^p \, d\mu)^{\frac{1}{p}}$ no longer satisfies the usual triangular inequality, it only satisfies a similar inequality with a multiplicative factor added to the second term. On the other hand, $\|f\|_p^p$ is no longer a norm, but it does satisfy the triangular inequality, hence it defines a metric [43].

In this paper we consider L^p-spaces endowed with this metric, for $p \in (0, 1)$.

3.3. Embedding a space with measured walls in a median space

Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a space with measured walls, and let x_0 be a base point in X.

Recall from Example 2.8(8) that $\mathcal{B}^{\mathcal{H}}_{\sigma_{x_0}}$ denotes the collection of subsets $A \subset \mathcal{H}$ s.t. $A \triangle \sigma_{x_0} \in \mathcal{B}$ and $\mu(A \triangle \sigma_{x_0}) < +\infty$, and that endowed with the pseudo-metric $\text{pdist}_\mu(A, B) = \mu(A \triangle B)$ this collection becomes a median pseudo-metric space. The map

$$\chi^{x_0} : \mathcal{B}^{\mathcal{H}}_{\sigma_{x_0}} \to \mathcal{S}^1(\mathcal{H}, \mu), \qquad \chi^{x_0}(A) = \chi_{A \triangle \sigma_{x_0}} \tag{2}$$

is an isometric embedding of $\mathcal{B}^{\mathcal{H}}_{\sigma_{x_0}}$ into the median subspace $\mathcal{S}^1(\mathcal{H}, \mu) \subset \mathcal{L}^1(\mathcal{H}, \mu)$, where $\mathcal{S}^1(\mathcal{H}, \mu) = \{\chi_B; \ B \text{ measurable and } \mu(B) < +\infty\}$.

The formula $A \triangle \sigma_{x_1} = (A \triangle \sigma_{x_0}) \triangle (\sigma_{x_0} \triangle \sigma_{x_1})$ and the fact that $\sigma_{x_0} \triangle \sigma_{x_1}$ is measurable with finite measure shows that the median pseudo-metric spaces $\mathcal{B}^{\mathcal{H}}_{\sigma_{x_0}}$ and $\mathcal{B}^{\mathcal{H}}_{\sigma_{x_1}}$ are identical: we simply denote this space by $\mathcal{B}^{\mathcal{H}}_X$. In particular $\sigma_x \in \mathcal{B}^{\mathcal{H}}_X$ for each $x \in X$.

For $x, y \in X$ we have $\text{pdist}_\mu(x, y) = \mu(\sigma_x \triangle \sigma_y)$, thus $x \mapsto \sigma_x$ is an isometric embedding of X into $(\mathcal{B}^{\mathcal{H}}_X, \text{pdist}_\mu)$. Composing with the isometry $\chi^{x_0} : \mathcal{B}^{\mathcal{H}}_X \to \mathcal{S}^1(\mathcal{H}, \mu)$, we get the following well-known result stating that a wall pseudo-distance is of type 1, in the terminology of [20, Troisième partie, §2]:

Lemma 3.12. *Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a space with measured walls, and $x_0 \in X$ a base point in it. Then the map $x \mapsto \chi_{\mathcal{W}(x|x_0)}$ defines an isometry from X to $L^1(\mathcal{W}, \mu)$. Thus if the wall pseudo-distance is a distance then (X, dist_μ) is isometric to a subset of $L^1(\mathcal{W}, \mu)$, and so it is submedian.*

We could probably define the median space associated to a space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$ to be the median hull of the isometric image of X inside $L^1(\mathcal{W}, \mu)$ (and then perhaps take the closure in order to get a complete median space). We give here an alternative construction which is more intrinsic.

Notation 3.13. We denote by $\overline{\mathcal{M}}(X)$ the set of admissible sections, and by $\mathcal{M}(X)$ the intersection $\overline{\mathcal{M}}(X) \cap \mathcal{B}^{\mathcal{H}}_X$.

Every section σ_x belongs to $\mathcal{M}(X)$, thus X isometrically embeds in $\mathcal{M}(X)$. We denote by $\iota : X \to \mathcal{M}(X)$ this isometric embedding.

Proposition 3.14. *Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a space with measured walls.*

(i) *The space $\mathcal{M}(X)$ defined as above is a median subspace of $\mathcal{B}^{\mathcal{H}}{}_X$.*

(ii) *Any homomorphism $\phi : X \to X'$ between X and another space with measured walls $(X', \mathcal{W}', \mathcal{B}', \mu')$ induces an isometry $\mathcal{M}(X) \to \mathcal{M}(X')$.*

(iii) *In particular the group of automorphisms of $(X, \mathcal{W}, \mathcal{B}, \mu)$ acts by isometries on $\mathcal{M}(X)$.*

Proof. (i) Given an arbitrary triple $(\sigma_1, \sigma_2, \sigma_3) \in \mathcal{M}(X)^3$, let us denote by $m(\sigma_1, \sigma_2, \sigma_3)$ the set of half-spaces h such that there exist at least two distinct indices $i, j \in \{1, 2, 3\}$ with $h \in \sigma_i$, $h \in \sigma_j$. In other words $m(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 \cap \sigma_2) \cup (\sigma_1 \cap \sigma_3) \cup (\sigma_2 \cap \sigma_3)$ (see also Example 4.6).

Clearly $m = m(\sigma_1, \sigma_2, \sigma_3)$ belongs to $\overline{\mathcal{M}}(X)$. Fix a point x_0 in X and take $\chi_0 = \chi^{x_0}$ the function defined in (2). We want to show that $\chi_0(m) = m(\chi_0(\sigma_1), \chi_0(\sigma_2), \chi_0(\sigma_3))$. This will prove that $m \in \mathcal{B}^{\mathcal{H}}{}_X$ and that m is a median point of $\sigma_1, \sigma_2, \sigma_3$.

For our set-theoretical calculation it is convenient to treat characteristic functions as maps from $\mathcal{H}$ to $\mathbb{Z}/2\mathbb{Z}$. We may then use the addition (mod 2) and pointwise multiplication on these functions. We get

$$\chi_{A \cap B} = \chi_A \chi_B, \qquad \chi_{A \triangle B} = \chi_A + \chi_B, \qquad \chi_{A \cup B} = \chi_A + \chi_B + \chi_A \chi_B.$$

It follows easily that for any three subsets A, B, C we have

$$\chi_{(A \cap B) \cup (A \cap C) \cup (B \cap C)} = \chi_A \chi_B + \chi_A \chi_C + \chi_B \chi_C.$$

Thus $\chi_{[(A \cap B) \cup (A \cap C) \cup (B \cap C)] \triangle D} = \chi_A \chi_B + \chi_A \chi_C + \chi_B \chi_C + \chi_D$. On the other hand $\chi_{((A \triangle D) \cap (B \triangle D)) \cup ((A \triangle D) \cap (C \triangle D)) \cup ((B \triangle D) \cap (C \triangle D))} = (\chi_A + \chi_D)(\chi_B + \chi_D) + (\chi_A + \chi_D)(\chi_C + \chi_D) + (\chi_B + \chi_D)(\chi_C + \chi_D) = \chi_A \chi_B + \chi_A \chi_C + \chi_B \chi_C + 2\chi_A \chi_D + 2\chi_B \chi_D + 2\chi_C \chi_D + 3\chi_D = \chi_A \chi_B + \chi_A \chi_C + \chi_B \chi_C + \chi_D$. We have thus checked that $[(A \cap B) \cup (A \cap C) \cup (B \cap C)] \triangle D$ coincides with $[(A \triangle D) \cap (B \triangle D)] \cup [(A \triangle D) \cap (C \triangle D)] \cup [(B \triangle D) \cap (C \triangle D)]$. Applying this to $A = \sigma_1$, $B = \sigma_2$, $C = \sigma_3$, $D = \sigma_{x_0}$ yields the desired result.

(ii) Consider a homomorphism of spaces with measured walls $\phi : X \to X'$. It is easily seen that the surjective map $\phi^* : \mathcal{W}' \to \mathcal{W}$ induces a surjective map $\phi^* : \mathcal{H}' \to \mathcal{H}$ such that for every $B \in \mathcal{B}^{\mathcal{H}}$, $(\phi^*)^{-1}(B) \in \mathcal{B}^{\mathcal{H}'}$ and $\mu'((\phi^*)^{-1}(B)) = \mu(B)$.

Let σ denote any admissible section. Set $\phi_*(\sigma) = (\phi^*)^{-1}(\sigma) = \{h' \in \mathcal{H}'; \phi^{-1}(h') \in \sigma\}$. Since ϕ is a homomorphism, $\phi_*(\sigma)$ is an admissible section of $(X', \mathcal{W}', \mathcal{B}', \mu')$. Note that $\phi_*(\sigma_x) = \sigma_{\phi(x)}$ and that $\phi_*(\sigma \triangle \sigma') = \phi_*(\sigma) \triangle \phi_*(\sigma')$. This implies that ϕ_* defines a map from $\mathcal{M}(X)$ to $\mathcal{M}(X')$. Moreover $\mathrm{pdist}_{\mathcal{M}(X')}(\phi_*(\sigma), \phi_*(\sigma')) = \mu'(\phi_*(\sigma) \triangle \phi_*(\sigma')) = \mu'(\phi_*(\sigma \triangle \sigma')) = \mu'((\phi^*)^{-1}(\sigma \triangle \sigma')) = \mu(\sigma \triangle \sigma') = \mathrm{pdist}_{\mathcal{M}(X)}(\sigma, \sigma')$. Thus ϕ_* is an isometry.

The statement (iii) is an immediate consequence of (ii). $\quad\square$

The results in Proposition 3.14 justify the following terminology.

Definition 3.15. We call $\mathcal{M}(X)$ *the median space associated with $(X, \mathcal{W}, \mathcal{B}, \mu)$.*

The first part of Theorem 1.1 is proved.

Remark 3.16. The median space $\mathcal{M}(X)$ has measured walls. Indeed for each $h \in \mathcal{H}$ define $h_{\mathcal{M}}$ to be the set of $\sigma \in \mathcal{M}(X)$ such that $h \in \sigma$. The complement of $h_{\mathcal{M}}$ in $\mathcal{M}(X)$ is the set of

$\sigma \in \mathcal{M}(X)$ such that $h \notin \sigma$, or equivalently by the properties of admissible sections $h^c \in \sigma$. In other words $(h_\mathcal{M})^c = (h^c)_\mathcal{M}$. Thus $\{h_\mathcal{M}\}_{h \in \mathcal{H}}$ is a collection of half-spaces – which we will denote by $\mathcal{H}_\mathcal{M}$. We denote by $\mathcal{W}_\mathcal{H}$ the associated set of walls on $\mathcal{M}(X)$. Using the bijection $\mathcal{W} \to \mathcal{W}_\mathcal{H}$ induced by $h \mapsto h_\mathcal{M}$ we define on $\mathcal{W}_\mathcal{H}$ a σ-algebra $\mathcal{B}_\mathcal{H}$ and a measure $\mu_\mathcal{M}$. Note that $\iota : X \to \mathcal{M}(X)$ is a homomorphism. Note also that the distance on $\mathcal{M}(X)$ coincides with the distance induced by the measured walls structure.

It is easy to check that the medianized space associated with $\mathcal{M}(X)$ endowed with this structure of space with measured walls is $\mathcal{M}(X)$ itself.

Remark 3.17. One cannot hope to define a median space $(\mathcal{M}(X), \mathrm{dist})$ associated to a space with measured walls such that there exists an isometric map $\iota : (X, \mathrm{pdist}_\mu) \to (\mathcal{M}(X), \mathrm{dist})$ with the universality property that any isometric map from (X, pdist_μ) to a median space factors through ι. This was explained in Remark 2.10.

4. A review of median algebras

The notion of median algebra appeared as a common generalization of trees and lattices (in the ordered structure sense of the word). We recall here some basic definitions and properties related to median algebras. For proofs and further details we refer the reader to the books [68, 69], the surveys [10,41], as well as the papers [16,65,66,60].

4.1. Definitions, examples

Definition 4.1 (*Median algebra, first definition*). A *median algebra* is a set X endowed with a ternary operation $(a, b, c) \mapsto m(a, b, c)$ such that:

(1) $m(a, a, b) = a$;
(2) $m(a, b, c) = m(b, a, c) = m(b, c, a)$;
(3) $m(m(a, b, c), d, e) = m(a, m(b, d, e), m(c, d, e))$.

Property (3) can be replaced by $(3')$ $m(a, m(a, c, d), m(b, c, d)) = m(a, c, d)$.

The element $m(a, b, c)$ is the *median of the points* a, b, c. In a median algebra (X, m), given any two points a, b the set $I(a, b) = \{x; \ x = m(a, b, x)\}$ is called *the interval of endpoints* a, b. This defines a map $I : X \times X \to \mathcal{P}(X)$. We say that a point $x \in I(a, b)$ is *between a and b*.

A *homomorphism* of median algebras is a map $f : (X, m_X) \to (Y, m_Y)$ such that $m_Y(f(x), f(y), f(z)) = f(m_X(x, y, z))$. Equivalently, f is a homomorphism if and only if it preserves the betweenness relation. If moreover f is injective (bijective) then f is called *embedding* or *monomorphism* (respectively *isomorphism*) of median algebras.

The following are straightforward properties that can be found in the literature (see for instance [65] and [60, §2]).

Lemma 4.2. *Let (X, m) be a median algebra. For $x, y, z \in X$ we have that*

(1) $I(x, x) = \{x\}$;
(2) $I(x, y) \cap I(x, z) = I(x, m(x, y, z))$;
(3) $I(x, y) \cap I(x, z) \cap I(y, z) = \{m(x, y, z)\}$;

(4) *if $a \in I(x, y)$ then for any t, $I(x, t) \cap I(y, t) \subseteq I(a, t)$ (equivalently $m(x, y, t) \in I(a, t)$);*
(5) *if $x \in I(a, b)$ and $y \in I(x, b)$ then $x \in I(a, y)$.*

A sequence of points $(a_1, a_2, \ldots, a_n)$ is *geodesic* in the median algebra (X, m) if $a_i \in I(a_1, a_{i+1})$ for all $i = 2, \ldots, n-1$. This is equivalent, by Lemma 4.2, point (5), to the condition that $a_{i+1} \in I(a_i, a_n)$ for all $i = 1, 2, \ldots, n-2$.

Lemma 4.3. *If (x, t, y) is a geodesic sequence, then:*

(1) $I(x, t) \cup I(t, y) \subseteq I(x, y)$;
(2) $I(x, t) \cap I(t, y) = \{t\}$.

According to [65,66] there is an alternative definition of median algebras, using intervals.

Definition 4.4 *(Median algebra, second definition).* A *median algebra* is a set X endowed with a map $I : X \times X \to \mathcal{P}(X)$ such that:

(1) $I(x, x) = \{x\}$;
(2) if $y \in I(x, z)$ then $I(x, y) \subset I(x, z)$;
(3) for every x, y, z in X the intersection $I(x, y) \cap I(x, z) \cap I(y, z)$ has cardinality 1.

Example 4.5. Let (X, dist) be a median space. Then the metric intervals $I(x, y)$ satisfy the properties in Definition 4.4, and thus the metric median $(x, y, z) \mapsto m(x, y, z)$ defines a structure of median algebra on X.

Example 4.6. Here is the set-theoretic generalization of Example 2.8(8). For any set X, the power set $\mathcal{P}(X)$ is a median algebra when endowed with the Boolean median operation

$$m(A, B, C) = (A \cap B) \cup (A \cap C) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \cap (B \cup C). \qquad (3)$$

The median algebra $(\mathcal{P}(X), m)$ is called a *Boolean median algebra*. One easily sees that in this case

$$I(A, B) = \{C; \ A \cap B \subset C \subset A \cup B\}. \qquad (4)$$

In what follows we use the notation $Bm(A, B, C)$ to designate the Boolean median defined in (3) and $BI(A, B)$ to designate the Boolean interval defined in (4).

It appears that Example 4.6 is in some sense the typical example of median algebra. More precisely, according to Corollary 4.11, any median algebra is a subalgebra of a Boolean median algebra, up to isomorphism.

4.2. Convexity

Definition 4.7. A *convex subset* A in a median algebra is a subset such that for any $a, b \in A$, $I(a, b) \subset A$; equivalently it is a subset such that for every $x \in X$, and a, b in A the element $m(a, x, b)$ is in A.

A subset h in a median space (X, m) is called *a convex half-space* if itself and the complementary set h^c are convex. The pair $\{h, h^c\}$ is called a *convex wall*. We denote by $\mathcal{H}_c(X)$ the set of convex half-spaces in X and by $\mathcal{W}_c(X)$ the set of convex walls in X. When there is no possibility of confusion we simply use the notations $\mathcal{H}_c$ and $\mathcal{W}_c$.

The above algebraic notion of convexity coincides with the metric notion of convexity introduced in Definition 2.11, in the case of the median algebra associated with a median space (see Example 4.5).

The following result shows that there are plenty of convex walls in a median algebra.

Theorem 4.8. *Let X be a median algebra, and let A, B be two convex non-empty disjoint subsets of X. Then there exists a convex wall separating A and B.*

A proof of Theorem 4.8 when A is a singleton can be found in [53]; in its most general form it follows from [67, Theorem 2.5]. Other proofs can be found in [11, §5.2] and [60, §2].

Corollary 4.9. *Given any two distinct points x, y in a median space (X, dist) there exists a convex wall $w = \{h, h^c\}$ with $x \in h$, $y \in h^c$.*

Definition 4.10. Given a median algebra X, one can define the map

$$\sigma : X \to \mathcal{P}(\mathcal{H}_c), \qquad \sigma(x) = \sigma_x = \{h \in \mathcal{H}_c; \ x \in h\}.$$

A consequence of Theorem 4.8 is the following.

Corollary 4.11. *The map σ is an embedding of median algebras.*

5. Median spaces have measured walls

The aim of this section is to prove the following.

Theorem 5.1. *Let (X, dist) be a median space. Let $\mathcal{W}$ be the set of convex walls, and let $\mathcal{B}$ be the σ-algebra generated by the following subset of $\mathcal{P}(\mathcal{W})$:*

$$\mathcal{U} = \big\{ \mathcal{W}(x|y); \ x, y \text{ points of } X \big\}.$$

Then there exists a measure μ on $\mathcal{B}$ such that:

(1) *$\mu(\mathcal{W}(x|y)) = \mathrm{dist}(x, y)$; consequently the 4-tuple $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a space with measured walls;*
(2) *any isometry of (X, dist) is an automorphism of the space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$.*

Remark 5.2. According to Carathéodory's theorem, a measure μ on the σ-algebra $\mathcal{B}$ is not uniquely defined by the condition (1) in Theorem 5.1. It is uniquely defined if there exists say a sequence of points (x_n) in X such that $\mathcal{W} = \bigcup_{n,m} \mathcal{W}(x_n|x_m)$. This happens for instance if there exists a countable subset in X whose convex hull is the entire X. Uniqueness is also guaranteed when for some topology on $\mathcal{W}$ the measure μ is borelian and $\mathcal{W}$ is locally compact second countable.

Combining Theorem 5.1 above and Lemma 3.12 we get the following:

Corollary 5.3. *Let (X, dist) be a median space. Then X isometrically embeds in $L^1(\mathcal{W}, \mu)$, where $(\mathcal{W}, \mu)$ are as in Theorem 5.1.*

More precisely, given any $x_0 \in X$, the space X is isometric to $\{\chi_{\mathcal{W}(x|x_0)}; \ x \in X\} \subset L^1(\mathcal{W}, \mu)$ endowed with the induced metric.

The fact that median spaces embed isometrically into L^1-spaces was known previously, though not via a construction of an embedding as above, but using Assouad's result that a space is embeddable into an L^1-space if and only if any finite subset of the space is [3,4,6]. That finite median spaces can be embedded into ℓ^1-spaces seems to be well known in graph theory; all proofs usually refer to finite median graphs only, but can be adapted to work for finite median spaces (see for instance [46]). There exist even algorithms which isometrically embed a given median graph into an ℓ^1-space; the same method yields algorithms in sub-quadratic time recognizing median graphs [38]. The statement that finite median spaces can be embedded into ℓ^1 was explicitly stated and proved for the first time in [69, Theorem V.2.3].

Corollary 5.4. *A metric space (X, dist) is submedian in the sense of Definition 2.7 if and only if it admits a structure of space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$ such that $\mathrm{dist} = \mathrm{dist}_\mu$. Moreover all walls in $\mathcal{W}$ may be assumed to be convex.*

Proof. The direct part follows from Theorem 5.1 and Lemma 3.9.

The converse part follows from Lemma 3.12. □

Remark 5.5. Corollary 5.4 for finite metric spaces was already known. We recall this version here as it will prove useful further on.

More precisely, according to [3] and [6] a finite metric space (X, dist) is isometrically ℓ^1-embeddable if and only if

$$\mathrm{dist} = \sum_{S \subseteq X} \lambda_S \delta_S,$$

where λ_S are non-negative real numbers, and $\delta_S(x, y) = 1$ if $x \neq y$ and $S \cap \{x, y\}$ has cardinality one, $\delta_S(x, y) = 0$ otherwise.

Theorem 5.1 together with Proposition 3.14 show that the natural dual category of median pseudo-metric spaces is the category of spaces with measured walls. Precise results on duality of categories for particular categories of median algebras and spaces with walls can be found in [60] and [11].

Remark 5.6. According to the construction in Section 3.2, a space with measured walls X has a natural embedding into a median space $\mathcal{M}(X)$; moreover $\mathcal{M}(X)$ has an induced structure of space with measured walls, and its metric coincides with the metric induced by the measured walls structure (Remark 3.16).

We note here that the above structure of space with measured walls on $\mathcal{M}(X)$ does not in general agree with the structure described in this section. In general the first structure does not have convex walls, as the walls on X may not be convex. In a forthcoming paper we will show

that both structures on $\mathcal{M}(X)$ are equivalent, in the sense that they induce the same structure of measured spaces with walls on finite subsets.

The strategy of the proof of Theorem 5.1 is to use Proposition 3.4. We first show that for any pair of finite non-empty sets F, G in X, $\mathcal{W}(F|G)$ is equal to $\mathcal{W}(a|b)$ for some pair of points a, b. In order to do this we need the following intermediate results.

Lemma 5.7. *Let (x, y, z) be a geodesic sequence. Then we have the following decomposition as a disjoint union:*

$$\mathcal{W}(x|z) = \mathcal{W}(x|y) \sqcup \mathcal{W}(y|z).$$

Proof. First notice that by convexity of half-spaces, the intersection $\mathcal{W}(x|y) \cap \mathcal{W}(y|z)$ is empty. Then the inclusion $\mathcal{W}(x|z) \subseteq \mathcal{W}(x|y) \cup \mathcal{W}(y|z)$ is clear because if a half-space h contains x but does not contain z, then either h contains y (in which case the wall $\{h, h^c\}$ separates y from z) or h^c contains y (in which case the wall $\{h, h^c\}$ separates x from y). The inclusion $\mathcal{W}(x|y) \cup \mathcal{W}(y|z) \subseteq \mathcal{W}(x|z)$ holds because if h contains x and $y \notin h$, again by convexity we cannot have $z \in h$ and hence $\{h, h^c\}$ separates x from z. $\square$

As an immediate consequence we get the following:

Corollary 5.8. *For any geodesic sequence $(x_1, x_2, \ldots, x_n)$ we have the following decomposition:*

$$\mathcal{W}(x_1|x_n) = \mathcal{W}(x_1|x_2) \sqcup \cdots \sqcup \mathcal{W}(x_{n-1}|x_n).$$

Corollary 5.9. *If (x, y) and (x', y') are parallel pairs then*

$$\mathcal{W}(x|y) = \mathcal{W}(x'|y') = \mathcal{W}(x, x'|y, y')$$

and

$$\mathcal{W}(x|y') = \mathcal{W}(x'|y) = \mathcal{W}(x|y) \sqcup \mathcal{W}(x|x').$$

Lemma 5.10. *Given three points x, y, z with median point m, we have $\mathcal{W}(x|y, z) = \mathcal{W}(x|m)$.*

Proof. According to Lemma 5.7 we have that $\mathcal{W}(x|y) = \mathcal{W}(x|m) \sqcup \mathcal{W}(m|y)$ and that $\mathcal{W}(x|z) = \mathcal{W}(x|m) \sqcup \mathcal{W}(m|z)$. It follows that

$$\mathcal{W}(x|y, z) = \mathcal{W}(x|y) \cap \mathcal{W}(x|z) = \mathcal{W}(x|m) \sqcup \big(\mathcal{W}(m|y) \cap \mathcal{W}(m|z)\big).$$

But by convexity of the walls $\mathcal{W}(m|y) \cap \mathcal{W}(m|z) = \emptyset$, and we are done. $\square$

We will use intensively the following two operations:

Definition 5.11 *(Projection and straightening).* Let $(x, y), (a, b)$ be two pairs of points of a median space X.

The *projection of* (x, y) *with target* (a, b) is the pair (x', y') defined by $x' = m(x, a, b)$, $y' = m(y, a, b)$.

If furthermore $x, y \in I(a, b)$ we also consider the *straightening of the path* (a, x, y, b), which by definition is the path (a, p, q, b), where the pair (p, q) is defined by $p = m(a, x, y)$, $q = m(b, x, y)$.

Observe that given two pairs of points (x, y), (a, b), the central rectangle $[x', a', y', b']$ associated with $[x, a, y, b]$ (as defined in Definition 2.27) is obtained by first projecting (x, y) with target (a, b) – this yields the pair (x', y') – and then straightening (a, x', y', b) – which yields the pair (a', b'). We now give some properties of both procedures.

Lemma 5.12. *Let* (x, y), (a, b) *be two pairs of points.*

(1) *Let* (x', y') *be the projection of* (x, y) *with target* (a, b). *Then*

$$\mathcal{W}(x'|y') = \mathcal{W}(x|y) \cap \mathcal{W}(a|b).$$

(2) *Assume* $x, y \in I(a, b)$, *and let* (p, q) *be the projection of* (a, b) *with target* (x, y). *Then* $[p, x, q, y]$ *is a rectangle,* $\mathcal{W}(p|q) = \mathcal{W}(x|y)$, *and* (a, p, q, b) *is a geodesic sequence (thus* (a, x, y, b) *has really been straightened to a geodesic).*
(3) *Let* $[x', a', y', b']$ *be the central rectangle associated with* $[x, a, y, b]$. *Then*

$$\mathcal{W}(x'|y') = \mathcal{W}(x|y) \cap \mathcal{W}(a|b), \quad \mathcal{W}(x'|y') = \mathcal{W}(a'|b').$$

Proof. Since the central rectangle is in fact obtained by composing the projecting and straightening operations, it is enough to prove statement (3).

The equality $\mathcal{W}(x'|y') = \mathcal{W}(a'|b')$ follows by Corollary 5.9.

By Lemma 5.10 we have $\mathcal{W}(x|x') = \mathcal{W}(x|a, b)$. In particular $\mathcal{W}(x|x') \cap \mathcal{W}(a|b) = \emptyset$. And similarly $\mathcal{W}(y|y') \cap \mathcal{W}(a|b) = \emptyset$.

Consider now a half-space h such that $x \in h$, $y \notin h$ and $\{h, h^c\} \in \mathcal{W}(a|b)$. Since $\mathcal{W}(x|x') \cap \mathcal{W}(a|b) = \emptyset$, we deduce that $x' \in h$. Similarly we have $y' \in h^c$. We have thus proved that $\mathcal{W}(x|y) \cap \mathcal{W}(a|b) \subset \mathcal{W}(x'|y')$.

On the other hand, since $\mathcal{W}(x'|y') = \mathcal{W}(a'|b')$ and (a, a', b', b) is a geodesic, it follows that $\mathcal{W}(x'|y') \subset \mathcal{W}(a|b)$.

According to Lemma 2.29, (x', y') is parallel to a pair (x'', y'') such that (x, x'', y'', y) is geodesic. This and Corollary 5.9 imply that $\mathcal{W}(x'|y') \subset \mathcal{W}(x|y)$. $\square$

Proposition 5.13. *Let* F *and* G *be two finite non-empty subsets in* X. *There exist two points* $p, q \in X$ *such that*

$$\mathcal{W}(F|G) = \mathcal{W}(p|q).$$

Proof. We use an inductive argument over $n = \text{card } F + \text{card } G$. For $n = 2$ the result is obvious, while for $n = 3$ it is Lemma 5.10.

Assume that the statement holds for n and let F, G be such that $\text{card } F + \text{card } G = n + 1 \geqslant 3$. Without loss of generality we may assume that $\text{card } F \geqslant 2$. Then $F = F_1 \sqcup \{x\}$, and $\mathcal{W}(F|G) = \mathcal{W}(F_1|G) \cap \mathcal{W}(x|G)$. The inductive hypothesis implies that $\mathcal{W}(F_1|G) = \mathcal{W}(a|b)$ and $\mathcal{W}(x|G) =$

$\mathcal{W}(c|d)$, for some points a, b, c, d. Hence $\mathcal{W}(F|G) = \mathcal{W}(a|b) \cap \mathcal{W}(c|d)$. We end up by applying Lemma 5.12. □

At this stage we have proven that the ring $\mathcal{R}$ defined in Proposition 3.4 coincides with the set of disjoint unions $\bigsqcup_{i=1}^n \mathcal{W}(x_i|y_i)$. It remains to show that there is a premeasure $\mu : \mathcal{R} \to \mathbb{R}^+$ on the ring $\mathcal{R}$ such that $\mu(\mathcal{W}(x|y)) = \operatorname{dist}(x, y)$. We first define μ as an additive function.

Lemma 5.14. *If* $\mathcal{W}(x|y) = \mathcal{W}(a|b)$ *then* $\operatorname{dist}(x, y) = \operatorname{dist}(a, b)$.

Proof. First let (x', y') be the projection of (x, y) with target (a, b). Then by Lemma 5.12(1) we have $\mathcal{W}(x'|y') = \mathcal{W}(x|y) \cap \mathcal{W}(a|b) = \mathcal{W}(a|b)$. By Corollary 2.15 the median map is 1-Lipschitz, thus $d(x', y') \leqslant d(x, y)$.

We now straighten (a, x', y', b) to (a, p, q, b) (thus (p, q) is the projection of (a, b) with target (x', y')). Then by Lemma 5.12(2) we have $\mathcal{W}(p|q) = \mathcal{W}(x'|y') = \mathcal{W}(a|b)$, and (a, p, q, b) is a geodesic. By Corollary 5.8 we deduce $\mathcal{W}(a|p) = \mathcal{W}(q|b) = \emptyset$, and thus $a = p$, $q = b$. It follows that $d(a, b) = d(p, q)$, and thus by Corollary 5.9 we have $d(a, b) = d(x', y') \leqslant d(x, y)$. We conclude by symmetry. □

Proposition 5.15. *Assume that for two points a, b the set of walls $\mathcal{W}(a|b)$ decomposes as $\mathcal{W}(a|b) = \bigsqcup_{j=1}^n \mathcal{W}(x_j|y_j)$. Then there exists a geodesic sequence $(a_1 = a, a_2, \ldots, a_{2^n} = b)$ and a partition $\{1, 2, \ldots, 2^n - 1\} = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$ such that:*

(1) *for each $j \in \{1, \ldots, n\}$ the set I_j has 2^{j-1} elements and we have a decomposition of $\mathcal{W}(x_j|y_j) = \bigsqcup_{i \in I_j} \mathcal{W}(a_i|a_{i+1})$;*
(2) *for each $j \in \{1, \ldots, n\}$ we have $\operatorname{dist}(x_j, y_j) = \sum_{i \in I_j} \operatorname{dist}(a_i, a_{i+1})$.*

In particular, $\operatorname{dist}(a, b) = \sum_j \operatorname{dist}(x_j, y_j)$.

We easily deduce the following:

Corollary 5.16. *There is a unique additive function $\mu : \mathcal{R} \to \mathbb{R}^+$ such that $\mu(\mathcal{W}(x|y)) = \operatorname{dist}(x, y)$.*

To prove Proposition 5.15 we need the following auxiliary result:

Lemma 5.17. *In a median space (X, dist), consider two geodesic sequences with common endpoints (a, p, q, b) and (a, p', q', b), such that $\mathcal{W}(p|q) \cap \mathcal{W}(p'|q') = \emptyset$. Let (s, t) be the projection of (p', q') with target (a, p). Similarly let (u, v) be the projection of (p', q') with target (q, b). Then $\operatorname{dist}(p', q') = \operatorname{dist}(s, t) + \operatorname{dist}(u, v)$.*

Proof. Consider two more points: $m = m(t, p', q'), n = m(u, p', q')$ (see Fig. 3). Let us check that $[s, t, m, p']$ is a rectangle. By construction (t, m, p') is a geodesic sequence. Since s, t are projection of p', q' onto the interval $I(a, p)$ we deduce that $(q', m, t, s), (p', s, t)$ are geodesic sequences. And since (x, p', q', y) is a geodesic sequence we see that (x, s, p', m, q', y) is geodesic.
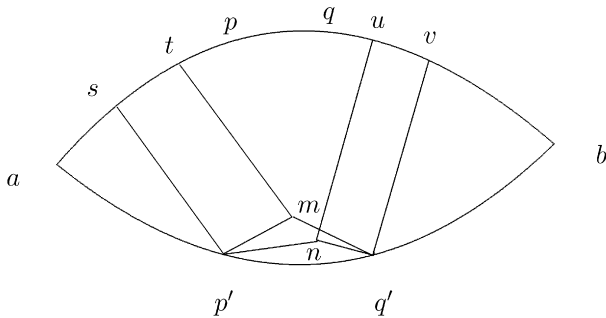
Fig. 3. The construction in Lemma 5.17.

We thus have $\mathrm{dist}(p', m) = \mathrm{dist}(s, t)$, and also $\mathcal{W}(p'|m) = \mathcal{W}(s|t)$ (by Corollary 5.9). Hence $\mathcal{W}(p'|m) = \mathcal{W}(a|p) \cap \mathcal{W}(p'|q')$ (by Lemma 5.12(1)). Similarly we get $\mathrm{dist}(n, q') = \mathrm{dist}(u, v)$, and $\mathcal{W}(n|q') = \mathcal{W}(q|b) \cap \mathcal{W}(p'|q')$.

We claim that $\mathcal{W}(m|q') = \mathcal{W}(q|b) \cap \mathcal{W}(p'|q')$. Indeed applying several times Lemma 5.7 we get

$$\mathcal{W}(p'|m) \sqcup \mathcal{W}(m|q') = \mathcal{W}(p'|q') \subset \mathcal{W}(a|b) = \mathcal{W}(a|p) \sqcup \mathcal{W}(p|q) \sqcup \mathcal{W}(q|b)$$

and the claim follows, since by assumption $\mathcal{W}(p|q) \cap \mathcal{W}(p'|q') = \emptyset$ and we already have $\mathcal{W}(p'|m) = \mathcal{W}(a|p) \cap \mathcal{W}(p'|q')$.

We deduce that $\mathcal{W}(m|q') = \mathcal{W}(n|q')$. This implies $\mathrm{dist}(m, q') = \mathrm{dist}(n, q') = \mathrm{dist}(u, v)$ by Lemma 5.14. Since (p', m, q') is a geodesic we get $\mathrm{dist}(p', q') = \mathrm{dist}(p', m) + \mathrm{dist}(m, q') = \mathrm{dist}(s, t) + \mathrm{dist}(u, v)$. $\square$

Proof of Proposition 5.15. We argue by induction on n. The case $n = 1$ follows by Lemma 5.14.

Now let us assume that $n > 1$ and that the lemma is true for partitions of any wall-interval into $n - 1$ wall-intervals. Notice first that, according to Lemma 5.12(1) and Lemma 5.14, modulo replacing (x_i, y_i) by its projection with target (a, b), we can assume that the x_i's and y_i's belong to the interval $I(a, b)$.

We straighten (a, x_1, y_1, b) to (a, p_1, q_1, b). Then by Lemma 5.12(2) the sequence (a, p_1, q_1, b) is geodesic, and we have $\mathcal{W}(x_1|y_1) = \mathcal{W}(p_1|q_1)$.

By Lemma 5.7 we have $\mathcal{W}(a|b) = \mathcal{W}(a|p_1) \sqcup \mathcal{W}(p_1|q_1) \sqcup \mathcal{W}(q_1|b)$. It follows that $\mathcal{W}(a|p_1) \sqcup \mathcal{W}(q_1|b) = \bigsqcup_{i=2}^{n} \mathcal{W}(x_i|y_i)$.

We now straighten each path (a, x_i, y_i, b) to (a, p_i, q_i, b) (when $i > 1$). Again we have $\mathcal{W}(x_i|y_i) = \mathcal{W}(p_i|q_i)$ and moreover $\mathrm{dist}(x_i, y_i) = \mathrm{dist}(p_i, q_i)$ (since $[x_i, p_i, y_i, q_i]$ is a rectangle). Now let us project the points p_i and q_i onto $I(x, p_1)$ and $I(q_1, y)$. So set $s_i = m(p_i, x, p_1)$, $t_i = m(q_i, x, p_1)$, $u_i = m(p_i, q_1, y)$ and $v_i = m(q_i, q_1, y)$.

Applying again Lemma 5.12(1) we get that $\mathcal{W}(p_i|q_i) \cap \mathcal{W}(a|p_1) = \mathcal{W}(s_i|t_i)$ and $\mathcal{W}(p_i|q_i) \cap \mathcal{W}(q_1|b) = \mathcal{W}(u_i|v_i)$. Thus $\mathcal{W}(p_i|q_i) = \mathcal{W}(s_i|t_i) \sqcup \mathcal{W}(u_i|v_i)$, and we get two decompositions: $\mathcal{W}(a|p_1) = \bigsqcup_{i=2}^{n} \mathcal{W}(s_i|t_i)$ and $\mathcal{W}(q_1|b) = \bigsqcup_{i=2}^{n} \mathcal{W}(u_i|v_i)$.

If we apply the induction hypothesis to the two decompositions above we see that we are done since Lemma 5.17 ensures that $\mathrm{dist}(p_i, q_i) = \mathrm{dist}(s_i, t_i) + \mathrm{dist}(u_i, v_i)$. $\square$

The following shows that the premeasure satisfies property (M_1'').

Proposition 5.18. *Let* (X, dist) *be a median space, endowed with convex walls. If* $(I_n)_{n \in \mathbb{N}}$ *is a non-increasing sequence of finite disjoint unions of wall-intervals such that* $\bigcap_n I_n = \emptyset$, *then* $I_k = \emptyset$ *for k large enough.*

Proof. In what follows we identify a half-space with its characteristic function. First note that the set of half-spaces bounding a convex wall (i.e. the set of convex subsets whose complement is convex as well) is a closed subset of $\{0, 1\}^X$. Then the set $\mathcal{H}(x|y)$ of half-spaces containing x but not y is a closed subset of the compact subset of $\{0, 1\}^X$ consisting in functions $f : X \to \{0, 1\}$ such that $f(x) = 1$, $f(y) = 0$. So $\mathcal{H}(x|y)$ is compact.

It is enough to argue when $I_0 = \mathcal{W}(x|y)$. Since $(I_n)_{n \in \mathbb{N}}$ is non-increasing for each n we have $I_n \subset \mathcal{W}(x|y)$. We then define H_n as the set of half-spaces h such that $\{h, h^c\} \in I_n$, and $x \in h$. It follows that $(H_n)_{n \in \mathbb{N}}$ is non-increasing, and has empty intersection. By projecting onto $I(x, y)$ we have $I_n = \bigsqcup \mathcal{W}(x_i|y_i)$ for some points $x_i, y_i \in I(x, y)$ (Lemma 5.12(1)). We know that $\mathcal{W}(x_i|y_i) = \mathcal{W}(p_i|q_i)$ for $p_i = m(x, x_i, y_i)$, $q_i = m(y, x_i, y_i)$, and furthermore (x, p_i, q_i, y) is a geodesic sequence. Thus $H_n = \bigsqcup \mathcal{W}(p_i|q_i)$ and H_n is compact. It follows that there exists k such that $H_k = \emptyset$, which implies that $I_k = \emptyset$. $\quad\square$

We now have all the ingredients to finish the proof of Theorem 5.1.

Proof of Theorem 5.1. That the premeasure μ is well-defined on $\mathcal{R}$ is the content of Proposition 5.15. It obviously satisfies properties (M_0) and (M_1'), while (M_1'') is proved in Proposition 5.18.

By Carathéodory's Theorem 3.1, μ^* restricted to $\mathcal{A}^*$ is a measure extending μ, hence its restriction to $\mathcal{B}$ is also a measure extending μ.

Obviously any isometry of (X, dist) defines a bijective transformation on $\mathcal{W}$ preserving $\mathcal{R}$ and the premeasure μ, hence the outer measure μ^* and $\mathcal{A}^*$, hence it defines an automorphism of the measured space $(\mathcal{W}, \mathcal{B}, \mu)$. $\quad\square$

6. Kernels, median spaces, properties (T) and Haagerup

6.1. Various types of kernels

A *kernel* on a set X is a symmetric map $\psi : X \times X \to \mathbb{R}_+$ such that $\psi(x, x) = 0$. For instance, a pseudo-metric is a kernel.

Let $f : X \to Y$ be a map and let ϕ be a kernel on Y. The *pull-back of ϕ under f* is the kernel $\psi(x, y) = \phi(f(x), f(y))$. Given a class of kernels $\mathcal{C}$, a kernel ψ on X is *of type $\mathcal{C}$* if ψ is the pull-back of some kernel in the class $\mathcal{C}$.

We will be particularly interested in *kernels of median type*, which are obtained by pulling back a median pseudo-distance. By considering the canonical median metric quotient, we see that any kernel of median type is also the pull-back of a median distance.

Properties (T) and Haagerup (a-T-menability) have often been described using conditionally negative definite kernels, the definition of which we now recall.

Definition 6.1. A kernel $\psi : X \times X \to \mathbb{R}_+$ is *conditionally negative definite* if for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i = 0$ the following holds:

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \psi(x_i, x_j) \leqslant 0.$$

Proposition 6.2. *(See [63].) If $\psi : X \times X \to \mathbb{R}_+$ is a conditionally negative definite kernel and $0 < \alpha \leqslant 1$ then ψ^α is a conditionally negative definite kernel.*

An example of conditionally negative definite kernel is provided by the following result.

Proposition 6.3. *(See [71, Theorem 4.10].) Let $(Y, \mathcal{B}, \mu)$ be a measured space. Let $0 < p \leqslant 2$, and let $E = L^p(Y, \mu)$ be endowed with the norm $\| \cdot \|_p$. Then $\psi : E \times E \to \mathbb{R}$, $\psi(x, y) = \|x - y\|_p^p$ is a conditionally negative definite kernel.*

In some sense, the example in Proposition 6.3 is universal for conditionally negative definite kernels, as the following statement shows.

Proposition 6.4. *(See [63].) A function $\psi : X \times X \to \mathbb{R}_+$ is a conditionally negative definite kernel if and only if there exists a map $f : X \to H$, where $(H, \| \cdot \|)$ is a Hilbert space, such that*

$$\psi(x, y) = \left\| f(x) - f(y) \right\|^2. \tag{5}$$

The discussion above suggests the following:

Definition 6.5. A function $\psi : X \times X \to \mathbb{R}_+$ is a *kernel of type p*, where $0 < p \leqslant 2$, if there exists a map $f : X \to L^p(Y, \mu)$, for some measured space $(Y, \mathcal{B}, \mu)$, such that

$$\psi(x, y) = \left\| f(x) - f(y) \right\|_p^p. \tag{6}$$

Proposition 6.6. *A function $\psi : X \times X \to \mathbb{R}_+$ is a kernel of type 1 if and only if it is of median type.*

Proof. Since $L^1(Y, \mu)$ is a median space (see Example 2.8(7)), a kernel of type 1 is of median type. Conversely, Corollary 5.3 shows that a median space embeds in some $L^1(W, \mu)$, so a kernel of median type, by composition with this embedding, will be of type 1. $\quad\square$

Remark 6.7. Clearly, the pull-back of a conditionally negative definite kernel (or of a kernel of type p) is also conditionally negative definite (respectively, of type p).

Proposition 6.4 states that conditionally negative definite kernels are the same thing as kernels of type 2. In order to investigate further the relationship between conditionally negative definite kernels and kernels of type p, we recall some results on isometric embeddings of L^p-spaces.

Theorem 6.8. *(See Theorems 1 and 7 in [20].) Let $1 \leqslant p \leqslant q \leqslant 2$.*

(1) *The normed space $(L^q(X, \mu), \| \cdot \|_q)$ can be embedded linearly and isometrically into $(L^p(X', \mu'), \| \cdot \|_p)$ for some measured space $(X', \mathcal{B}', \mu')$.*
(2) *If $L^p(X, \mu)$ has infinite dimension then $(L^p(X, \mu), \| \cdot \|_p^\alpha)$ can be embedded isometrically into $(L^q(X', \mu'), \| \cdot \|_q)$ for some measured space $(X', \mathcal{B}', \mu')$ if and only if $0 < \alpha \leqslant \frac{p}{q}$.*

Remark 6.9. Note that according to [42], the space l_p with $p > 2$ does not coarsely embed into a Hilbert space.

Remark 6.10. Theorem 6.8(1) implies that every metric space that can be isometrically embedded in a space $L^p(X, \mu)$ with $p \in [1, 2]$ (*metric space of type p* in the terminology of [20] and [31]) is a submedian space. See [31] for examples of such spaces.

Using these results we can now establish a more precise relationship between kernels of type p and conditionally negative definite.

Corollary 6.11.

(1) *If ψ is a kernel of type p for some $0 < p \leqslant 2$ then ψ is a conditionally negative definite kernel.*

(2) *If ψ is a conditionally negative definite kernel and $1 \leqslant p \leqslant 2$ then $\psi^{\frac{p}{2}}$ is a kernel of type p.*

Proof. Let X be an arbitrary space and let $\psi : X \times X \to \mathbb{R}_+$.

(1) follows from Proposition 6.3 and Remark 6.7.

(2) According to Proposition 6.4 there exists a map $g : X \to L^2(X, \mu)$ such that $\psi(x, y) = \|g(x) - g(y)\|_2^2$. By Theorem 6.8(1), there exists an isometric embedding $F : (L^2(X, \mu), \| \cdot \|_2) \to (L^p(X', \mu'), \| \cdot \|_p)$. Consequently $\psi(x, y) = \|g(x) - g(y)\|_2^2 = \|F(g(x)) - F(g(y))\|_p^2$, and $\psi^{p/2}$ is a kernel of type p. $\quad \square$

Remark 6.12.

(1) By Proposition 6.6, Corollary 6.11 and Proposition 6.4, every submedian space (X, dist) has the property that $(X, \text{dist}^{1/2})$ can be embedded isometrically in a Hilbert space. This can be refined ([5], [30, Proposition 2.5]) to the sequence of implications: (X, dist) submedian $\Rightarrow$ (X, dist) hypermetric $\Rightarrow$ $(X, \text{dist}^{1/2})$ spherically L^2-embeddable $\Rightarrow$ $(X, \text{dist}^{1/2})$ L^2-embeddable.

Recall that a kernel $\psi : X \times X \to \mathbb{R}$ (in particular a metric) is *hypermetric* if for any finite sequence $x_1, \ldots, x_n$ in X and any integers $\lambda_1, \ldots \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$, we have $\sum_{i,j=1}^n \lambda_i \lambda_j \psi(x_i, x_j) \leqslant 0$. A kernel is *spherical* if its restriction to any finite subset of X coincides with a pull-back of a metric on an Euclidean unit sphere. A metric space is called *spherically L^2-embeddable* if its distance is a spherical kernel.

It follows that any submedian space (X, dist) has the property that all its finite subsets endowed with the metric $\text{dist}^{1/2}$ are isometric to subsets on an Euclidean unit sphere. This holds even for submedian spaces of negative curvature, like $\mathbb{H}_{\mathbb{R}}^n$, as was first noticed by Robertson in [58, Corollary 3.2].

(2) The above implications can be reformulated in terms of kernels thus: ψ kernel of type 1 $\Rightarrow$ ψ hypermetric kernel $\Rightarrow$ $\psi^{1/2}$ spherical kernel $\Rightarrow$ ψ kernel of type 2.

Robertson and Steger defined in [59] an alternate type of kernels.

Definition 6.13. (See Robertson and Steger [59].) A *measure definite kernel* on a space X is a map $\psi : X \times X \to \mathbb{R}_+$ such that there exists a measured space $(\mathcal{M}, \mathcal{B}, \mu)$ and a map $S : X \to \mathcal{B}$, $x \mapsto S_x$, satisfying $\psi(x, y) = \mu(S_x \triangle S_y)$.

In [59] it is asked (Question (i)) whether measure definite kernels can be given an intrinsic characterization among the conditionally negative definite kernels. It turns out that measure defi-

nite kernels are very much related to structures of space with measured walls, as well as to median spaces (see Lemma 6.14). This relationship allows us to answer this question (Corollary 6.17).

Lemma 6.14. *A kernel $\psi : X \times X \to \mathbb{R}_+$ on a space X is measure definite if and only if it is of median type, in other words it is the pull-back of a median metric. Moreover when X is a topological space the kernel ψ is continuous if and only if the pull-back map f is continuous.*

Proof. Assume that ψ is a measure definite kernel on X. Then there exists a map $S : X \to \mathcal{B}$, $x \mapsto S_x$, where $(\mathcal{M}, \mathcal{B}, \mu)$ is a measured space, and $\psi(x, y) = \mu(S_x \vartriangle S_y)$. Fix some base point x_0 and endow $\mathcal{B}_{S_{x_0}}$ with the structure of median pseudo-metric space described in Example 2.8(8). Then ψ is the pull-back under S of this median pseudo-metric.

Conversely, consider a map f from X to a median space (Y, dist) such that $\psi(x, x') = \text{dist}(f(x), f(x'))$. By Theorem 5.1, there exists a set of convex walls $\mathcal{W}$ on Y, a σ-algebra $\mathcal{B}$ on $\mathcal{W}$ and a measure μ on $\mathcal{B}$ such that the 4-tuple $(Y, \mathcal{W}, \mathcal{B}, \mu)$ is a space with measured walls, and moreover $\text{dist}(y, y') = \mu(\mathcal{W}(y|y'))$.

We fix a point x_0 in X and we define the map $S : X \to \mathcal{B}$, $S_x = \mathcal{W}(f(x)|f(x_0))$. Then $\mu(S_a \vartriangle S_b) = \mu(\mathcal{W}(f(a)|f(x_0)) \vartriangle \mathcal{W}(f(b)|f(x_0))) = \mu(\mathcal{W}(f(a)|f(b))) = \text{dist}(f(a), f(b)) = \psi(a, b)$.

Obviously f continuous implies ψ continuous. Conversely, assume that ψ is continuous. If $y \in X$ is close to $x \in X$ then (x, y) is close to (x, x) hence $\psi(x, y) = \text{dist}(f(x), f(y))$ is close to $\psi(x, x) = 0$. $\square$

The following statement is an improvement of [59, Proposition 1.2] and [25, Proposition 2].

Lemma 6.15. *A map $\psi : X \times X \to \mathbb{R}_+$ on a space X is a measure definite kernel if and only if there exists a structure of space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$ on X such that $\psi(x, x') = \mu(\mathcal{W}(x|x'))$.*

Proof. The if part follows from Lemmata 3.12 and 6.14.

Conversely, assume that ψ is a measure definite kernel on X. By Proposition 6.14 the kernel ψ is the pull-back of a median distance: $\psi(x, y) = \text{dist}(f(x), f(y))$ for some map $f : X \to Y$ where (Y, dist) is a median space. Consider the structure of space with measured walls on Y given by Theorem 5.1. The pull-back structure of space with measured walls on X has ψ as wall pseudo-distance, according to Lemma 3.9. $\square$

Proposition 6.16. *A kernel is measure definite if and only if it is of type 1.*

Proof. Follows directly from Lemma 6.14 and Proposition 6.6. $\square$

Corollary 6.17. *A kernel $\psi : X \times X \to \mathbb{R}$ is measure definite if and only if ψ satisfies the triangular inequality, moreover for every finite subset F in X, $\psi|_{F \times F}$ is equal to $\sum_{S \subseteq F} \lambda_S \delta_S$ for some $\lambda_S \geqslant 0$, where $\delta_S(x, y) = 1$ if $\psi(x, y) > 0$ and $S \cap \{x, y\}$ is of cardinality 1, $\delta_S(x, y) = 0$ otherwise.*

Proof. It follows immediately from Proposition 6.16, from the fact that a metric space is isometrically embeddable into an L^1-space if any finite subset of it is [3,6], and from Remark 5.5. $\square$

Corollary 6.18.

(1) *Every measure definite kernel is conditionally negative definite.*
(2) *If ψ is a conditionally negative definite kernel then $\sqrt{\psi}$ is a measure definite kernel.*

Statement (1) in Corollary 6.18 has already been proved in [59], where it appears as Proposition 1.1, while statement (2) has been proved in [59, Proposition 1.4(i)] under the extra assumption that the set on which the kernel is defined is countable.

6.2. Properties (T) and Haagerup and actions on median, measured walls and L^p-spaces

Definition 6.19. A function $\Phi : G \to \mathbb{R}_+$ defined on a group is *conditionally negative definite* if the function $G \times G \to \mathbb{R}_+$, $(g, h) \mapsto \Phi(g^{-1}h)$, is a conditionally negative definite kernel.

Recall that a function Φ is called *proper* if $\lim_{g \to \infty} \Phi(g) = \infty$. Here $g \to \infty$ means that g leaves any compact subset.

If a conditionally negative definite kernel $\psi : G \times G \to \mathbb{R}_+$ is *left invariant*, i.e. $\psi(g_1, g_2) = \psi(hg_1, hg_2)$ for every h, g_1, g_2 in G, then the map $\Phi : G \to \mathbb{R}_+$ defined by $\Phi(g) = \psi(1, g)$ is a conditionally negative definite function. If Φ is proper we say that the kernel ψ is *proper*.

We also recall that a *second countable space* is a topological space satisfying the second axiom of countability, that is such that its topology has a countable base. A second countable space is *separable* (i.e. has a countable dense subset) and *Lindelöf* (i.e. every open cover has a countable sub-cover). The converse implications do not hold in general, but they do for metric spaces.

Characterizations of properties (T) and Haagerup (also called a-T-menability) using conditionally negative definite kernels are well-known and can be found in the literature. We recall here the relevant ones.

Theorem 6.20. *(See [29,36,1,28,24].) Let G be a second countable, locally compact group.*

(1) *The group G has property (T) if and only if every continuous conditionally negative definite function on G is bounded (equivalently, every continuous left invariant conditionally negative definite kernel on G is bounded).*
(2) *The group G has the Haagerup property if and only if there exists a continuous proper conditionally negative definite function on G (equivalently, there exists a continuous proper left invariant conditionally negative definite kernel on G).*

Theorem 6.20 and Corollary 6.11 imply the following.

Corollary 6.21. *Let G be a second countable, locally compact group.*

(1) *If the group G has property (T) then for every $p \in (0, 2]$, every continuous left invariant kernel of type p on G is bounded.*
(2) *The group G has the Haagerup property if for some $p \in (0, 2]$, there exists a continuous proper left invariant kernel of type p on G.*

Remark 6.22. For $p \in [1, 2]$ the converse statements in Corollary 6.21 immediately follow from Corollary 6.11(2).

Corollary 6.21 can be reformulated in terms of actions of the group on subsets of L^p-spaces, as follows.

Corollary 6.23. *(See [29,1,71].) Let G be a second countable, locally compact group.*

(1) *If G has property (T) then for every $p \in (0, 2]$, every continuous action by isometries of G on a subset of a space $L^p(X, \mu)$ has bounded orbits.*
(2) *The group G has the Haagerup property if there exists $p \in (0, 2]$, and a continuous proper action by isometries of G on a subset of some $L^p(X, \mu)$.*

Proof. Both (1) and (2) follow from the fact that if S is a subset of some $L^p(X, \mu)$ and there exists an action of G on S by isometries, $G \times S \to S$, $(g, s) \mapsto g \cdot s$, then for any $s \in S$ the map $\psi(g, h) = \|g \cdot s - h \cdot s\|_p^p$ is a continuous left invariant kernel of type p on G. □

Remark 6.24. In [54] the following result is stated: a second countable locally compact group has the Haagerup property if and only if for some (for all) $p \in (1, 2)$ the group has a proper affine isometric action on $L^p[0, 1]$. The proof in that paper has been completed in an updated version of his preprint appearing on arXiv at [55].

The converse statements in Corollary 6.23 (and their stronger versions, with "every $p \in (0, 2]$" replaced by "there exists $p \in (0, 2]$" in (1), and the opposite replacement done in (2)) follow immediately from the following fact. Given ψ a continuous proper left invariant kernel of type p on G, that is a map $\psi : G \times G \to \mathbb{R}_+$ defined by $\psi(g, h) = \|f(g) - f(h)\|_p^p$, where $f : G \to L^p(X, \mu)$ is continuous, one can define a continuous action by isometries of G on $f(G)$ by $g \cdot f(h) = f(gh)$.

Much stronger versions of the converse statements in Corollary 6.23 are provided by Corollary 1.5. But in order to obtain those, we first need to obtain improved converse statements for $p = 1$. Indeed, for this value of p, the sufficient condition to have property (T) can be weakened: it suffices to look at actions of G on median subspaces of L^1-spaces. Also, Haagerup property implies more for $p = 1$: the existence of a continuous proper action by isometries of G on a median subspace of some L^1-space. Both statements are straightforward consequences of the following result.

Theorem 6.25. *Let G be a separable topological group.*

(1) *If G acts continuously by isometries on a median space (X, dist) and $x \in X$ then $\psi : G \times G \to \mathbb{R}_+$, $\psi(g, g') = \text{dist}(g \cdot x, g' \cdot x)$ is a continuous left invariant kernel of type 1.*
(2) *If $\psi : G \times G \to \mathbb{R}_+$ is a continuous left invariant kernel which is the square root of a kernel of type 2 (hence ψ is a kernel of type 1) then there exists a continuous action by isometries of G on a median space (X, dist), and a point $x \in X$ such that*

$$\psi(g, g') = \text{dist}(g \cdot x, g' \cdot x).$$

We first need to establish equivariant versions of Lemmata 6.14 and 6.15 when X is a group G. In the particular case when the group is countable, Lemma 6.15 has the following equivariant version.

Proposition 6.26. *(See [59].) Let Γ be a countable group.*

(1) *If Γ is endowed with a left invariant structure of space with measured walls $(\Gamma, \mathcal{W}, \mathcal{B}, \mu)$ then $\psi : \Gamma \times \Gamma \to \mathbb{R}_+$ defined by $\psi(g, g') = \mu(\mathcal{W}(g|g'))$ is a left invariant measure definite kernel.*
(2) *If $\psi : \Gamma \times \Gamma \to \mathbb{R}_+$ is the square root of a left invariant conditionally negative definite kernel, then Γ can be endowed with a left invariant structure of space with measured walls $(\Gamma, \mathcal{W}, \mathcal{B}, \mu)$ such that $\psi(g, g') = \mu(\mathcal{W}(g|g'))$.*

Proof. (1) follows immediately from the definition of a measure definite kernel. It appears in [59] as Proposition 1.1.

(2) follows from Proposition 1.4 and the proof of Theorem 2.1 on p. 252 in [59]. □

This implies the following equivariant version of Lemma 6.14.

Lemma 6.27. *Let Γ be a countable group.*

(1) *If Γ acts by isometries on a median space (X, dist), and x is a point in X then $\psi : \Gamma \times \Gamma \to \mathbb{R}_+$, $\psi(g, g') = \mathrm{dist}(g \cdot x, g' \cdot x)$ is a left invariant kernel of type 1.*
(2) *If $\psi : \Gamma \times \Gamma \to \mathbb{R}_+$ is a left invariant kernel which is the square root of a conditionally negative definite kernel (hence ψ is of type 1) then there exists an action by isometries of Γ on a median space (X, dist), and a point $x \in X$ such that $\psi(g, g') = \mathrm{dist}(g \cdot x, g' \cdot x)$.*

Proof of Theorem 6.25. (1) follows from the fact that median spaces isometrically embed in L^1-spaces by Corollary 5.3.

(2) Let Γ be a countable dense subgroup in G. Restrict ψ to Γ and apply Lemma 6.27(2): there exists an action by isometries of Γ on a median space (X, dist) and $x \in X$ such that $\psi(\gamma, \gamma') = \mathrm{dist}(\gamma \cdot x, \gamma' \cdot x)$. The metric completion of (X, dist) is still median by Proposition 2.21, and any isometry of X extends uniquely to an isometry of the completion. We get an action of Γ on a complete median space that still induces the kernel ψ. Thus we may – and will – assume that the median space (X, dist) is already complete.

The map $f : \Gamma \to X$ sending γ to $\gamma \cdot x$ is uniformly continuous since ψ is continuous (we endow $\Gamma \subset G$ with the induced topology). Since X is complete it follows that f extends to a continuous map $G \to X$ still verifying $\psi(g, g') = \mathrm{dist}(f(g), f(g'))$ (for all g, g' in G). As usual the left invariance of ψ implies that $g \cdot f(h) = f(gh)$ defines an action of G by isometries on $f(G)$. And the continuity of ψ implies that the action is continuous.

To end the argument it suffices to prove that the above action of G on $f(G)$ extends to an action by isometries on the median hull of $f(G)$ in X, which we denote by M. Set $M_0 = f(G)$ and then define inductively $M_{i+1} = \{m(x, y, z) \mid (x, y, z) \in (M_i)^3\}$. Clearly M is the ascending union of the M_i's. Every element $g \in G$ defines an isometry $g : M_0 \to M_0$. We first note that there is at most one isometric embedding $\bar{g} : M \to X$ extending $g : M_0 \to M_0$, and $\bar{g}(M) \subset M$. Indeed, since isometries commute with the median map, $\bar{g}$ is completely determined on M_1 and

$\bar{g}(M_1) \subset M_1$, then $\bar{g}$ is completely determined on M_2 and $\bar{g}(M_2) \subset M_2$, and so on. We now prove the existence of such an isometric extension.

Choose a sequence γ_n of elements of Γ converging to g in the topological group G. Then for every $f(h) \in M_0$, $\gamma_n f(h) = f(\gamma_n h)$ converges to $f(gh) = gf(h)$ by continuity of f. We prove by induction on i that for any $m \in M_i$ the sequence $\gamma_n(m)$ converges. This is true for $m \in M_0$. Assume we know that $\gamma_n(p)$ converges for every $p \in M_i$, and let m denote an element of M_{i+1}. Write $m = m(x, y, z)$ with $(x, y, z) \in (M_i)^3$. Since γ_n acts by isometry on the whole space X we have $\gamma_n(m) = m(\gamma_n(x), \gamma_n(y), \gamma_n(z))$. By induction the three sequences $(\gamma_n(x))_{n \geqslant 0}, (\gamma_n(y))_{n \geqslant 0}(\gamma_n(z))_{n \geqslant 0}$ are convergent. By the continuity of the median map (Corollary 2.15) it follows that $(\gamma_n(m))_{n \geqslant 0}$ converges. Denote $\bar{g} : M \to X$ the pointwise limit of γ_n on M. It immediately follows that $\bar{g}$ is an isometric embedding which extends $g : M_0 \to M_0$. By the remarks above we have that $\bar{g}(M) \subset M$.

Using the uniqueness of the extension it is now straightforward to check that the maps $\bar{g}$ are isometries of M (with inverse $\overline{g^{-1}}$), and finally that $g \mapsto \bar{g}$ defines an action of G by isometries on M that extends the action of G on $f(G)$, and thus still induces the kernel ψ. $\square$

Theorem 6.25 allows to obtain some results concerning a structure of space with measured walls on the complex hyperbolic space.

Corollary 6.28 (*Walls in the complex hyperbolic space*). *The complex hyperbolic space* $\mathbb{H}^n_{\mathbb{C}}$ *admits a structure of space with measured walls such that:*

(1) *the induced wall metric is* dist$^{1/2}$, *where* dist *is the hyperbolic distance;*
(2) *the walls are all the convex walls with respect to the metric* dist$^{1/2}$;
(3) $SU(n, 1)$ *acts by isomorphisms on this structure.*

Proof. According to [31] the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ equipped with the metric dist$^{1/2}$ can be embedded into a Hilbert space. It follows, by Theorem 6.8(1), that $\psi : \mathbb{H}^n_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}} \to \mathbb{R}_+$, $\psi(x, y) = $ dist$^{1/2}(x, y)$, is the square root of a kernel of type 2, in the terminology of Definition 6.5. Obviously ψ is continuous and left-invariant with respect to the action of $G = SU(n, 1)$.

Via the identification of $\mathbb{H}^n_{\mathbb{C}}$ with G/K, where $K = SU(n)$, the kernel ψ induces a left invariant pull-back kernel $\psi_G : G \times G \to \mathbb{R}_+$. Theorem 6.25 implies that G acts by isometries on a median space $(X, $ dist$_X)$ such that $\psi_G(g, g') = $ dist$_X(g \cdot x, g' \cdot x)$ for some $x \in X$. It follows easily that the map $g \mapsto g \cdot x$ factorizes to a G-equivariant isometric embedding $gK \mapsto gx$ of $(\mathbb{H}^n_{\mathbb{C}}, $ dist$^{1/2})$ into $(X, $ dist$_X)$. All the required statements then follow from Lemma 3.9 and from Theorem 4.8. $\square$

We now prove the results stated in the introduction.

Proof of Theorem 1.2. By Corollary 6.21 and Remark 6.22, property (T) and a-T-menability for a group G are characterized by properties of continuous left invariant kernels of type 1. By Theorem 6.25(1), continuous actions of G on median spaces induce such kernels. On the other hand, a kernel of type 1 is by Corollary 6.11 also of type 2, hence its square root is defined by a continuous action on a median space, according to Theorem 6.25(2). Theorem 1.2 follows, since bounded kernels correspond to actions with bounded orbits, and proper kernels correspond to proper actions. $\square$

Proof of Theorem 1.3. If a group acts continuously on a median space by isometries then the group acts continuously by automorphisms on the structure of measured walls associated to it, by Theorem 5.1. This and Theorem 1.2 give the direct implication in (2) and the converse implication in (1).

On the other hand, a space with measured walls is a submedian space, by Corollary 5.4, hence a subspace of an L^1-space by Corollary 5.3. Corollary 6.23 then gives the direct implication in (1) and the converse implication in (2). $\quad\square$

Proof of Corollary 1.5. A continuous action of a group G on a space with measured walls $(X, \mathcal{W}, \mu)$ induces by Lemma 3.10 a continuous action by affine isometries on $L^p(\mathcal{H}, \mu_{\mathcal{H}})$ for any $p > 0$, defined by $g \cdot f = \pi_p(g)(f) + \chi_{\sigma_{gx}} - \chi_{\sigma_x}$, where x is an arbitrary point in X.

The hypothesis in case (1) implies that the orbit of the constant function zero, composed of the functions $\chi_{\sigma_{gx}} - \chi_{\sigma_x}, g \in G$, is bounded. This implies that the orbit of x is bounded. It remains to apply Theorem 1.3(1).

(2) If G is a-T-menable then we may assume by Theorem 1.3(2) that G acts on $(X, \mathcal{W}, \mathcal{B}, \mu)$ such that $\mathrm{pdist}_\mu(x, gx) \to \infty$ when $g \to \infty$. Hence the action of G on $L^p(\mathcal{H}, \mu_{\mathcal{H}})$ is proper. $\quad\square$

Proof of Theorem 1.7. The only if part of (1) and the if part of (2) immediately follow from Corollary 6.21 and Proposition 6.16.

The if part of (1) and the only if part of (2) follow from Theorem 1.2 and from Corollary 5.3. $\quad\square$

Acknowledgments

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